15.082 and 6.855

Multicommodity Flows 2

On the Multicommodity Flow Problem
O-D version

K origin-destination pairs of nodes
$(s_1, t_1), (s_2, t_2), \ldots, (s_K, t_K)$

Network G = (N, A)

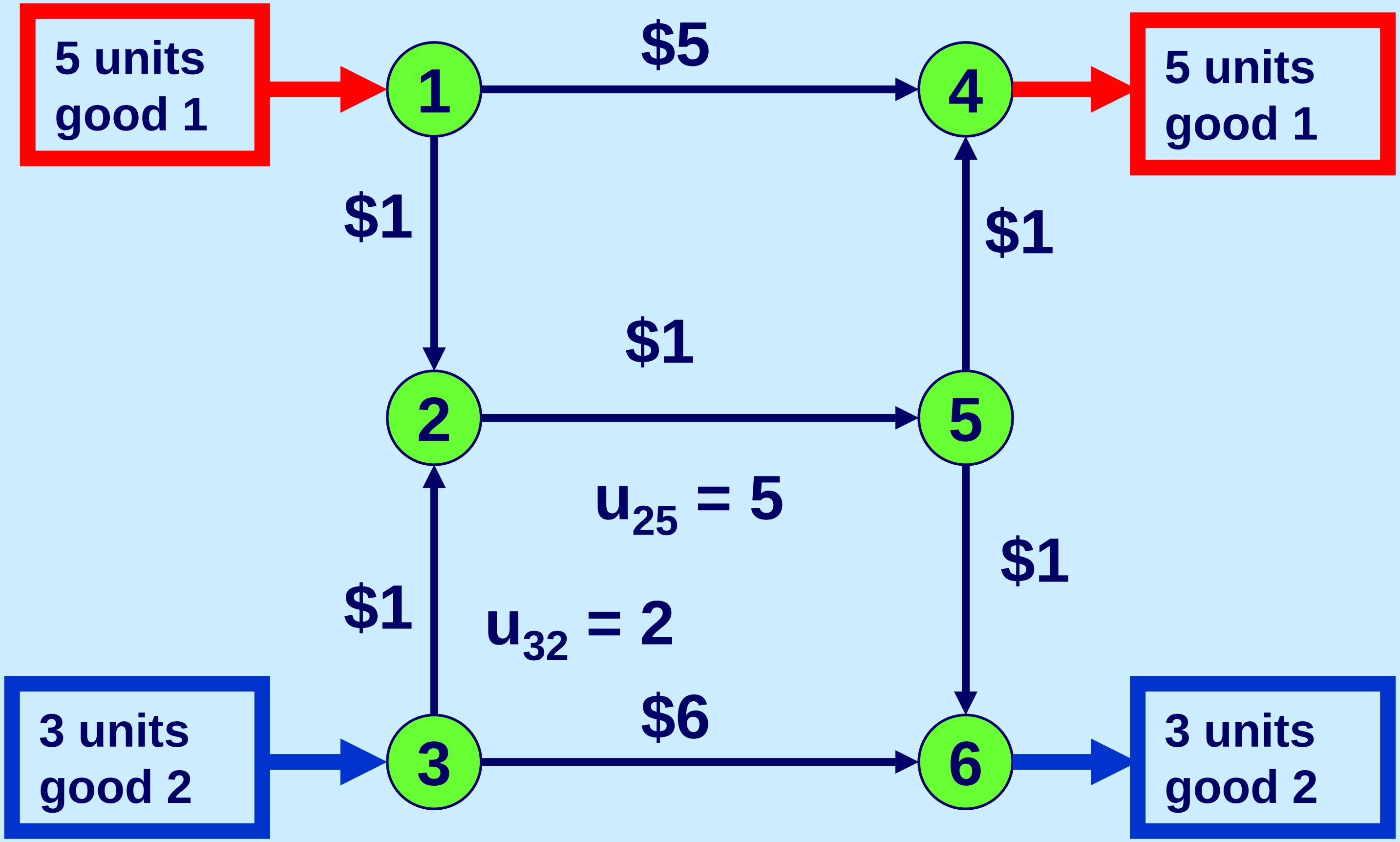
d_k = amount of flow that must be sent from s_k to t_k.

u_{ij} = capacity on (i,j) shared by all commodities

c_{ij}^k = cost of sending 1 unit of commodity k in (i,j)

x_{ij}^k = flow of commodity k in (i,j)

A Linear Multicommodity Flow Problem



The Multicommodity Flow LP

Min $$\sum_{(i,j)\in A} \sum_{k} c_{ij}^{k} x_{ij}^{k}$$

$$\sum_{j} x_{ij}^{k} - \sum_{j} x_{ji}^{k} = \begin{cases} d_k & \text{if } i = s_k \\ -d_k & \text{if } i \in t_k \\ 0 & \textit{otherwise} \end{cases}$$

Supply/ demand constraints

$$\sum_{k} x_{ij}^{k} \le u_{ij} \quad \text{for all } (i,j) \in A$$

Bundle constraints

$$x_{ij}^{k} \ge 0 \quad \forall (i,j) \in A,\ k \in K$$

Assumptions (for now)

- **Homogeneous goods.** Each unit flow of commodity k on (i,j) uses up one unit of capacity on (i,j).
- **No congestion.** Cost is linear in the flow on (i,j) until capacity is totally used up.
- **Fractional flows.** Flows are permitted to be fractional.
- **OD pairs.** Usually a commodity has a single origin and single destination.

Optimality Conditions: Partial Dualization

Theorem. The multicommodity flow $x = (x^k)$ is an optimal multicommodity flow for (17) if there exists non-negative prices $w = (w_{ij})$ on the arcs so that the following is true

1. If $w_{ij} > 0$, then $\sum_k x_{ij}^k = u_{ij}$

2. The flow x^k is optimal for the k-th commodity if c^k is replaced by $c^{w,k}$, where

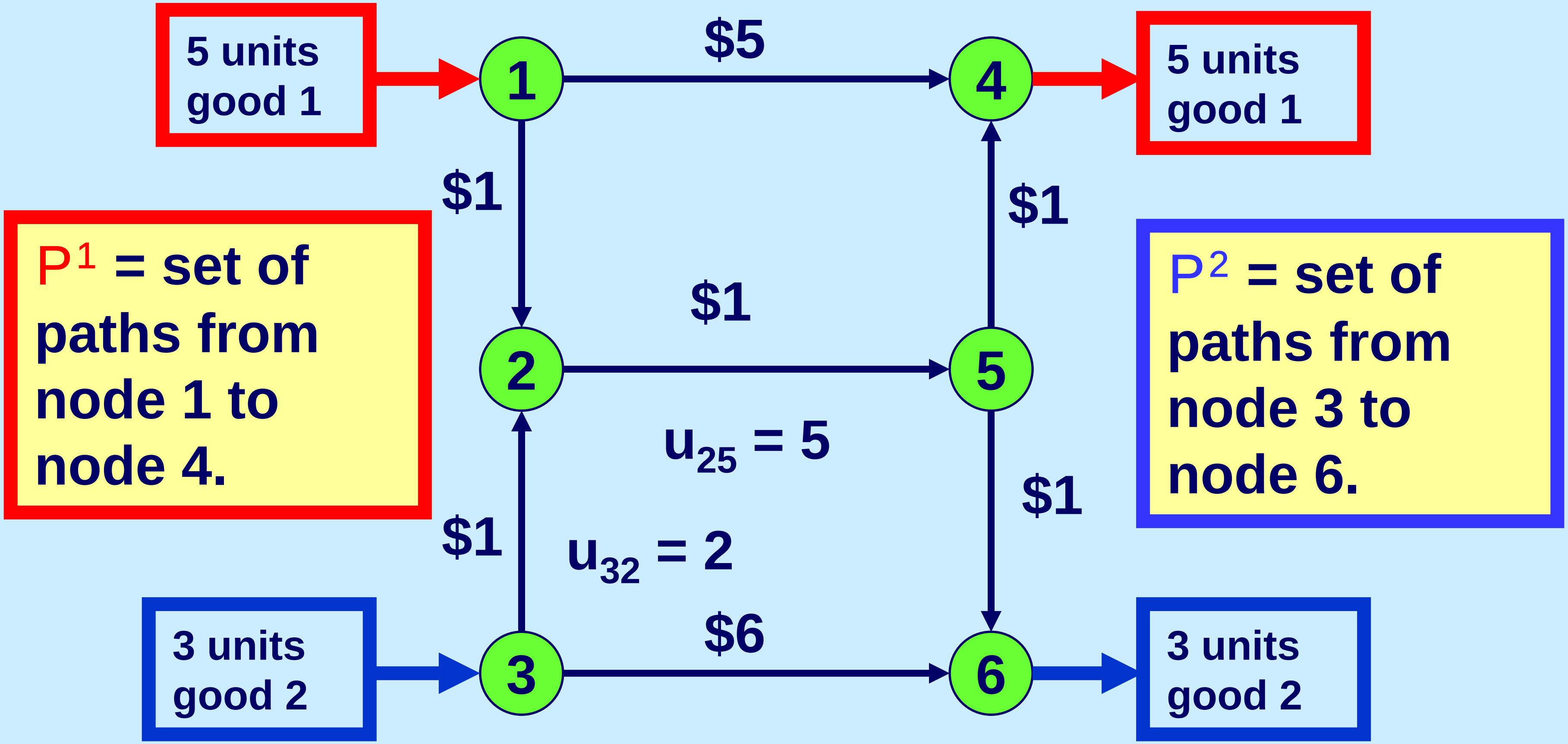
$$c_{ij}^{w,k} = c_{ij}^k + w_{ij}$$

Recall: x^k is optimal for the k-th commodity if there is no negative cost cycle in the kth residual network.

Another approach: path-based approach

- Represent flows from s_k to t_k as the sum of flows in paths.
- The resulting LP may have an exponential number of columns
- Use “column generation” to solve the LP.

A Linear Multicommodity Flow Problem



P^1 = set of paths from node 1 to node 4.

P^2 = set of paths from node 3 to node 6.

P^1 = {1-4, 1-2-5-4}

P^2 = {3-6, 3-2-5-6}

A path based formulation

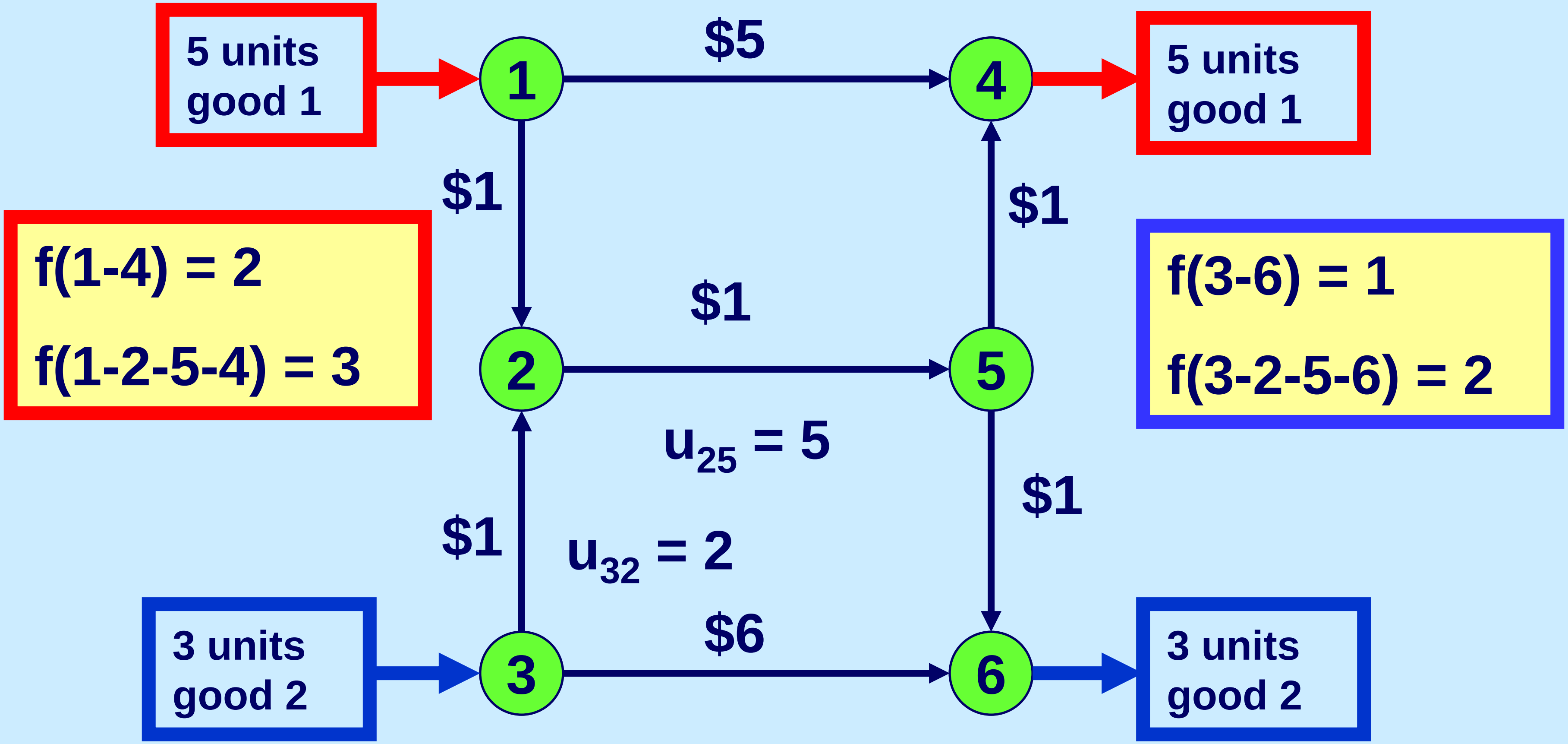
f(P) = flow in path P

c(P) = cost of path P

c(1-4) = 5
c(1-2-5-4) = 3
c(3-6) = 6
c(3-2-5-6) = 3

$$
\begin{array}{lrcl}
\text{Minimize} & \multicolumn{3}{l}{5\,f(1\text{-}4) + 3\,f(1\text{-}2\text{-}5\text{-}4) + 6\,f(3\text{-}6) + 3\,f(3\text{-}2\text{-}5\text{-}6)} \\
\text{subject to} & f(1\text{-}4) + f(1\text{-}2\text{-}5\text{-}4) & = & 5 \\
 & f(3\text{-}6) + f(3\text{-}2\text{-}5\text{-}6) & = & 3 \\
 & f(1\text{-}2\text{-}5\text{-}4) + f(3\text{-}2\text{-}5\text{-}6) & \le & u_{25} = 5 \\
 & f(3\text{-}2\text{-}5\text{-}6) & \le & u_{32} = 2 \\
 & \multicolumn{3}{c}{f(P) \ge 0 \text{ for all paths } P}
\end{array}
$$

Optimal solution for the path based version



5 units
good 1

$f(1\text{-}4) = 2$

$f(1\text{-}2\text{-}5\text{-}4) = 3$

5 units
good 1

$f(3\text{-}6) = 1$

$f(3\text{-}2\text{-}5\text{-}6) = 2$

$u_{25} = 5$

$u_{32} = 2$

3 units
good 2

3 units
good 2

The path based LP can be solved using the simplex method.

General formulation for the path based version

Let P^k = set of directed paths from s_k to t_k

Let $c^k(P)$ = cost of path $P \in P^k$.

$$\text{Let } \delta_{ij}(P) = \begin{cases} 1 & \text{if } (i,j) \in P \\ 0 & \text{otherwise} \end{cases}$$

Let f(P) = flow on path P.

Master Problem

$$\text{Minimize} \quad \sum_k \sum_{P \in P^k} c^k(P) f(P)$$

$$\sum_k \sum_{P \in P^k} \delta_{ij}(P) f(P) \le u_{ij} \qquad \text{for all } (i,j) \in A$$

$$\sum_{P \in P^k} f(P) = d^k \qquad \text{for } k = 1 \text{ to } K$$

$$f(P) \ge 0 \qquad \text{for } P \in \bigcup_{k=1}^{K} P^k$$

$$\text{Minimize} \quad \sum_{k} \sum_{P \in P^k} c^k(P) f(P)$$

$$\sum_{k} \sum_{P \in P^k} \delta_{ij}(P) f(P) \le u_{ij} \qquad \text{for all } (i, j) \in A$$

$$\sum_{P \in P^k} f(P) = d^k$$

$$f(P) \ge 0 \qquad \text{for } P \in \bigcup_{k=1}^{K} P^k$$

__bundle constraints__: one for each capacitated arc.

__supply demand constraints__: one for commodity.

__variables:__ one for each path from origin to destination

Column Generation Approach: Generate paths as needed

Let S^k = subset of P^k = directed paths from s_k to t_k

Let $c^k(P)$ = cost of path $P \in S^k$.

Let f(P) = flow on path P.

$$\text{Let } \delta_{ij}(P) = \begin{cases} 1 & \text{if } (i,j) \in P \\ 0 & \text{otherwise} \end{cases}$$

Restricted Master Problem

$$\text{Minimize} \quad \sum_{k} \sum_{P \in S^k} c^k(P) f(P)$$

$$\sum_{k} \sum_{P \in S^k} \delta_{ij}(P) f(P) \le u_{ij} \qquad \text{for all } (i,j) \in A$$

$$\sum_{P \in S^k} f(P) = d^k \qquad \text{for } k = 1 \text{ to } K$$

$$f(P) \ge 0 \qquad \text{for } P \in \bigcup_{k=1}^{K} S^k$$

Solving the Master Problem

1. **Initialize S^k for each k.**
2. **Solve the restricted master problem for paths in $S = \cup_k S^k$ obtaining solution $x = (x^k)$.**
3. **Check to see if x is optimal for the master problem. If not, find new paths to add to S and return to step 2.**

Towards Optimality conditions for the master and restricted master

Restricted Master Problem

$$\text{Minimize} \quad \sum_{k} \sum_{P \in S^k} c^k(P) f(P)$$

$$\sum_{k} \sum_{P \in S^k} \delta_{ij}(P) f(P) \leq u_{ij} \qquad \text{for all } (i,j) \in A$$

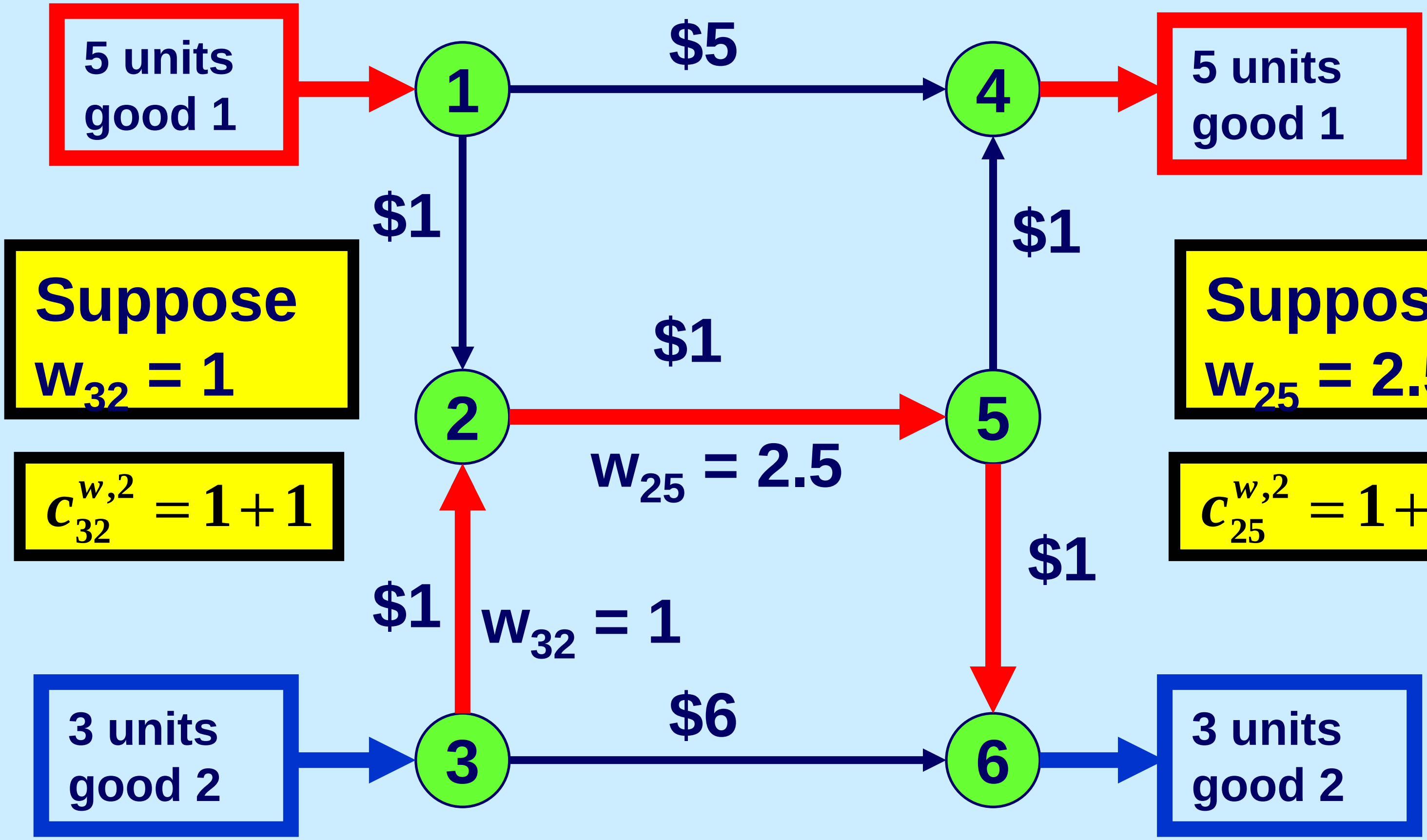
$$\sum_{P \in S^k} f(P) = d^k$$

$$f(P) \geq 0 \qquad \text{for } P \in \bigcup_{k=1}^{K} S^k$$

Let w = (w_{ij}) be a set of non-negative tolls on the arcs.

$$c_{ij}^{w,k} = c_{ij}^{k} + w_{ij} \qquad\qquad c^{w,k}(P) = \sum_{(i,j) \in P} c_{ij}^{w,k}$$

Illustration of definitions



Suppose
$w_{32} = 1$

$$c_{32}^{w,2} = 1 + 1$$

Suppose
$w_{25} = 2.5$

$$c_{25}^{w,2} = 1 + 2.5 = 3.5$$

$$c^{w,2}(3-2-5-6) = 2 + 3.5 + 1 = 6.5$$

Optimality conditions for the restricted master

A flow f is optimal for the restricted master if it is feasible, and if there is a non-negative vector w of tolls on arc so that the following is true:

1. $w_{ij} > 0 \Rightarrow \sum_{k} \sum_{P \in S^k} \delta_{ij}(P) f(P) = u_{ij}$ for all (i,j) $\in$ A

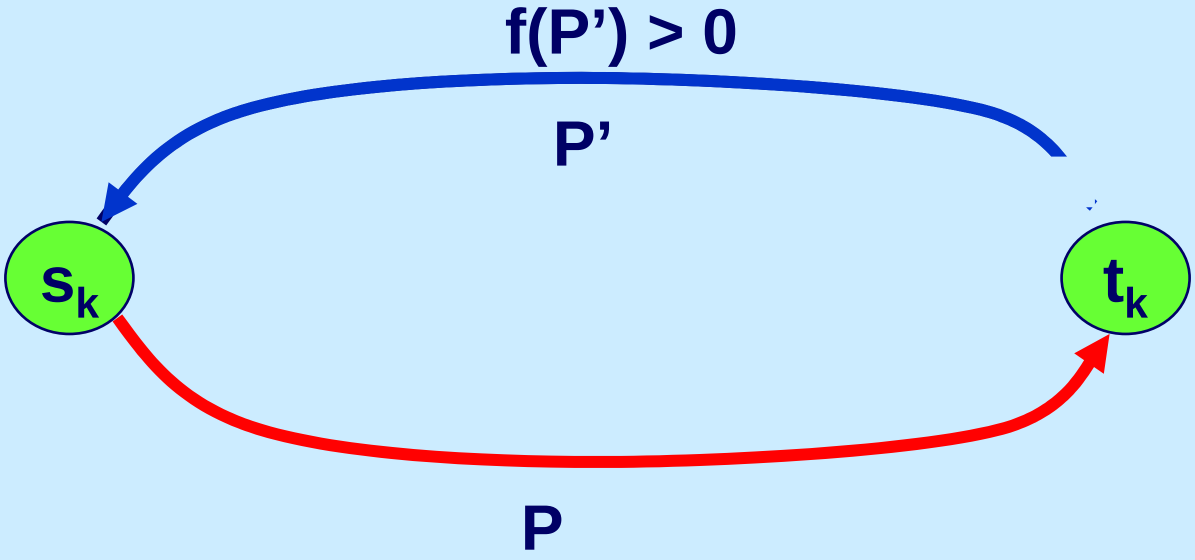
2. $f(P') > 0$ for P' $\in$ S^k

$$\Rightarrow c^{w,k}(P') = \min (c^{w,k}(P) : P \in S^k)$$

The tolls w are produced by the LP solution to the restricted master.

Optimality for the master problem is the same except that S^k is replaced by P^k.

On the optimality conditions



Suppose that the optimality conditions are not satisfied, and that $c(P') > c(P)$.

The residual network $G(x^k)$ has $Rev(P')$.
$Rev(P') + P$ is a circulation with negative cost.

Thus $G(x^k)$ has a negative cost cycle.

Adding paths or proving optimality

Let f be optimal for the restricted master

1. $w_{ij} > 0 \Rightarrow \sum_{k} \sum_{P \in S^k} \delta_{ij}(P) f(P) = u_{ij}$ for all $(i,j) \in A$

2. $f(P') > 0$ for $P' \in S^k$

$\Rightarrow c^{w,k}(P') = \min (c^{w,k}(P) : P \in S^k)$

Let P^k be the shortest path from s_k to t_k using $c^{w,k}$.

If $P^k \in S^k$ for each k, then f is optimal for the master problem because condition 1 and condition 2 are both satisfied when we replace S^k by P^k for each k.

Otherwise, add P^k to S^k for each k, and solve the new restricted master problem.

More on optimality conditions

We used partial dualization. That is, we had optimality conditions that used w, but not node potentials.

In usual column generation, one would use LP optimality conditions, which are more detailed, but follow the same general approach.

Restricted Master Problem 1

f(P) = flow in path P

c(P) = cost of path P

c(1-4) = 5

c(3-6) = 6

Minimize 5 f(1-4) + 6 f(3-6)

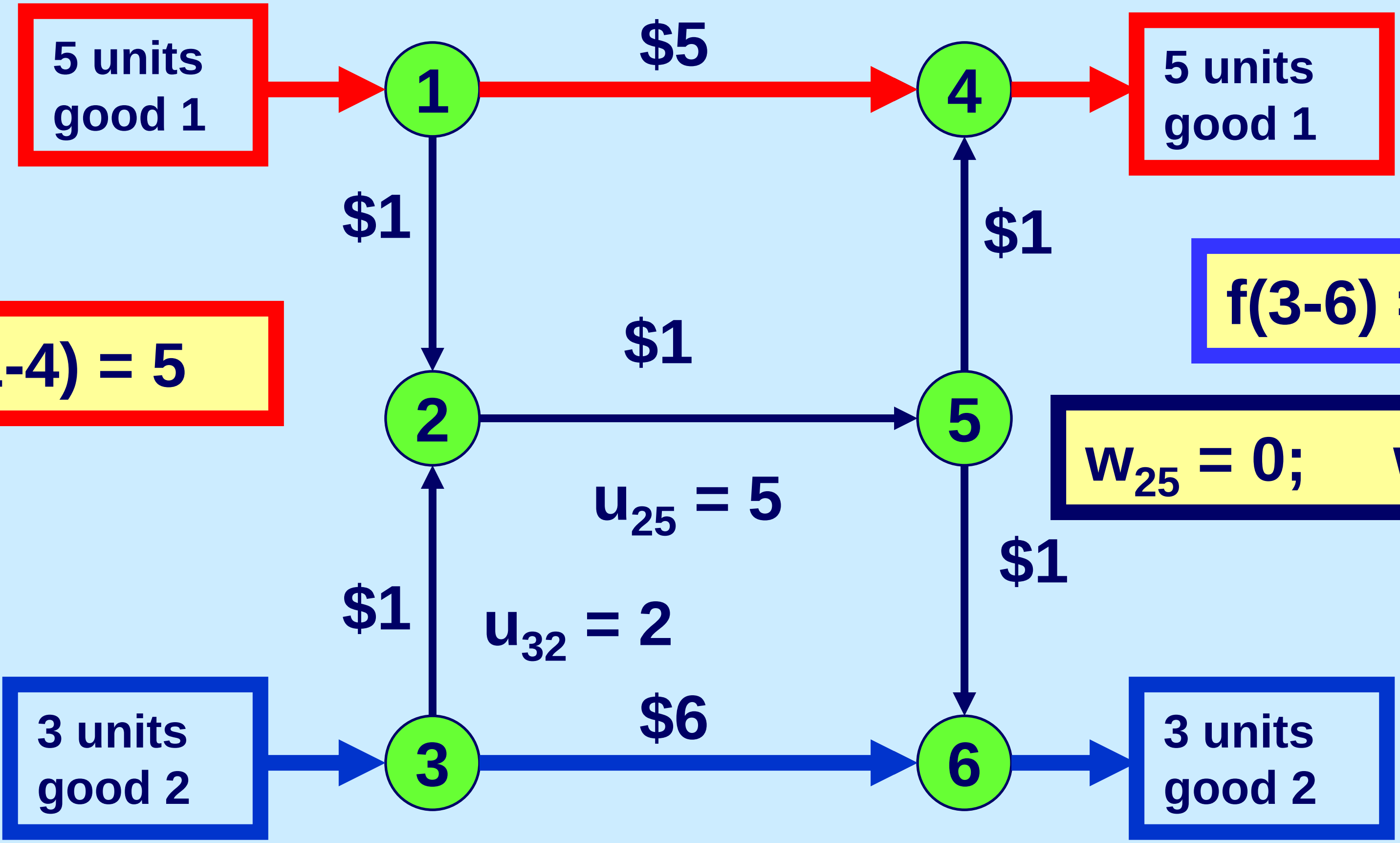
subject to

f(1-4) = 5

f(3-6) = 3

f(P) ≥ 0 for all paths P

Optimal solution for restricted master 1



$f(1\text{-}4) = 5$

$f(3\text{-}6) = 3$

$u_{25} = 5$

$u_{32} = 2$

$w_{25} = 0; \quad w_{32} = 0$

The unique shortest path for commodity 1 is 1-2-5-4.

The unique shortest path for commodity 2 is 3-2-5-6.

Restricted Master Problem 2

Suppose we add path 3-2-5-6 to the restricted master

f(P) = flow in path P

c(P) = cost of path P

c(1-4) = 5

c(3-6) = 6
c(3-2-5-6) = 3

Minimize $5\ f(1\text{-}4) + 6\ f(3\text{-}6) + 3\ f(3\text{-}2\text{-}5\text{-}6)$

subject to

$$f(1\text{-}4) = 5$$

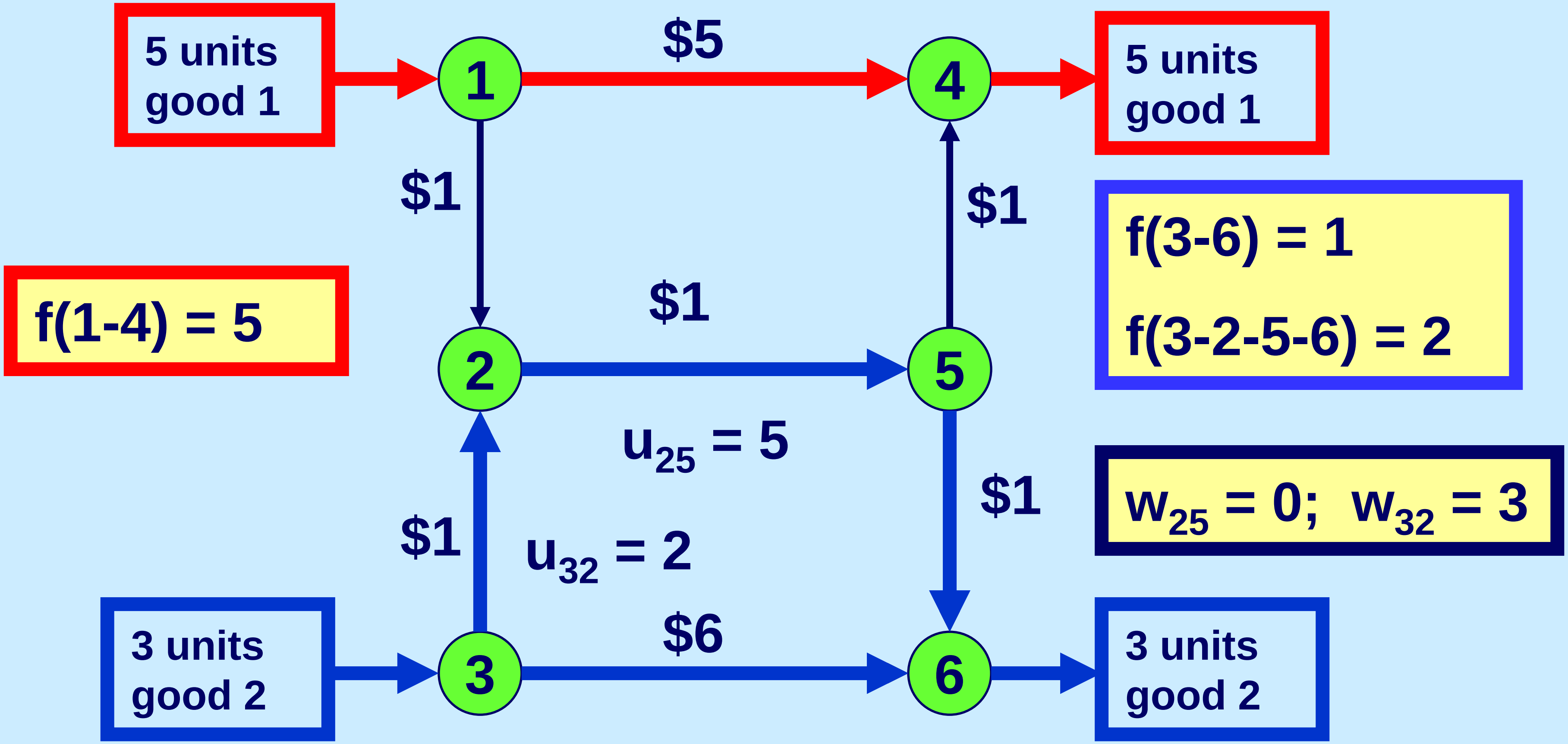
$$f(3\text{-}6) + f(3\text{-}2\text{-}5\text{-}6) = 3$$

$$f(3\text{-}2\text{-}5\text{-}6) \leq u_{25} = 5$$

$$f(3\text{-}2\text{-}5\text{-}6) \leq u_{32} = 2$$

$f(P) \geq 0$ for all paths P

Optimal solution for restricted master 2



5 units
good 1

5 units
good 1

3 units
good 2

3 units
good 2

$f(1\text{-}4) = 5$

$f(3\text{-}6) = 1$

$f(3\text{-}2\text{-}5\text{-}6) = 2$

$u_{25} = 5$

$u_{32} = 2$

$w_{25} = 0$; $w_{32} = 3$

The unique shortest path for commodity 1 is 1-2-5-4.

The shortest paths for commodity 2 are 3-2-5-6 and 3-6

Restricted Master Problem 3

We next add path 1-2-5-4 to the restricted master

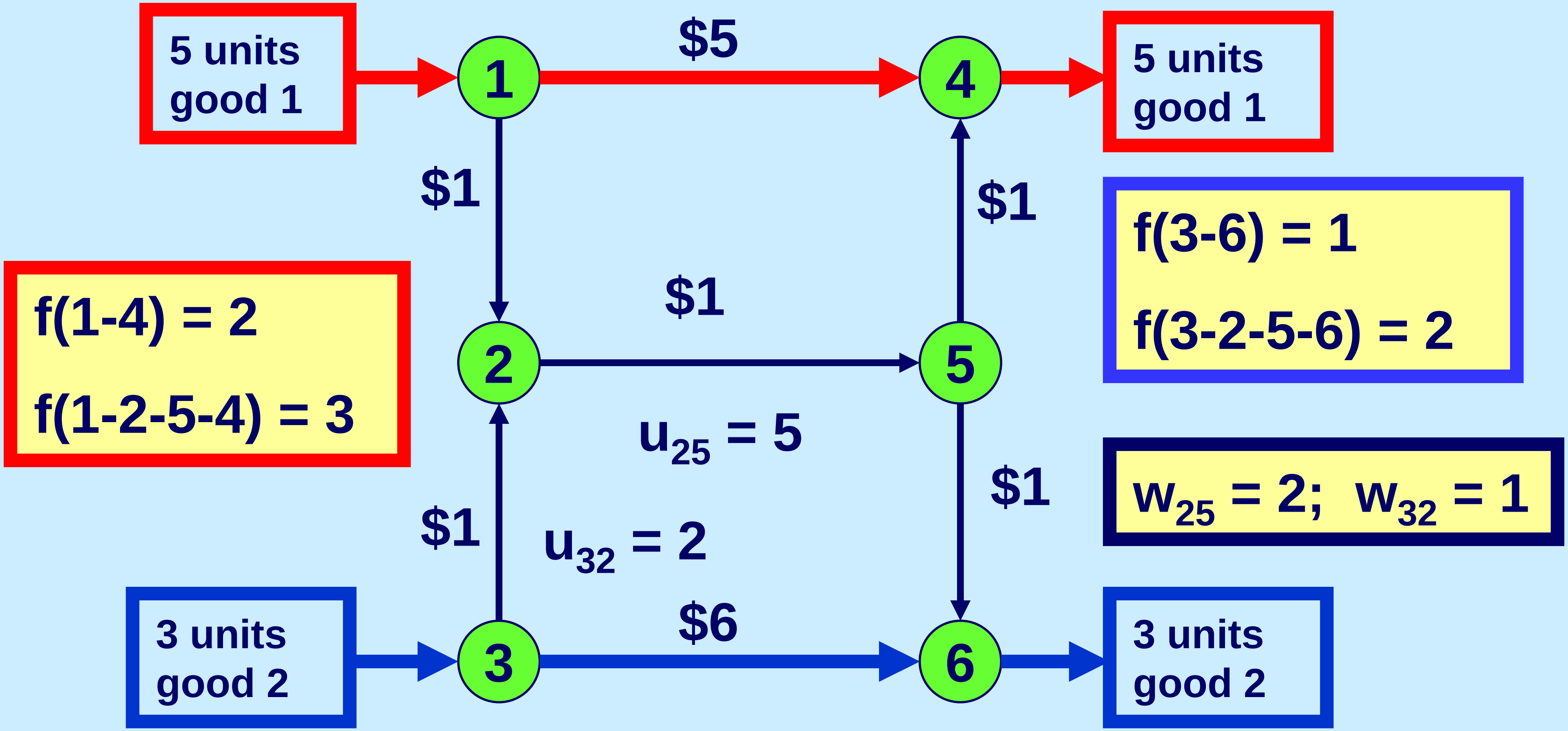
f(P) = flow in path P

c(P) = cost of path P

c(1-4) = 5
c(1-2-5-4) = 3
c(3-6) = 6
c(3-2-5-6) = 3

$$
\begin{array}{lrcl}
\text{Minimize} & \multicolumn{3}{l}{5\,f(1\text{-}4) + 3\,f(1\text{-}2\text{-}5\text{-}4) + 6\,f(3\text{-}6) + 3\,f(3\text{-}2\text{-}5\text{-}6)} \\
\text{subject to} & f(1\text{-}4) + f(1\text{-}2\text{-}5\text{-}4) & = & 5 \\
 & f(3\text{-}6) + f(3\text{-}2\text{-}5\text{-}6) & = & 3 \\
 & f(1\text{-}2\text{-}5\text{-}4) + f(3\text{-}2\text{-}5\text{-}6) & \leq & u_{25} = 5 \\
 & f(3\text{-}2\text{-}5\text{-}6) & \leq & u_{32} = 2 \\
 & \multicolumn{3}{c}{f(P) \geq 0 \text{ for all paths } P}
\end{array}
$$

Optimal solution for the path based version



f(1-4) = 2

f(1-2-5-4) = 3

f(3-6) = 1

f(3-2-5-6) = 2

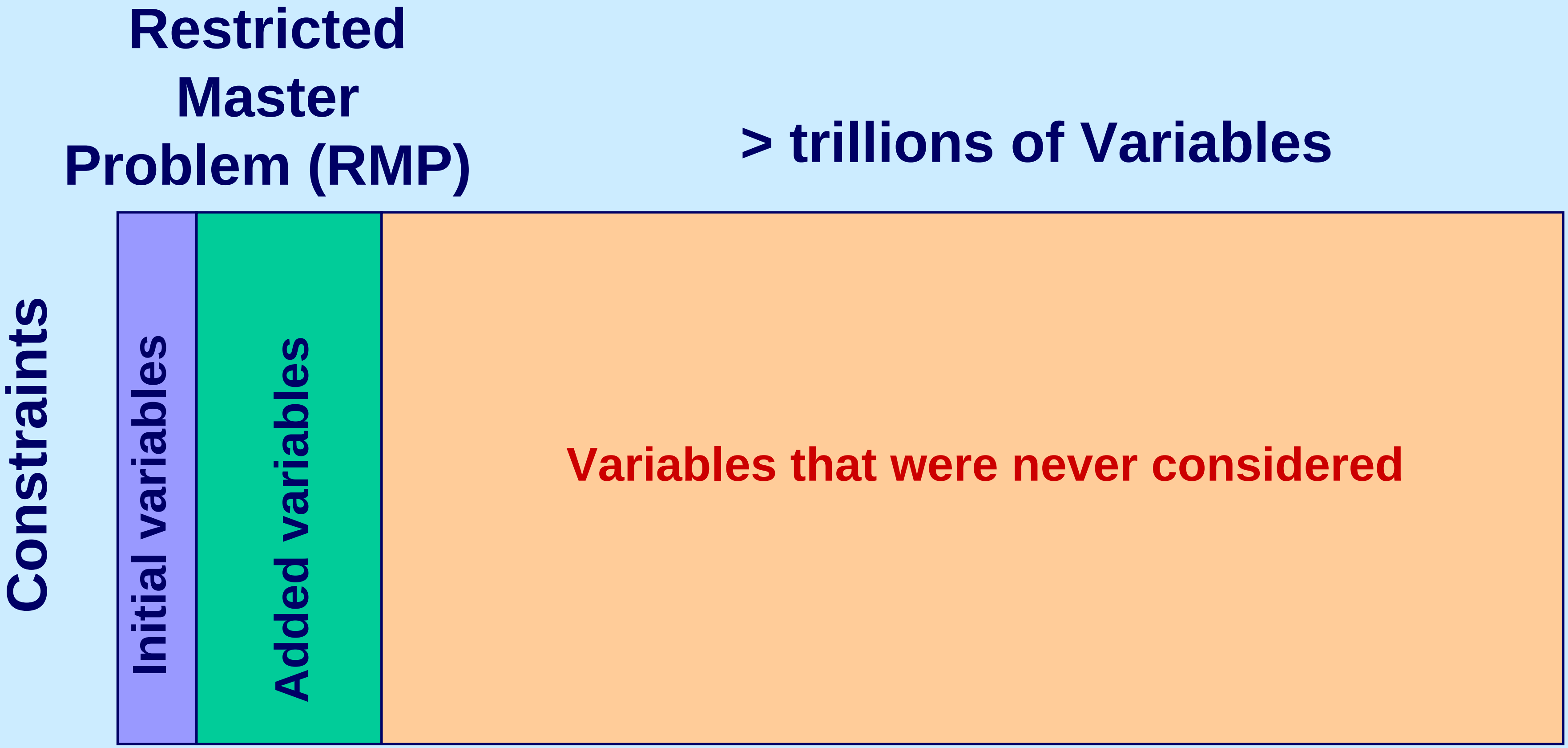
$u_{25} = 5$

$u_{32} = 2$

$w_{25} = 2$; $w_{32} = 1$

The solution is optimal for the entire problem.

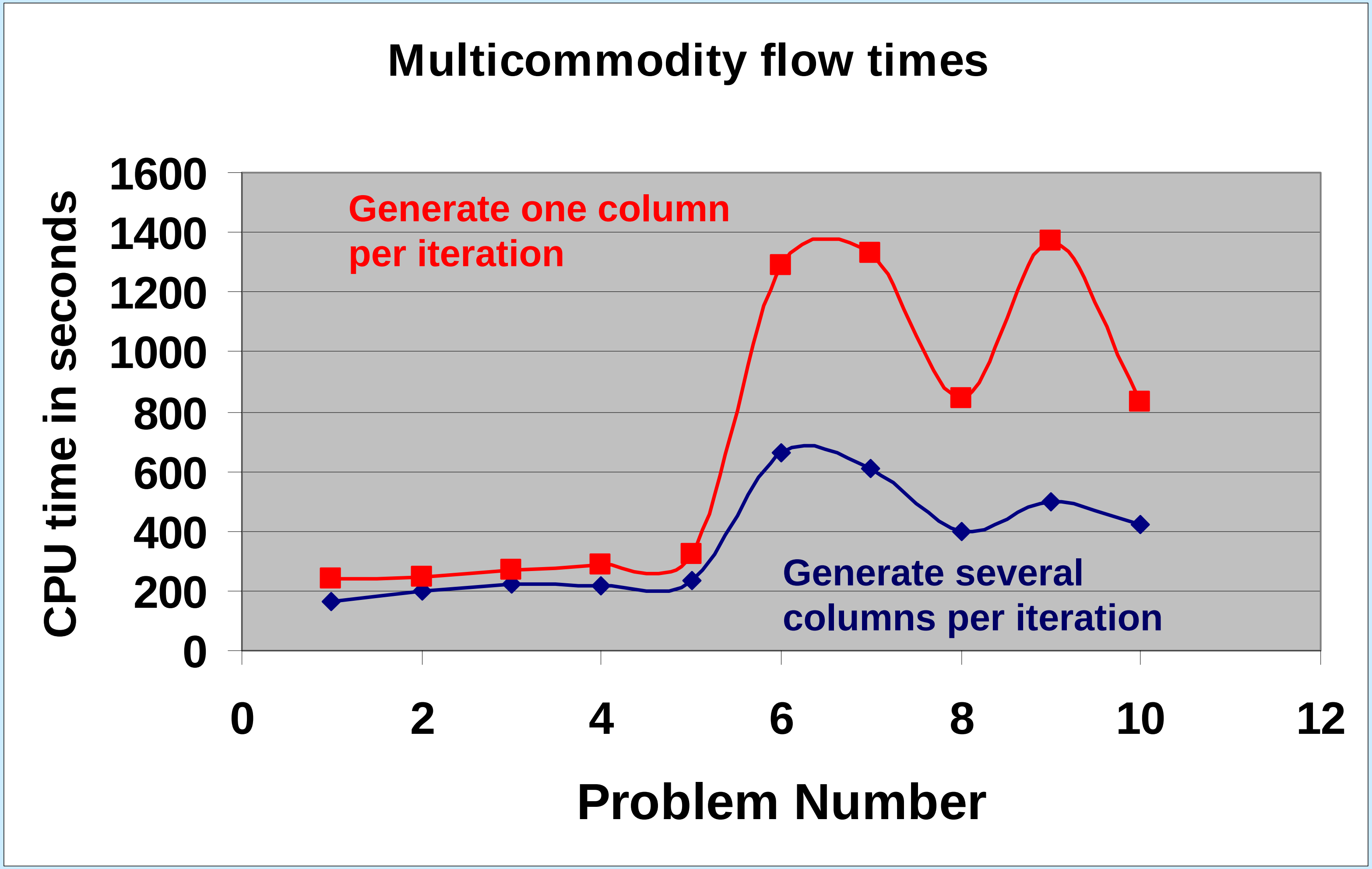
Column Generation



Choices in running column generation

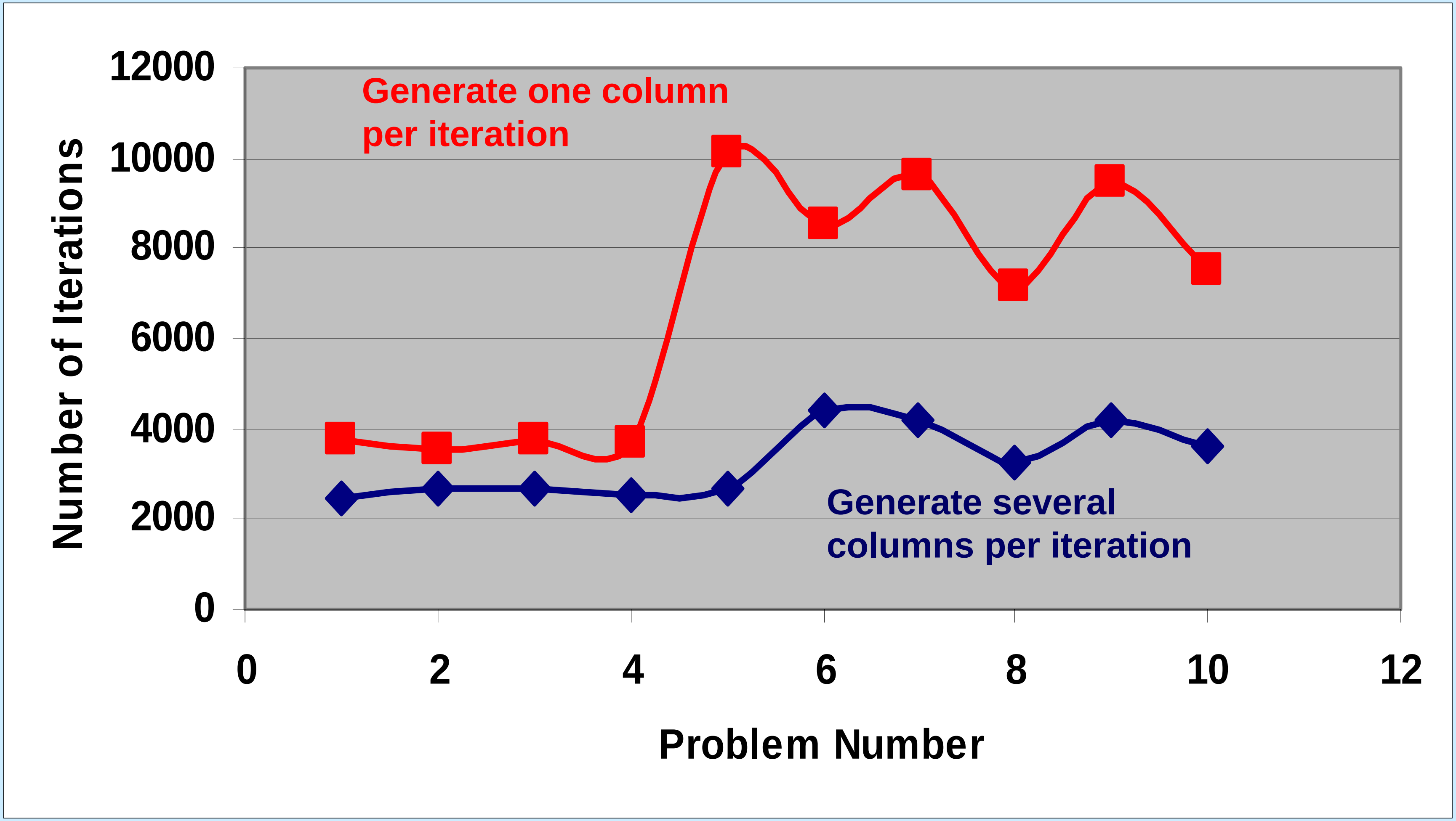
- **Starting columns**
- **How many columns to generate at a time**
- **Which LP solver to use**
- **and more**

Some running times

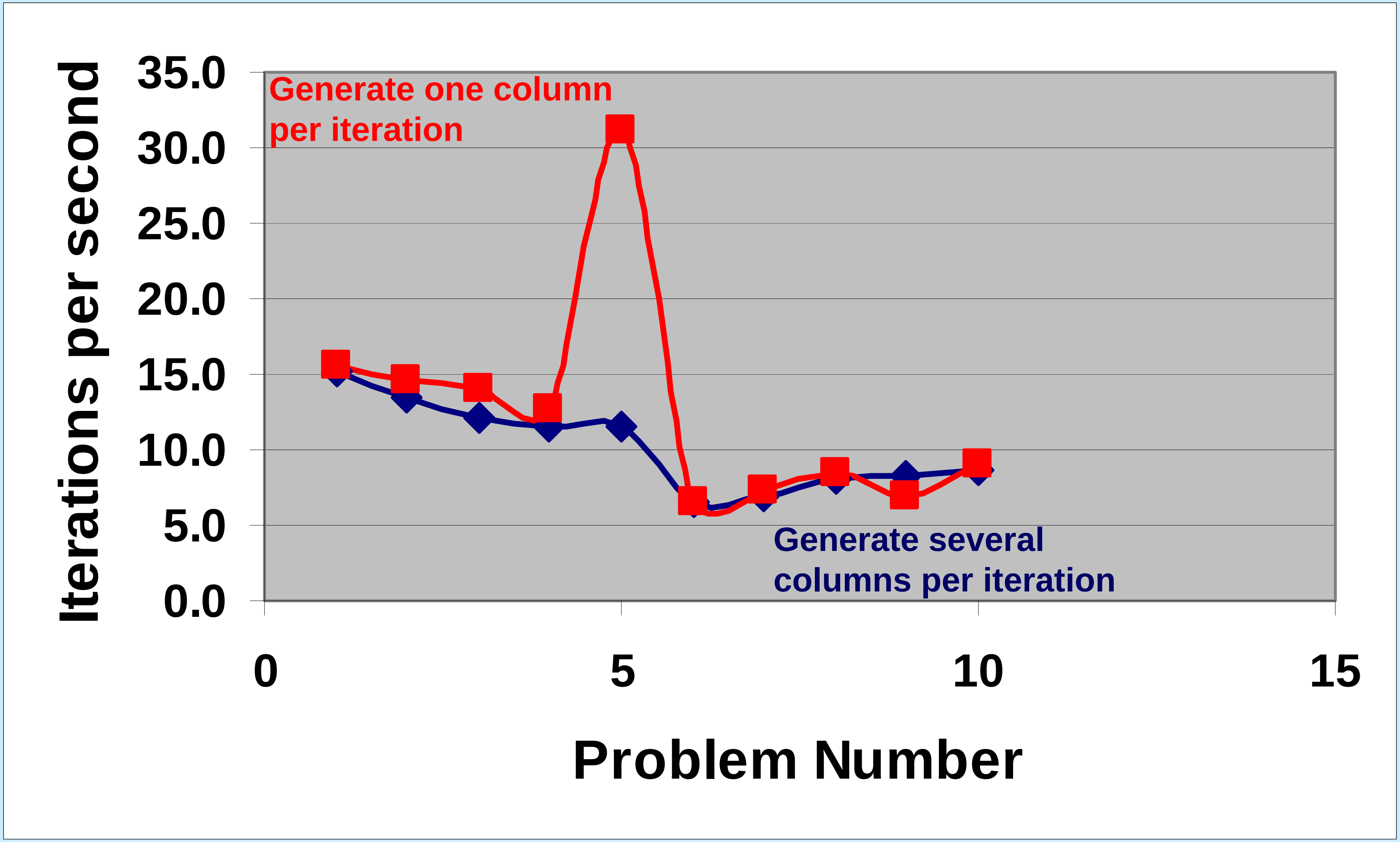


301 nodes, 497 arcs, 1320 commodities.
Times are on an IBM RS6000/590.

The number of iterations per problem



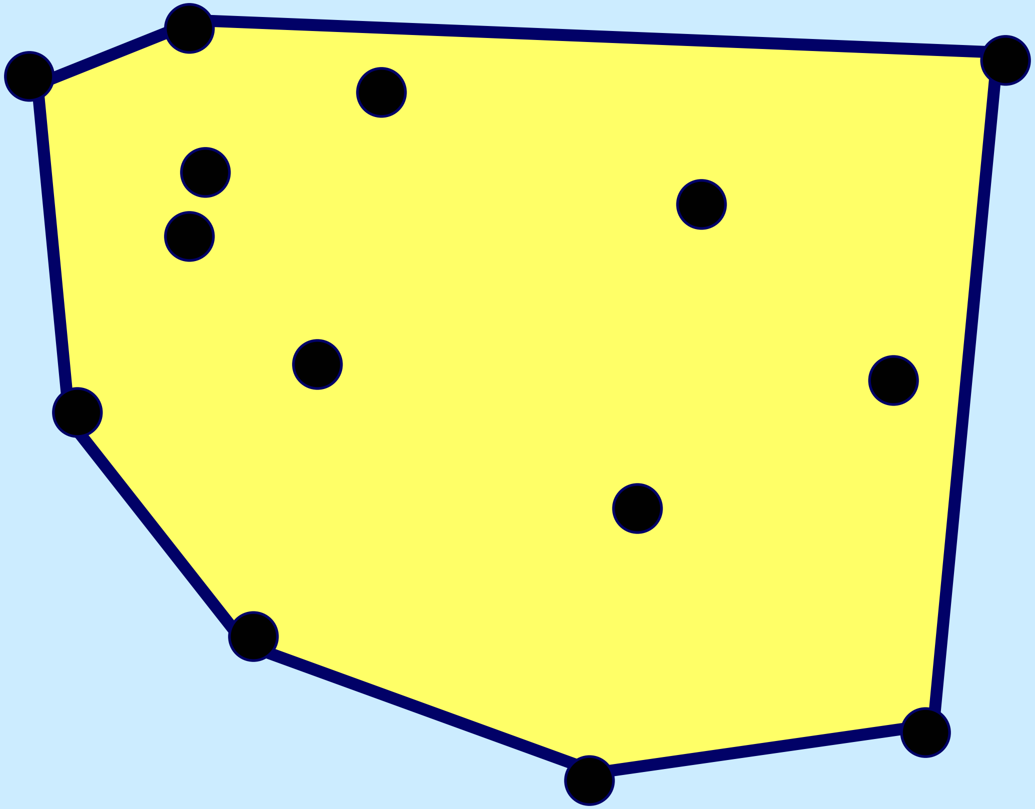
Number of iterations per second



Decomposition in LPs

- **The following slides will generalize our approach here to solving "staircase structure LPs"**
- **Here we relied on the fact that a flow of d_k units from s_k to t_k can be decomposed into the sum of flows along paths.**
- **However, for the more general problem, we will rely on the fact that any feasible solution x in a bounded polyhedron can be expressed as a convex combination of its corner points.**

Convex Hulls in two dimensions

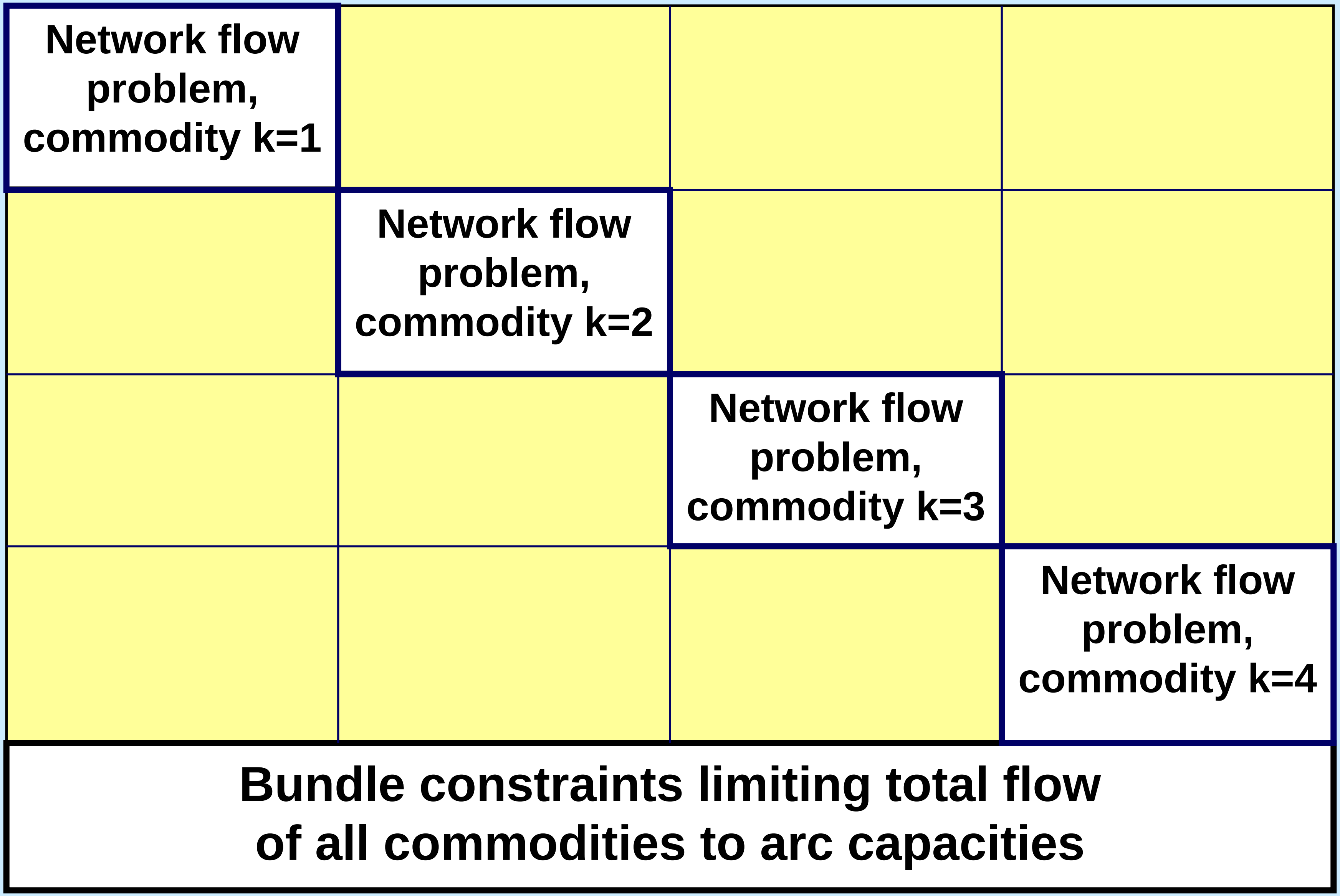


Each feasible point in an LP may be expressed as a convex combination of its corner points.

Staircase structure LPs

$$
\begin{array}{|c|c|c|c|c|c|c|c|c|}
-z & x^1 & x^2 & x^3 & x^4 & \cdots & x^K & & \\
1 & c^1 & c^2 & c^3 & c^4 & \cdots & c^K & = & -z_0 \\
0 & N & 0 & 0 & 0 & \cdots & 0 & = & b^1 \\
0 & 0 & N & 0 & 0 & \cdots & 0 & = & b^2 \\
0 & 0 & 0 & N & 0 & \cdots & 0 & = & b^3 \\
0 & 0 & 0 & 0 & N & & 0 & = & b^4 \\
 & & & \cdots & & & & & \\
0 & 0 & 0 & 0 & 0 & \cdots & N & = & b^K \\
0 & A^1 & A^2 & A^3 & A^4 & \cdots & A^K & \leq & u
\end{array}
$$

Another view of staircase structures



Network flow problem, commodity k=1

Network flow problem, commodity k=2

Network flow problem, commodity k=3

Network flow problem, commodity k=4

Bundle constraints limiting total flow of all commodities to arc capacities

On Staircase LPs

These staircase structures are superb for trying Lagrangian relaxation.

They are also good for Dantzig-Wolfe decomposition.

In Dantzig-Wolfe decomposition, we will represent each x^k as the convex combination of corner points of $\{x: Nx^k = b^k \text{ and } x \geq 0\}$, assuming the polyhedron is bounded.

Dantzig Wolfe Decomposition

$$Nx^k = b^k \quad \text{for } k = 1, 2, \ldots, k \qquad (17.1c)$$

Any solution to 17.1c can be expressed as the convex combination of corner points.

Make the replacement in the original problem. The new variables are the corner points of 17.1c for each k.

The constraints $Nx^k = b^k$ are all replaced.

The remaining constraints are the bundle constraints.

Linear MCF Problem Solution

- **G = (N, A) with K commodities. |N| = n, |A| = m.**

- **A standard solution approach**
  - **LP Solver such as CPLEX**

- **Difficulty**
  - **Node-arc formulation:**
    - **Constraints: nK +m**
    - **Variables: mK**
  - **Path formulation:**
    - **Constraints: m + K**
    - **Variables: |Paths for *ALL* commodities|**

Summary

- Multicommodity Flows
- Column generation approach
  - Path-based generalization
  - Generalization to Dantzig-Wolfe Decomposition
- Relies on optimality conditions
- Solving restricted master problems