15.082 and 6.855

The Multicommodity Flow Problem

On the Multicommodity Flow Problem O-D version

K origin-destination pairs of nodes

$(s_1, t_1), (s_2, t_2), \ldots, (s_K, t_K)$

Network $G = (N, A)$

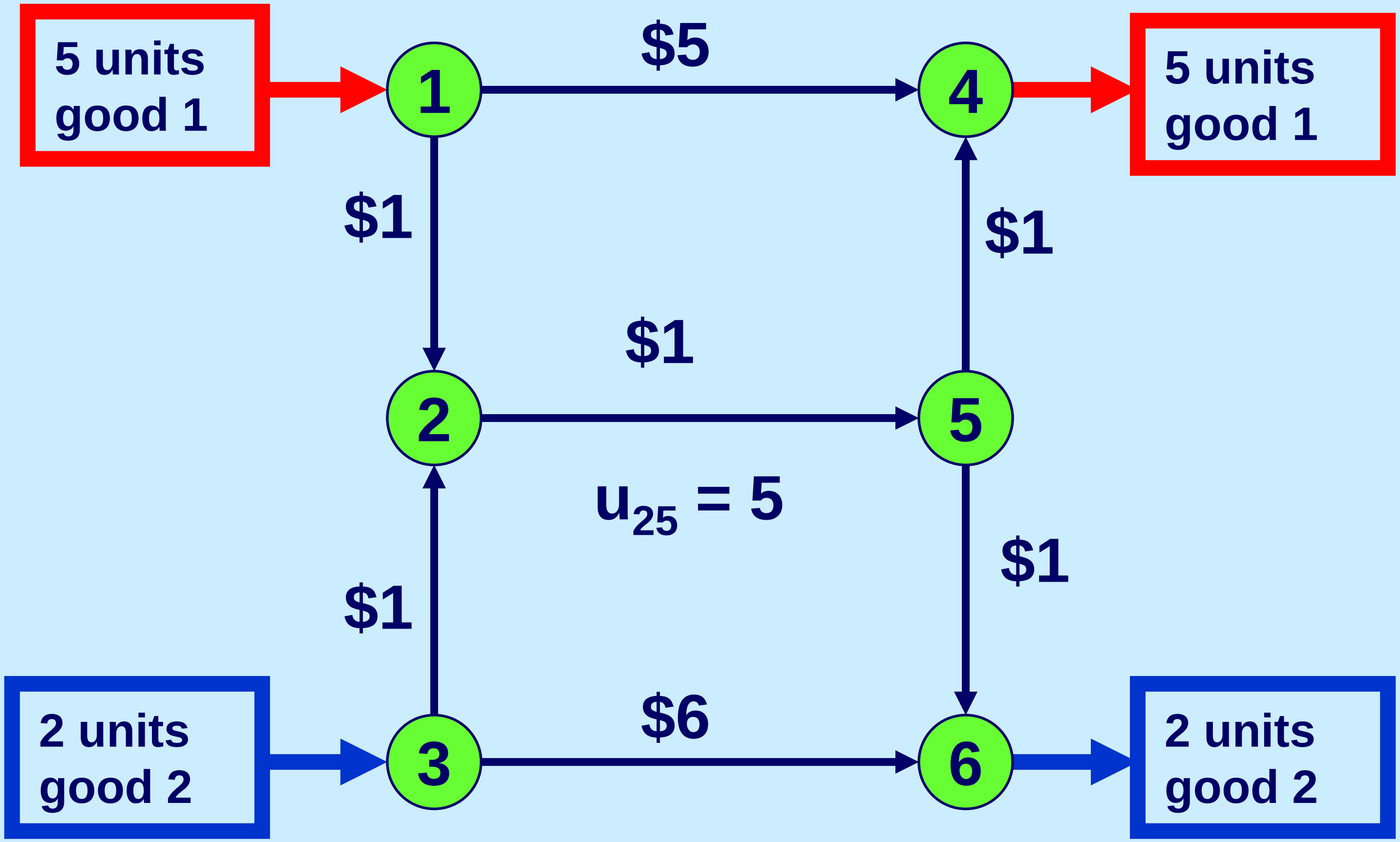
d_k = amount of flow that must be sent from s_k to t_k.

u_{ij} = capacity on (i,j) shared by all commodities

c_{ij}^k = cost of sending 1 unit of commodity k in (i,j)

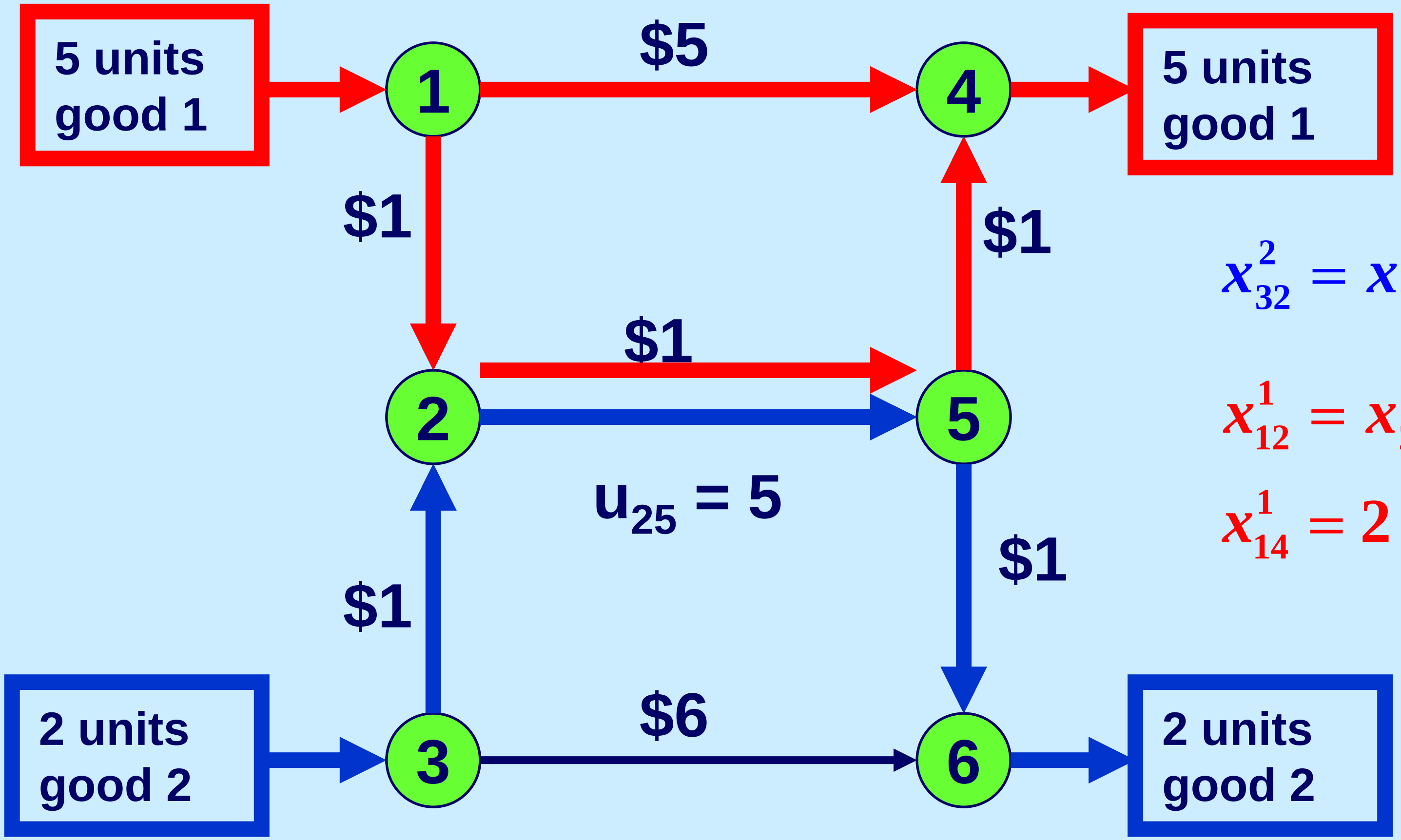
x_{ij}^k = flow of commodity k in (i,j)

A Linear Multicommodity Flow Problem



Quick exercise: determine the optimal multicommodity flow.

A Linear Multicommodity Flow Problem



$$x_{32}^2 = x_{25}^2 = x_{56}^2 = 2$$

$$x_{12}^1 = x_{25}^1 = x_{54}^1 = 3$$

$$x_{14}^1 = 2$$

The Multicommodity Flow LP

Min $\sum_{(i,j)\in A} \sum_{k} c_{ij}^k x_{ij}^k$

$$\sum_j x_{ij}^k - \sum_j x_{ji}^k = \begin{cases} d_k & \text{if } i = s_k \\ -d_k & \text{if } i \in t_k \\ 0 & \textit{otherwise} \end{cases}$$

Supply/ demand constraints

$$\sum_k x_{ij}^k \le u_{ij} \quad \text{for all } (i,j) \in A$$

Bundle constraints

$$x_{ij}^k \ge 0 \quad \forall (i,j) \in A,\ k \in K$$

Assumptions (for now)

- **Homogeneous goods.** Each unit flow of commodity k on (i,j) uses up one unit of capacity on (i,j).

- **No congestion.** Cost is linear in the flow on (i,j) until capacity is totally used up.

- **Fractional flows.** Flows are permitted to be fractional.

- **OD pairs.** Usually a commodity has a single origin and single destination.

Application areas

Type of Network	Nodes	Arcs	Flow
Communic. Networks	O-D pairs for messages	Transmission lines	message routing
Computer Networks	storage dev. or computers	Transmission lines	data, messages
Railway Networks	yard and junction pts.	Tracks	Trains
Distribution Networks	plants warehouses,...	highways railway tracks etc.	trucks, trains, etc

On Fractional Flows

- **In general, multicommodity flow problems have fractional flows, even if all data is integral.**

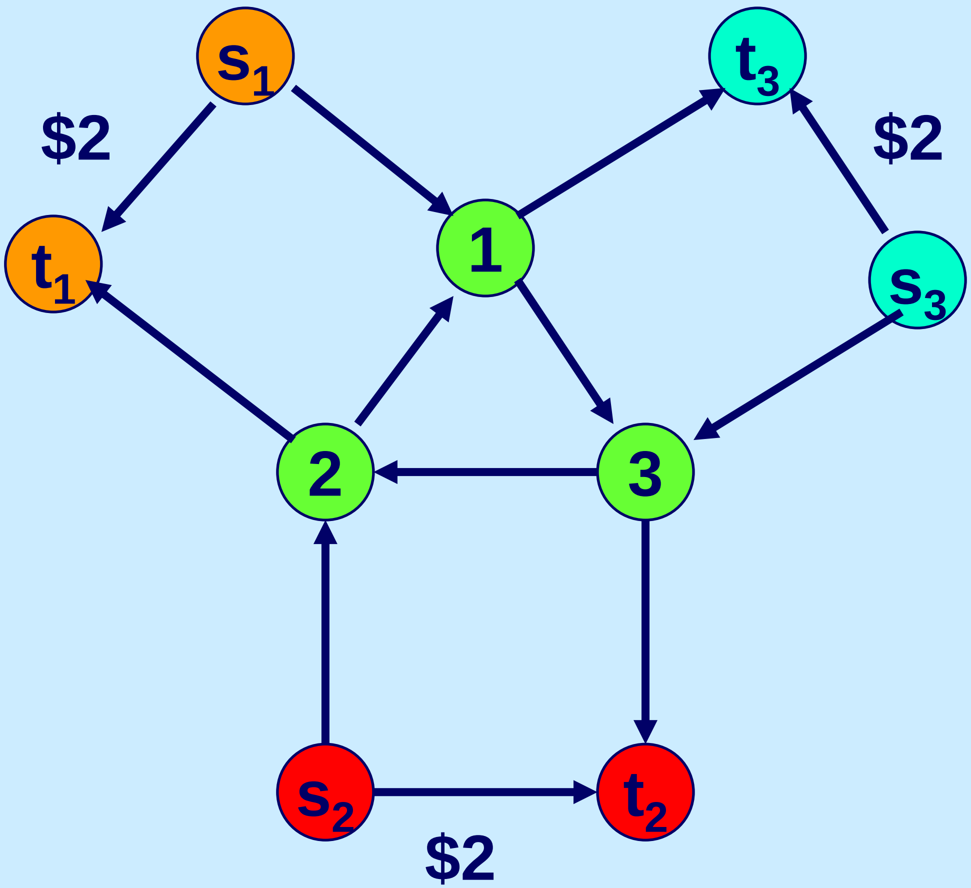
- **The integer multicommodity flow problem is difficult to solve to optimality.**

A fractional multicommodity flow

u_{ij} = 1 for all arcs

c_{ij} = 0 except as listed.

1 unit of flow must be sent from s_i to t_i for i = 1, 2, 3.

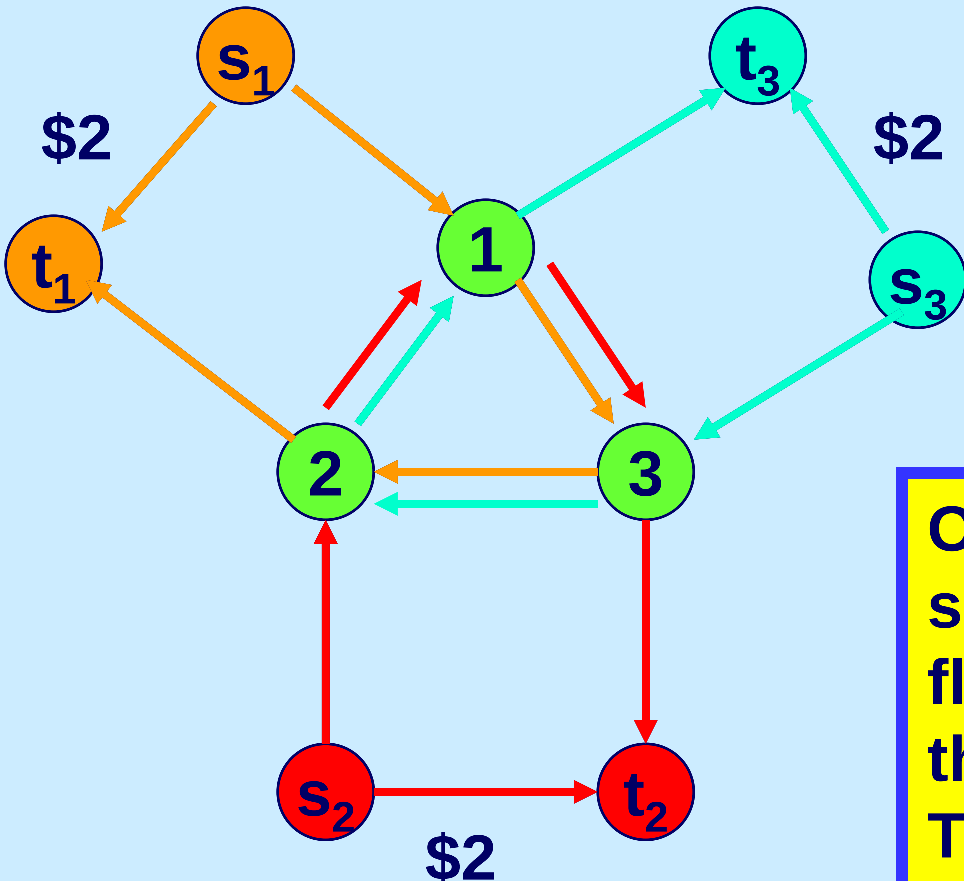


A fractional multicommodity flow

$u_{ij} = 1$ for all arcs

$c_{ij} = 0$ except as listed.

1 unit of flow must be sent from s_i to t_i for i = 1, 2, 3.



Optimal solution: send ½ unit of flow in each of these 15 arcs. Total cost = $3.

Decomposition based approaches

Price directed decomposition.

Focus on prices or tolls on the arcs. Then solve the problem while ignoring the capacities on arcs.

Resource directive decomposition.

Allocate flow capacity among commodities and solve

Simplex based approaches

Try to speed up the simplex method by exploiting the structure of the MCF problem.

A formulation without OD pairs

Minimize $\sum_{1 \le k \le K} c^k x^k$ (17.1a)

subject to $\sum_{1 \le k \le K} x_{ij}^k \le u_{ij}$ for all $(i, j) \in A$ (17.1b)

$Nx^k = b^k$ for $k = 1, 2, ..., k$ (17.1c)

$0 \le x_{ij}^k \le u_{ij}^k$ for all $(i, j) \in A$ for $k = 1, 2, ..., k$ (17.1d)

Optimality Conditions: Partial Dualization

Theorem. The multicommodity flow $x = (x^k)$ is an optimal multicommodity flow for (17) if there exists non-negative prices $w = (w_{ij})$ on the arcs so that the following is true

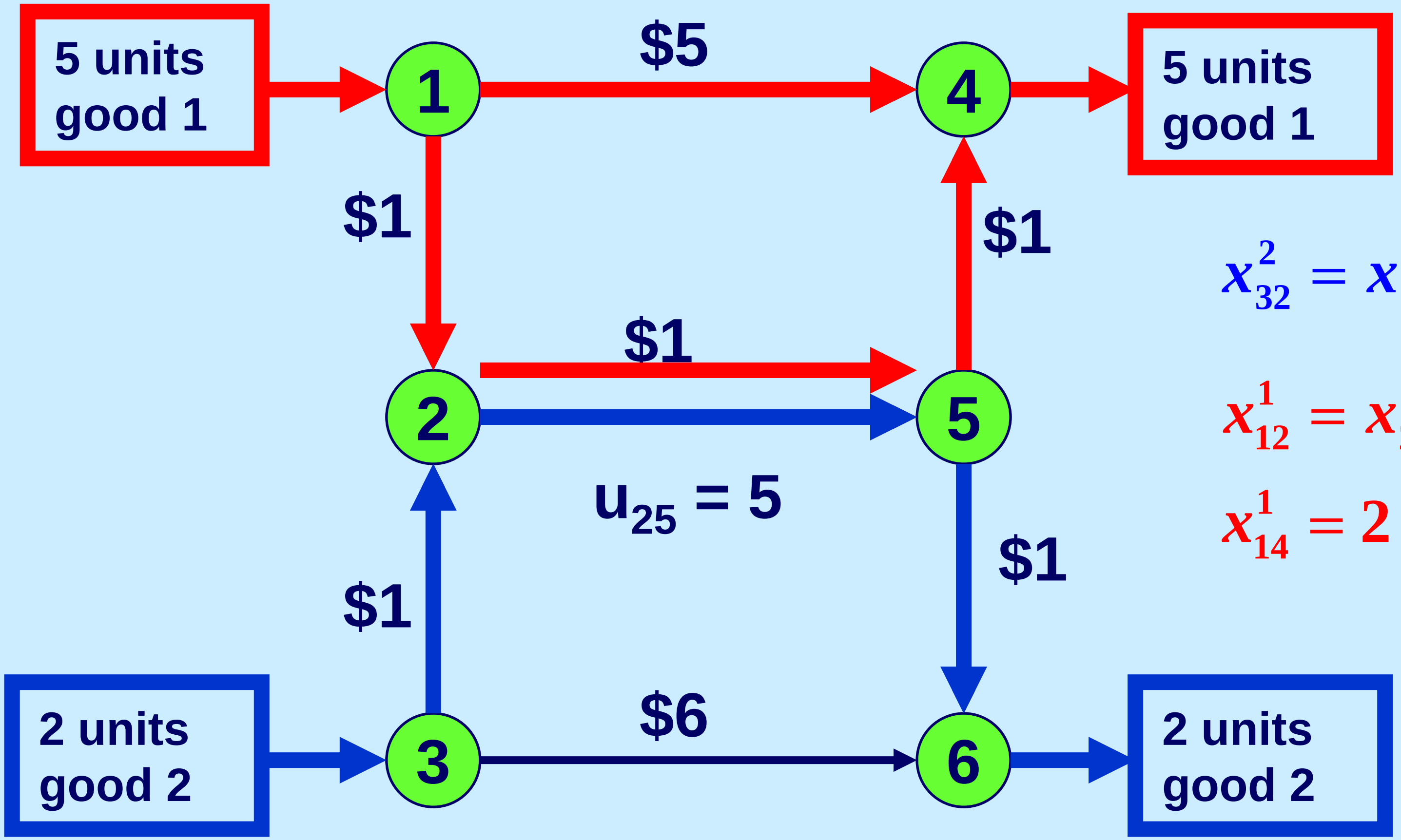
1. If $w_{ij} > 0$, then $\sum_k x_{ij}^k = u_{ij}$

2. The flow x^k is optimal for the k-th commodity if c^k is replaced by $c^{w,k}$, where

$$c_{ij}^{w,k} = c_{ij}^k + w_{ij}$$

Recall: x^k is optimal for the k-th commodity if there is no negative cost cycle in the kth residual network.

A Linear Multicommodity Flow Problem



$x^2_{32} = x^2_{25} = x^2_{56} = 2$

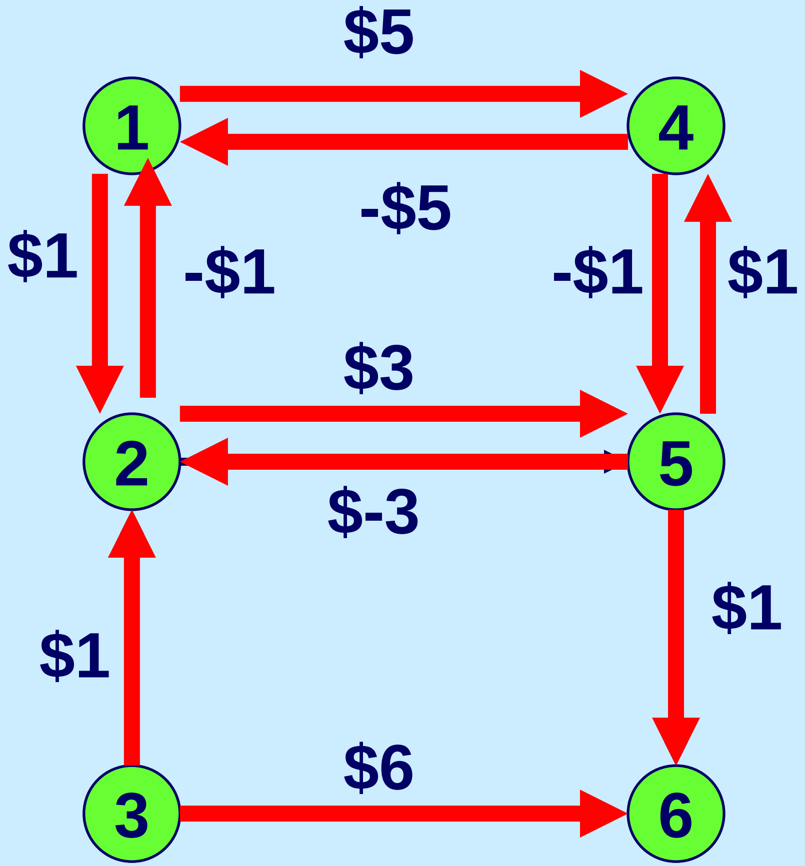
$x^1_{12} = x^1_{25} = x^1_{54} = 3$

$x^1_{14} = 2$

Set $w_{2,5} = 2$

Create the residual networks

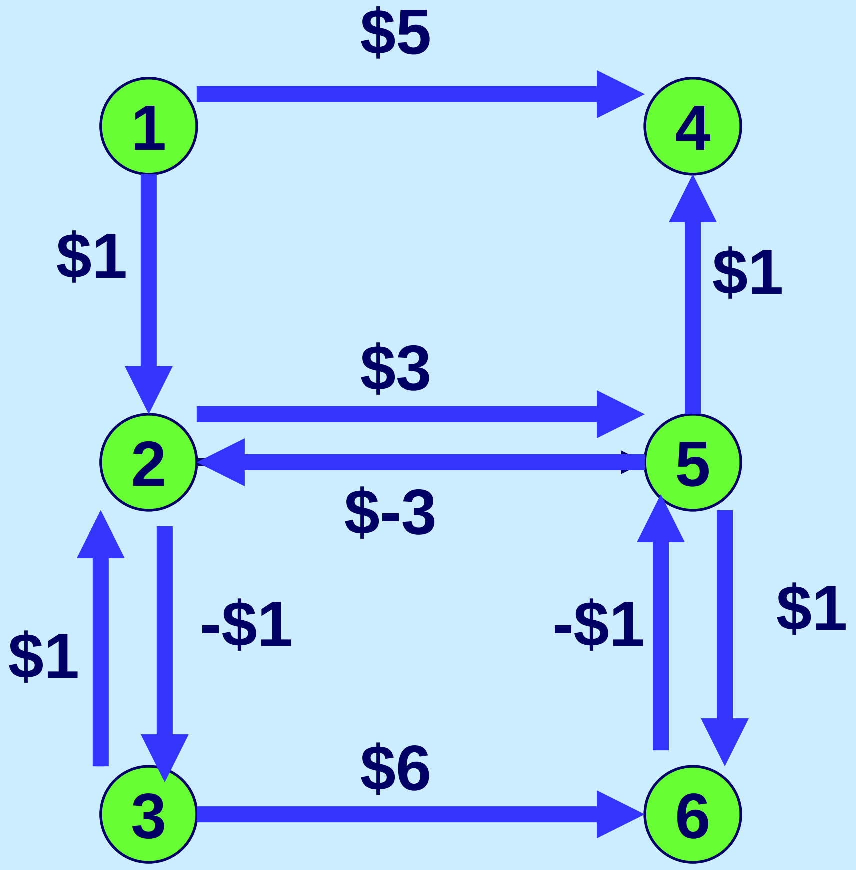
The residual network for commodity 1



Set $w_{2,5}$ = $2

There is no negative cost cycle.

The residual network for commodity 2



Set $w_{2,5}$ = $2

There is no negative cost cycle.

Optimality Conditions: full dualization

One can also define node potentials π so that the reduced cost

$$c_{ij}^{\pi,k} = c_{ij}^{k} + w_{ij} - \pi_i^k + \pi_j^k \geq 0$$

$$\text{for all } (i,j) \in A \text{ and } k = 1, \ldots, K$$

This combines optimality conditions for min cost flows with the partial dualization optimality conditions for multicommodity flows.

Lagrangian relaxation for multicommodity flows

$$\text{Min} \quad \sum_{(i,j)\in A} \sum_k c_{ij}^k x_{ij}^k$$

$$\sum_j x_{ij}^k - \sum_j x_{ji}^k = \begin{cases} d_k & \text{if } i = s_k \\ -d_k & \text{if } i \in t_k \\ 0 & \textit{otherwise} \end{cases}$$

Supply/ demand constraints

$$\sum_k x_{ij}^k \le u_{ij} \quad \text{for all } (i,j) \in A$$

Bundle constraints

$$x_{ij}^k \ge 0 \quad \forall (i,j) \in A, \; k \in K$$

Lagrangian relaxation for multicommodity flows

Min $\sum_{(i,j)\in A} \sum_{k} c_{ij}^{k} x_{ij}^{k} + \sum_{(i,j)\in A} w_{ij} (\sum_{k} x_{ij}^{k} - u_{ij})$

$$\sum_{j} x_{ij}^{k} - \sum_{j} x_{ji}^{k} = \begin{cases} d_k & \text{if } i = s_k \\ -d_k & \text{if } i \in t_k \\ 0 & \text{otherwise} \end{cases}$$

Supply/ demand constraints

$$x_{ij}^{k} \geq 0 \quad \forall (i,j) \in A,\ k \in K$$

Penalize the bundle constraints.

Relax the bundle constraints.

Lagrangian relaxation for multicommodity flows

$$L(w) = \text{Min} \quad \sum_{(i,j)\in A} \sum_{k} (c_{ij}^k + w_{ij}) x_{ij}^k \; - \sum_{(i,j)\in A} w_{ij} u_{ij}$$

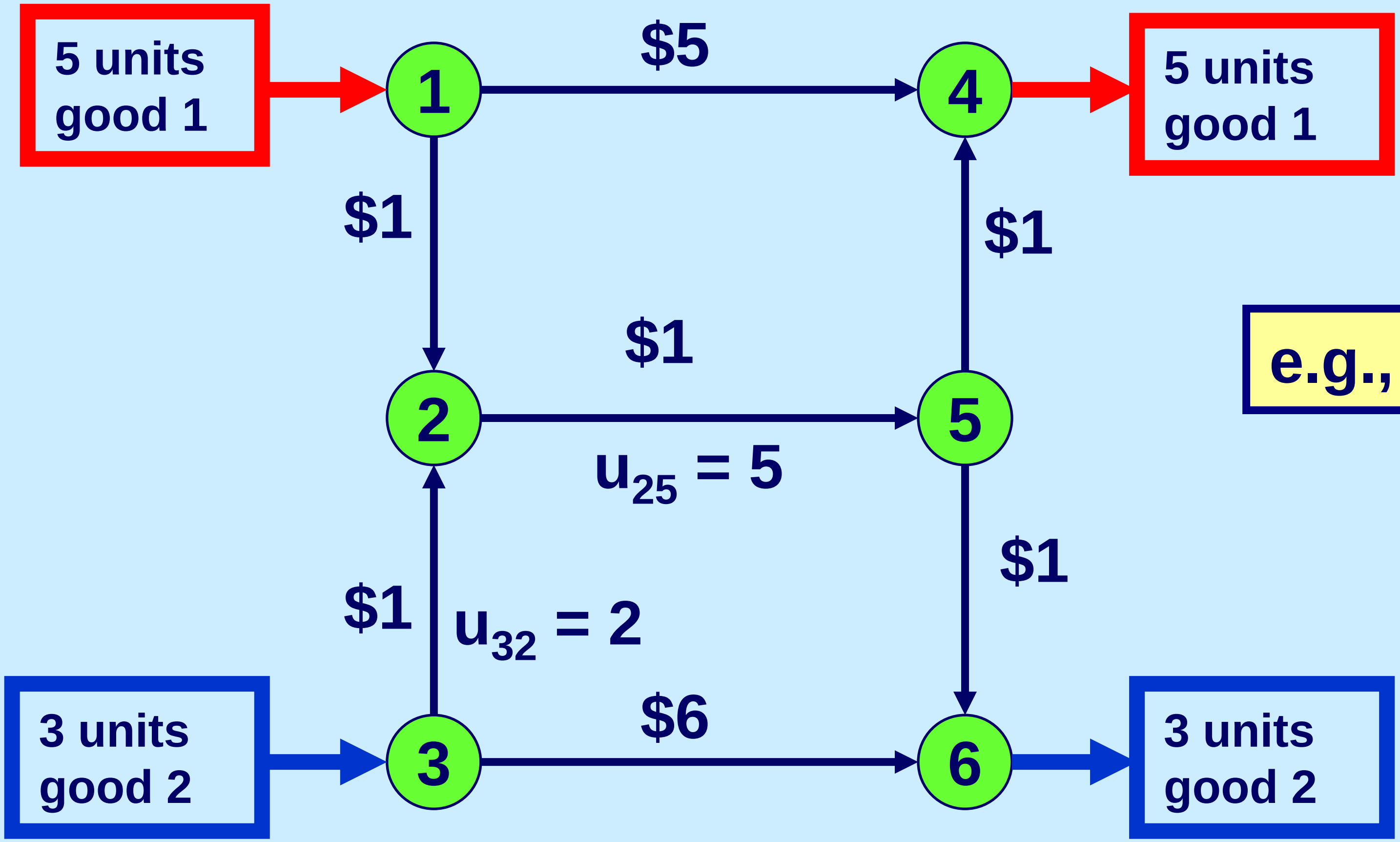
$$\sum_{j} x_{ij}^k - \sum_{j} x_{ji}^k = \begin{cases} d_k & \text{if } i = s_k \\ -d_k & \text{if } i \in t_k \\ 0 & \text{otherwise} \end{cases}$$

Supply/ demand constraints

$$x_{ij}^k \geq 0 \quad \forall (i,j) \in A, \; k \in K$$

Simplify the objective function.

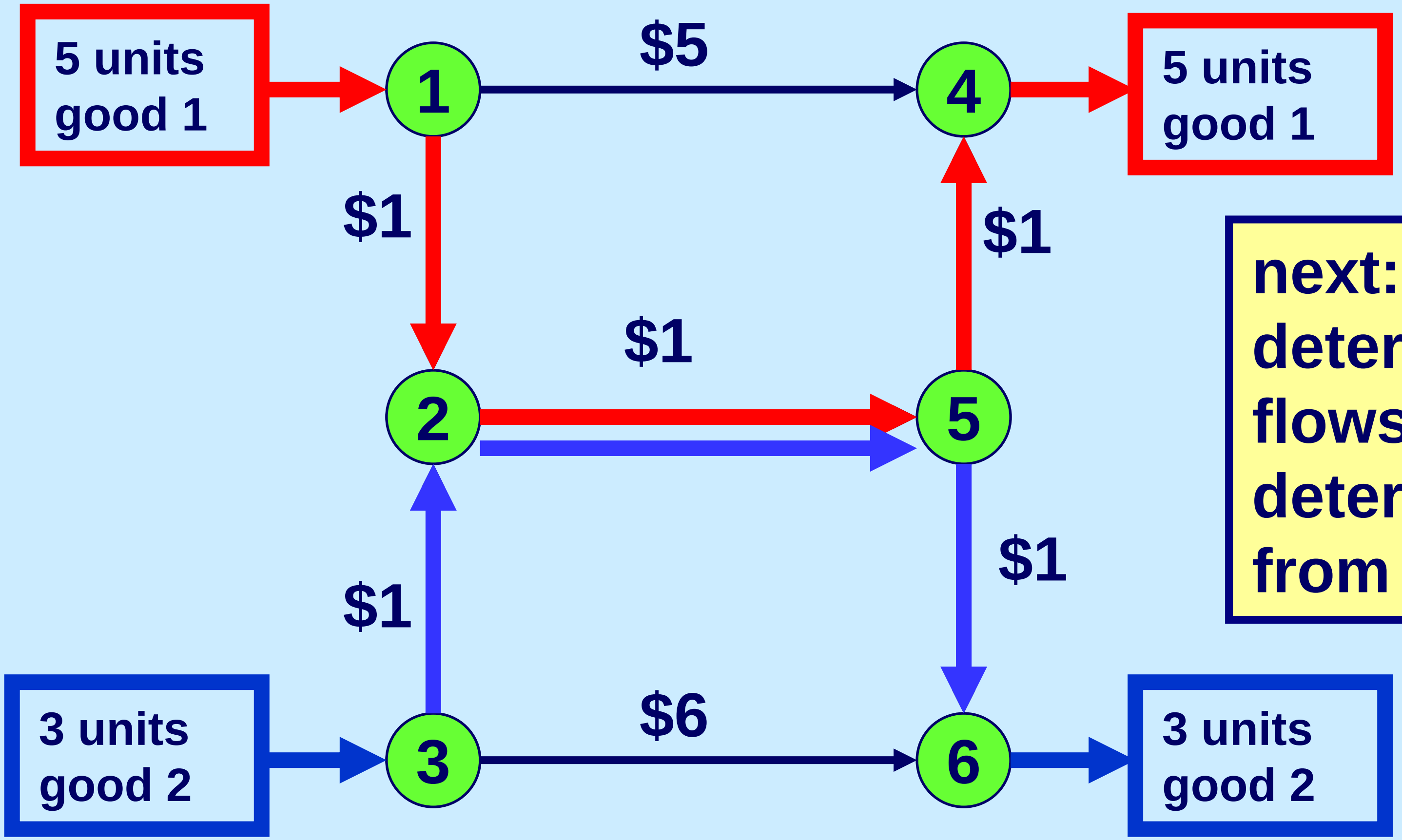
Subgradient Optimization for solving the Lagrangian Multiplier Problem



e.g., set $w^0 = 0$.

Choose an initial value w^0 of the "tolls" w, and find the optimal solution for L(w).

Subgradient Optimization for solving the Lagrangian Multiplier Problem



next: determine the flows, and then determine w^1 from w^0

The flow on (2,5) = 8 > u_{25} = 5.

The flow on (3,2) = 3 > u_{32} = 2.

Choosing a search direction

$$r^{+} = \max(0, r)$$

$$y_{ij} = \sum_k x_{ij}^k$$ = flow in arc (i,j)

$$w_{ij}^{q+1} = [w_{ij}^q + \theta_q (y_{ij} - u_{ij})]^+$$

$(y-u)^+$ is called the ***search direction***.

$$w_{25}^1 = [w_{25}^0 + \theta_0 (8 - 5)]^+ = 3\theta_0$$

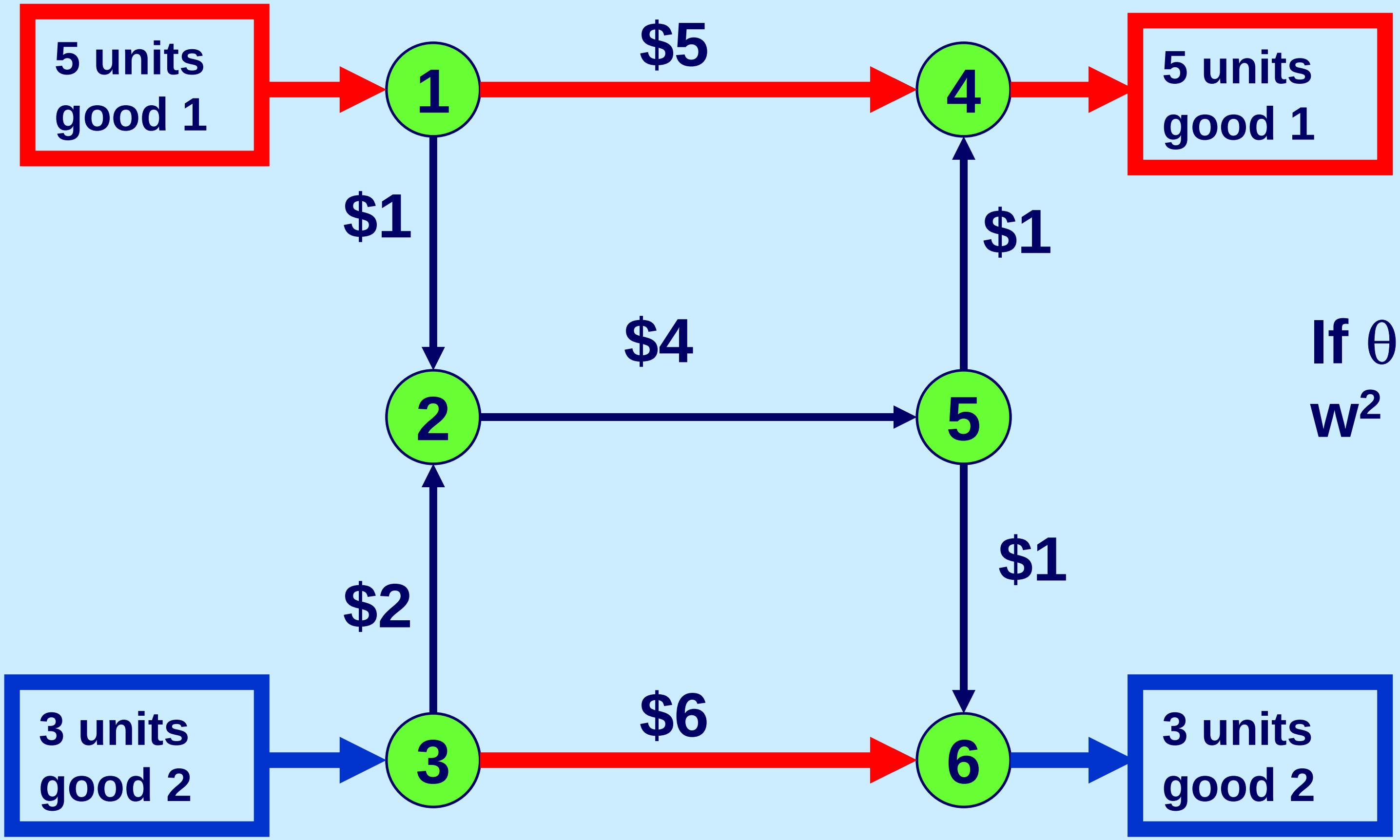
θ_q is called the ***step size***.

$$w_{32}^1 = [w_{32}^0 + \theta_0 (3 - 2)]^+ = \theta_0$$

So, if we choose θ_0 = 1, then $w_{25}^1 = 3$ and $w_{32}^1 = 1$

Then solve $L(w^1)$.

Solving $L(w^1)$



If $\theta^1 = 1$, then $w^2 = 0$.

$$w_{25}^2 = [w_{25}^1 + \theta_1(0-5)]^+ = [3-5\theta_1]^+$$

$$w_{32}^2 = [w_{32}^1 + \theta_1(0-2)]^+ = [1-2\theta_1]^+$$

Comments on the step size

- The search direction is a good search direction.
- But the step size must be chosen carefully.
- Too large a step size and the solution will oscillate and not converge
- Too small a step size and the solution will not converge to the optimum.

On choosing the step size

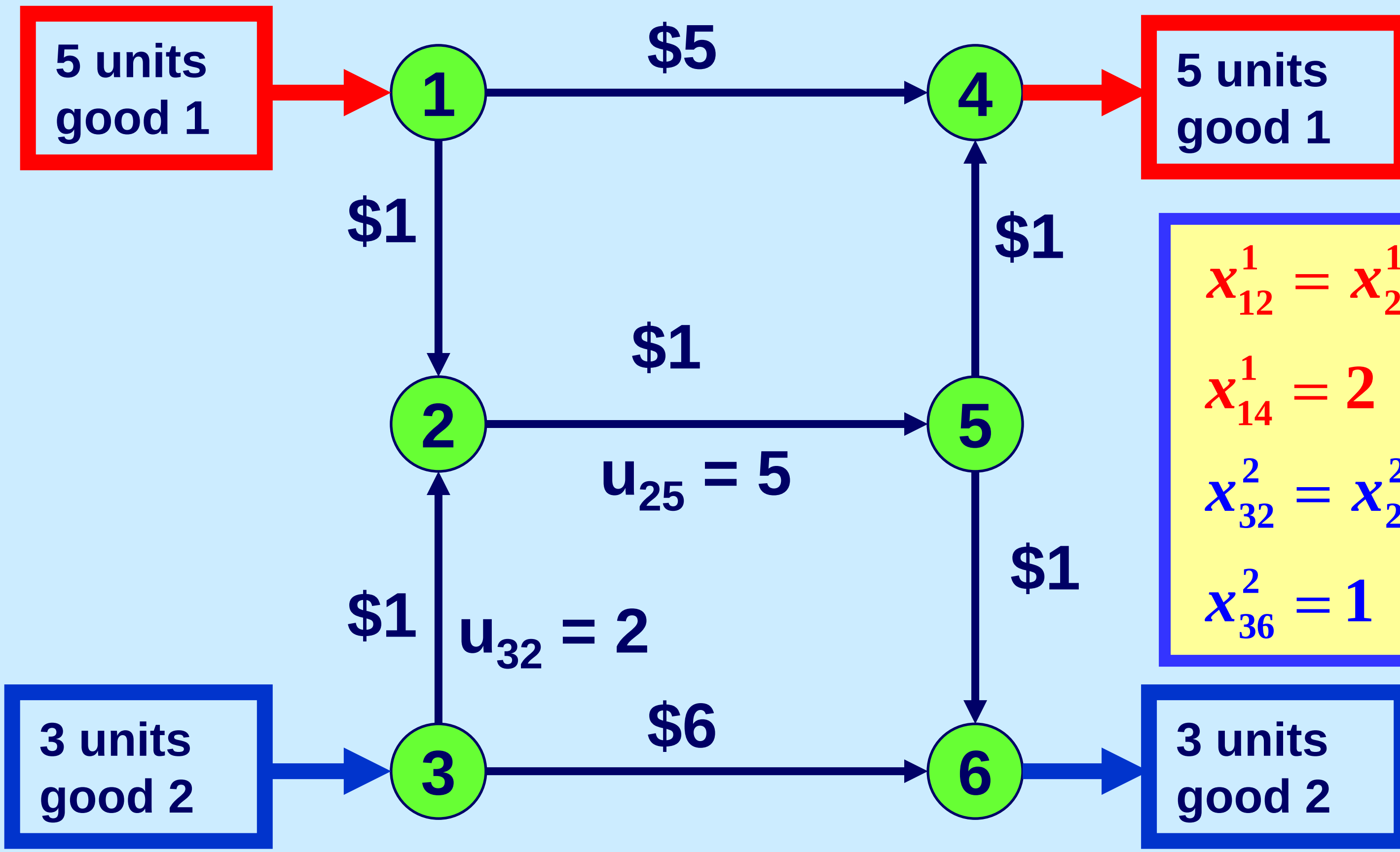
The step size θ_q should be chosen so that

$$\lim_{q \to \infty} \theta_q = 0 \quad \text{and} \quad \sum_{q=1}^{\infty} \theta_q = \infty \qquad (1)$$

e.g., take θ_q = 1/q.

***Theorem*. If the step size is chosen as on the previous slides, and if (θ_q) satisfies (1), then the w^q converges to the optimum for the Lagrangian dual.**

The optimal multipliers and flows.



$$x^1_{12} = x^1_{25} = x^1_{54} = 3$$

$$x^1_{14} = 2$$

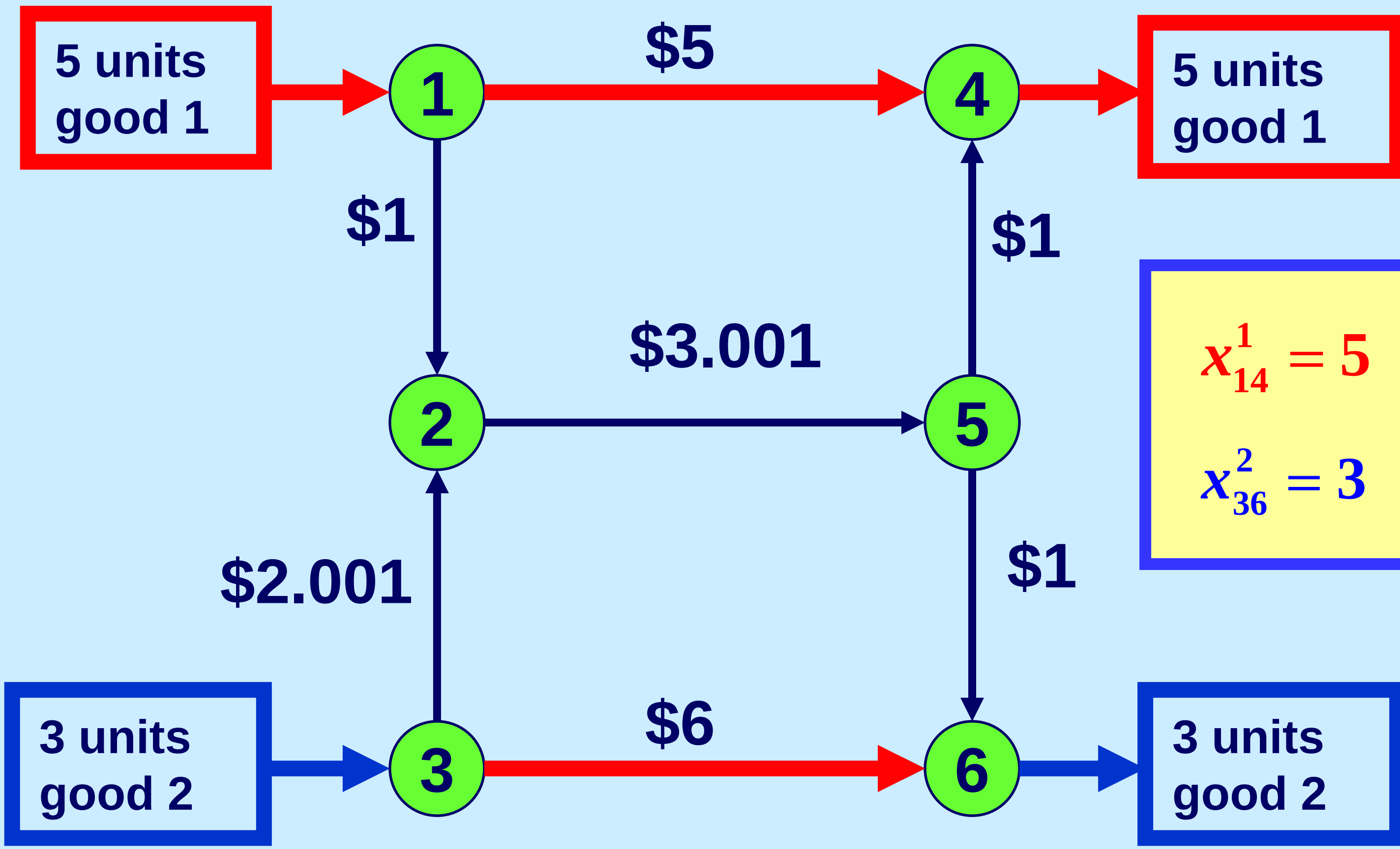
$$x^2_{32} = x^2_{25} = x^2_{56} = 2$$

$$x^2_{36} = 1$$

$$\lim_{q \to \infty} w^q_{32} = 1$$

$$\lim_{q \to \infty} w^q_{25} = 2$$

Suppose that $w_{32} = 1.001$ and $w_{25} = 2.001$



$$x_{14}^{1} = 5$$

$$x_{36}^{2} = 3$$

Conclusion: **Near Optimal Multipliers do not always lead to near optimal (or even feasible) flows.**

Summary of MCF

- Applications
- Optimality Conditions
- Lagrangian Relaxation
  - subgradient optimization
- Next Lecture: Column Generation and more