15.082 and 6.855J

Review of Linear Programming

Overview

Describe LP and IP

min cost flow as an LP

Graphical solution

Basic feasible solutions.

Simplex Method

Basic feasible solutions in matrix form

Duality

Note: this will cover lots of material. We will also have a recitation.

The Minimum Cost Flow Problem

u_{ij} = capacity of arc (i,j).

c_{ij} = unit cost of shipping flow from node i to node j on (i,j).

x_{ij} = amount shipped on arc (i,j)

Minimize $\sum_{(i,j)\in A} c_{ij}x_{ij}$

$$\sum_j x_{ij} - \sum_k x_{ki} = b_i \quad \text{for all } i \in N.$$

and $0 \leq x_{ij} \leq u_{ij}$ for all $(i,j) \in A$.

Terminology

x_{ij} = ***Decision variable.*** Describes a decision to be made

Minimize $\sum_{(i,j)\in A} c_{ij}x_{ij}$ ***Objective Function***

$\sum_j x_{ij} \quad - \quad \sum_k x_{ki} = b_i$ for all $i \in N$.

Constraints

and $0 \le x_{ij} \le u_{ij}$ for all $(i,j) \in A$.

Terminology

Minimize $\sum_{(i,j)\in A} c_{ij}x_{ij}$ ***Objective Function***

$\sum_j x_{ij} \quad - \quad \sum_k x_{ki} = b_i$ for all $i \in N$.

Constraints

and $0 \le x_{ij} \le u_{ij}$ for all $(i,j) \in A$.

In a linear program, the objective function and the constraints are all linear.

Typically, but not always, the variables are constrained to be non-negative.

If variables are constrained to be integers, it is called an integer program.

Some Applications of LPs + IPs

Production Planning: Given several products with varying production requirements and cost structures, determine how much of each product to produce in order to maximize profits.

Scheduling: Given a staff of people, determine an optimal work schedule that maximizes worker preferences while adhering to scheduling rules.

Portfolio Management: Determine bond portfolios that maximize expected return subject to constraints on risk levels and diversification.

And an incredible number more.

Graphing 2-Dimensional LPs

Example 1:

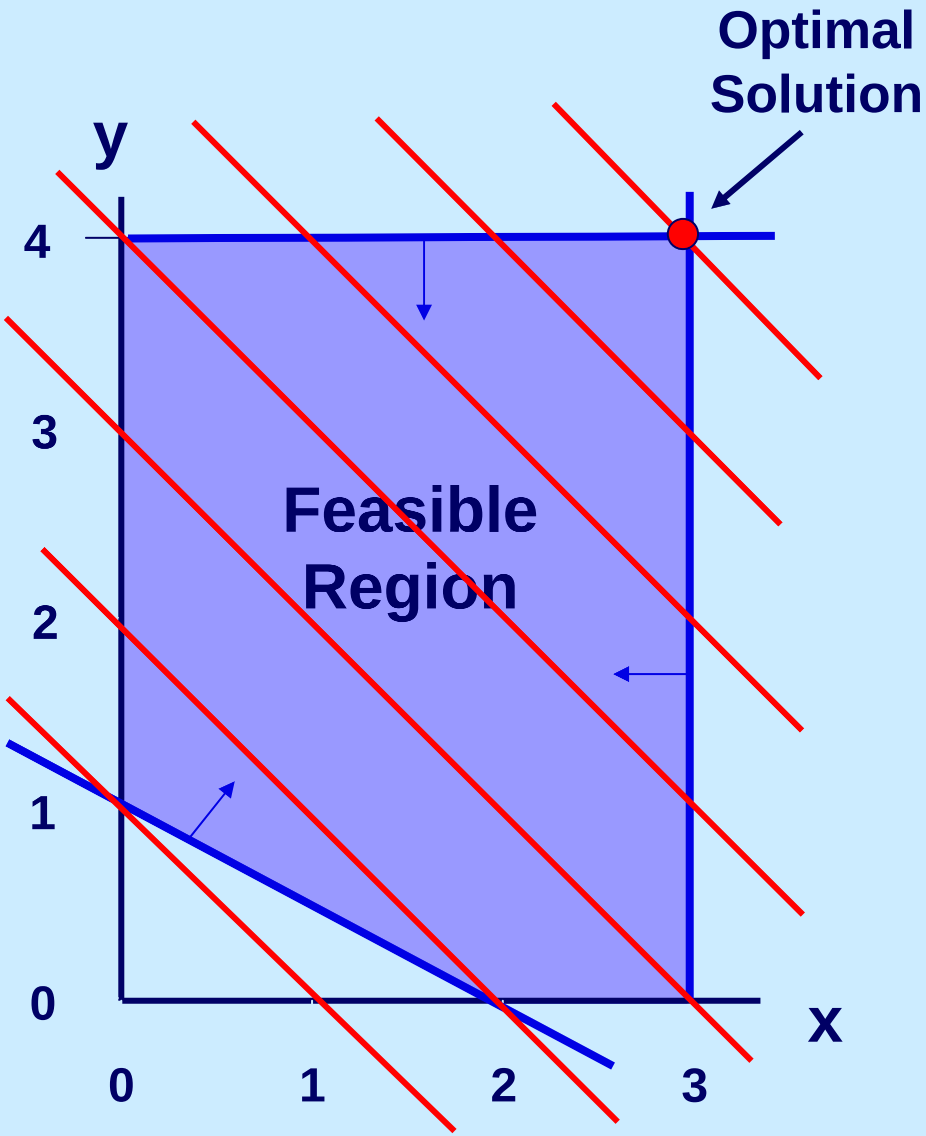
Maximize $x + y$

Subject to: $x + 2y \geq 2$

$x \leq 3$

$y \leq 4$

$x \geq 0 \quad y \geq 0$



These LP animations were created by Keely Crowston.

Graphing 2-Dimensional LPs

Example 2:

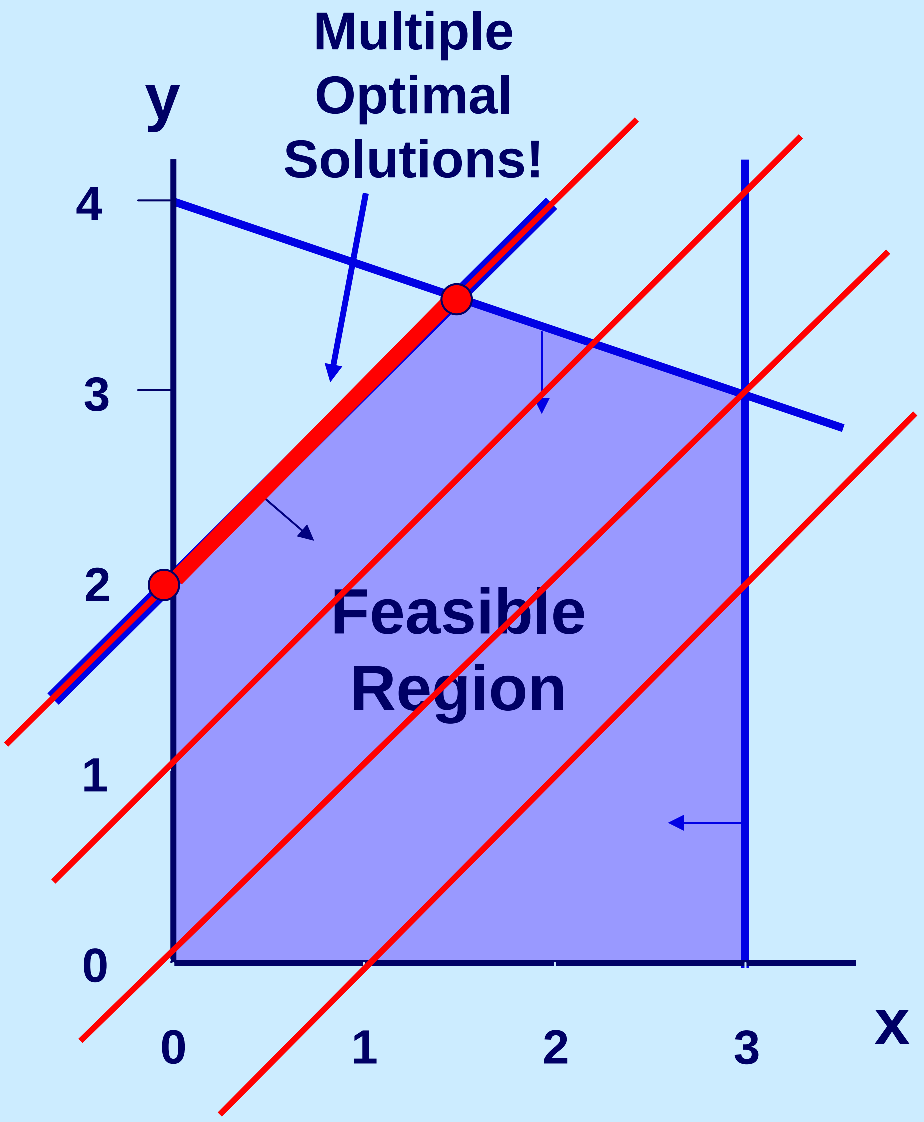
Minimize ** x - y

Subject to: $\frac{1}{3}x + y \leq 4$

$-2x + 2y \leq 4$

$x \leq 3$

$x \geq 0 \quad y \geq 0$



Graphing 2-Dimensional LPs

Example 3:

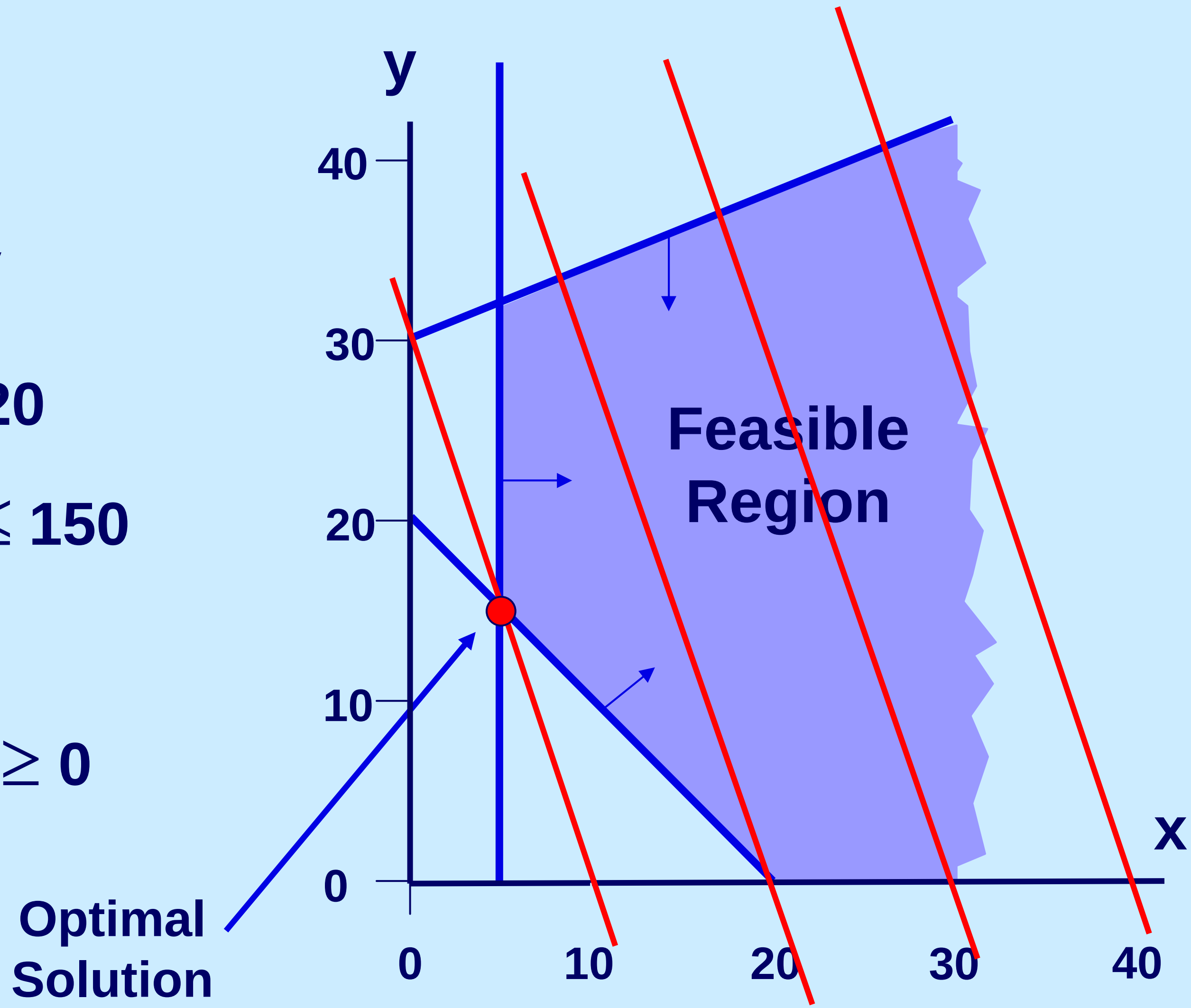
Minimize $x + 1/3\, y$

Subject to: $x + y \geq 20$

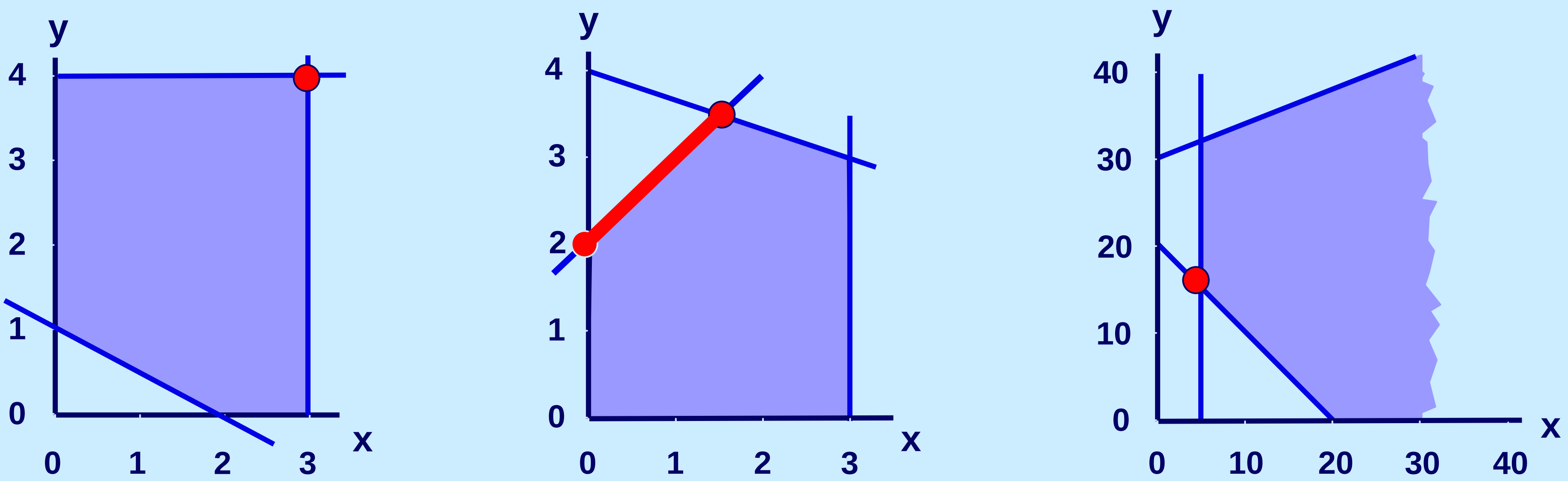
$-2x + 5y \leq 150$

$x \geq 5$

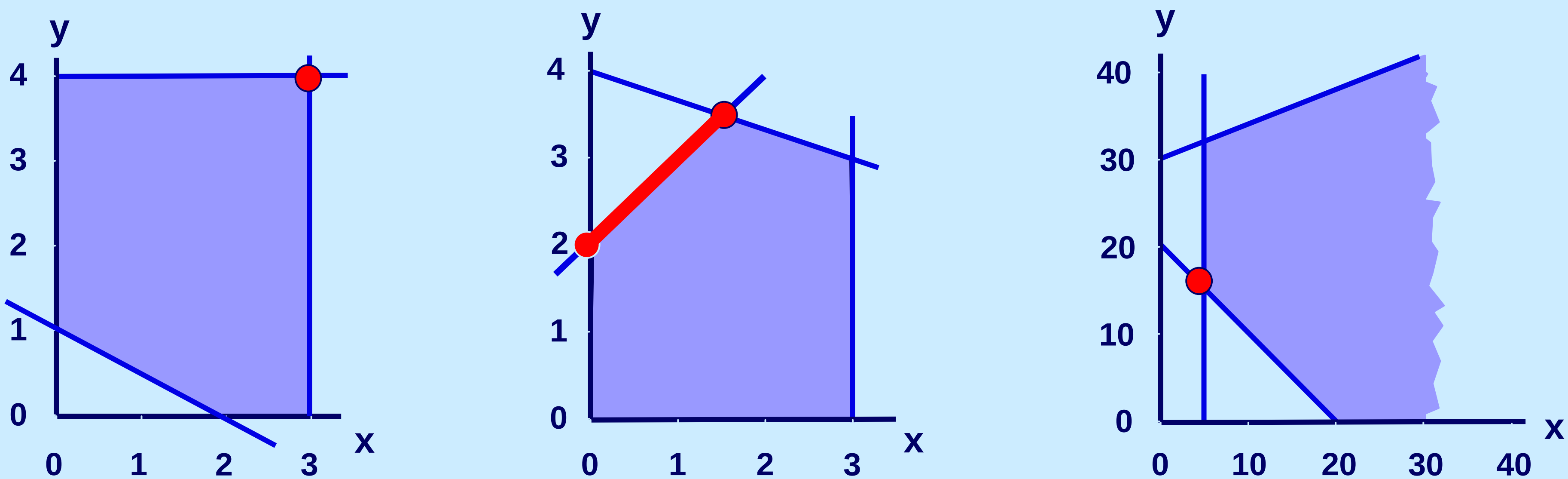
$x \geq 0 \quad y \geq 0$



Do We Notice Anything From These 3 Examples?



A Fundamental Point



If an optimal solution exists, there is always a corner point optimal solution!

Graphing 2-Dimensional LPs

Example 1:

Maximize $x + y$

Subject to: $x + 2y \geq 2$

$x \leq 3$

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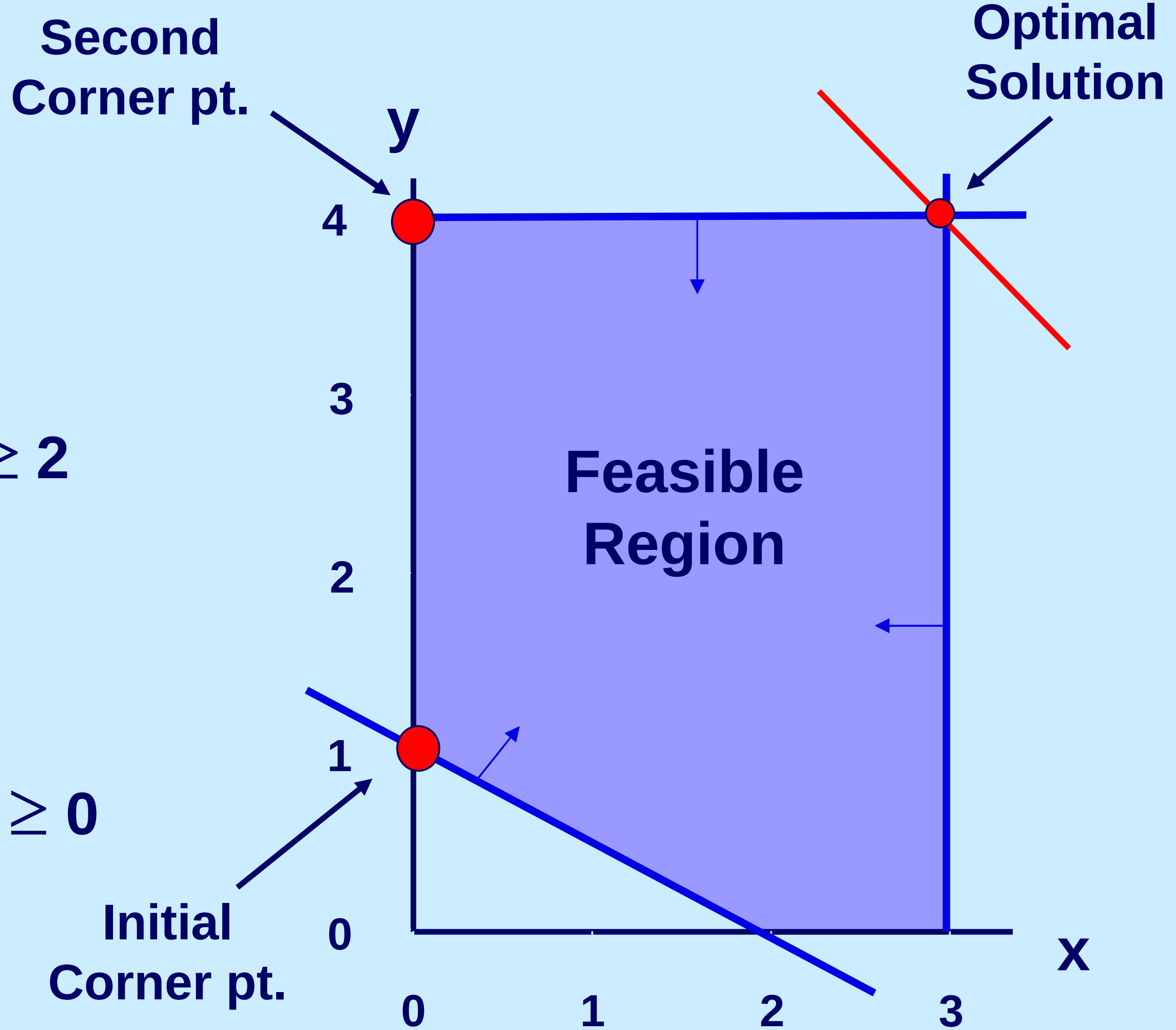
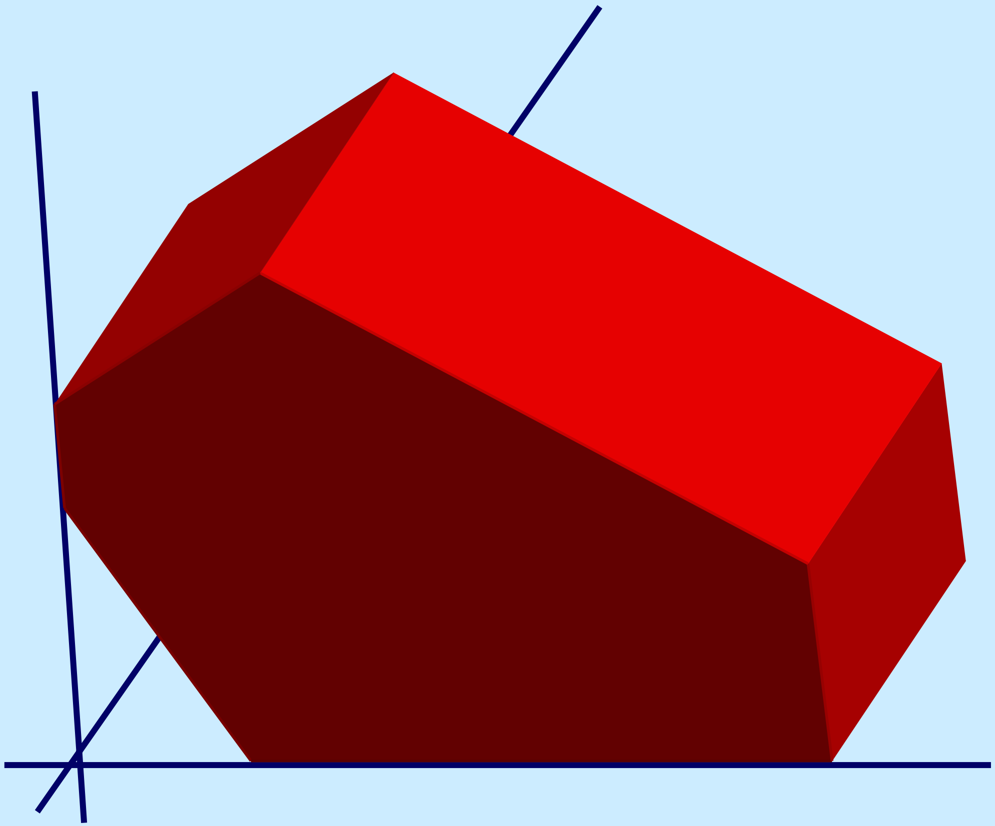
$x \geq 0 \quad y \geq 0$



And We Can Extend this to Higher Dimensions

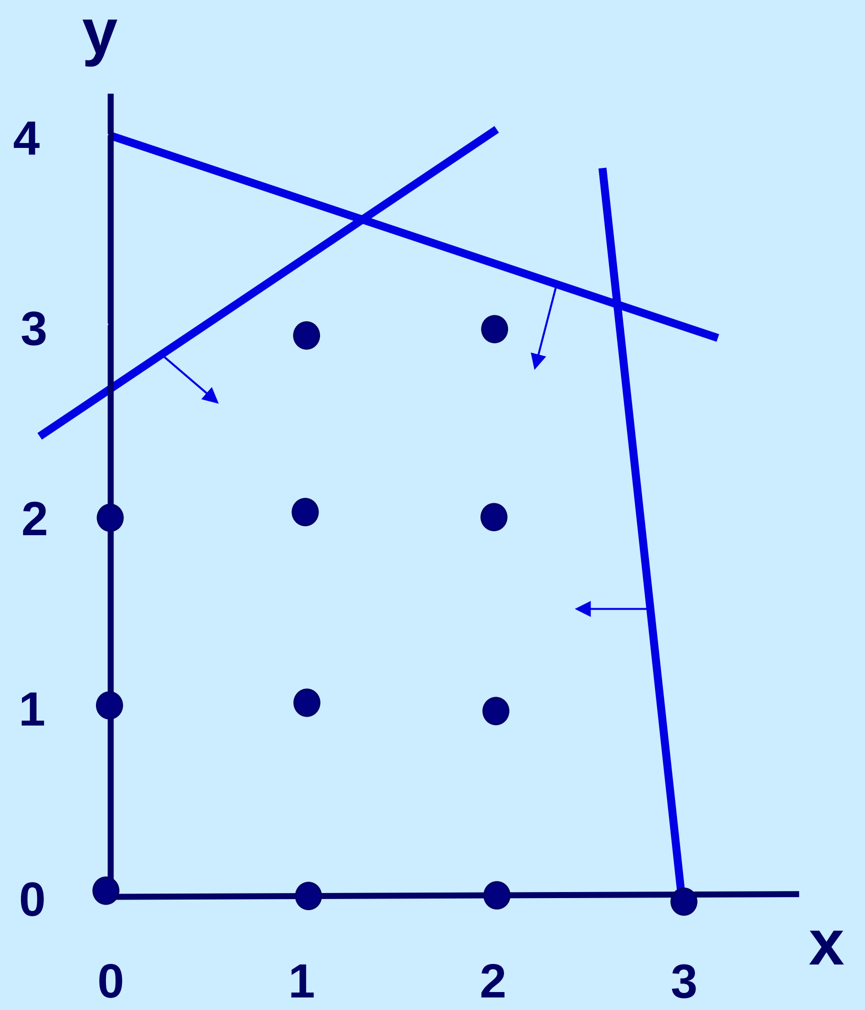


Then How Might We Solve an LP?

- The constraints of an LP give rise to a geometrical shape - we call it a polyhedron.
- If we can determine all the corner points of the polyhedron, then we can calculate the objective value at these points and take the best one as our optimal solution.
- The *Simplex Method* intelligently moves from corner to corner until it can prove that it has found the optimal solution.

But an Integer Program is Different

- Feasible region is a set of discrete points.
- Can't be assured a corner point solution.
- There are no "efficient" ways to solve an IP.
- Solving it as an LP provides a relaxation and a bound on the solution.



Linear Programs in higher dimensions

$$
\begin{aligned}
\textbf{maximize} \quad & z = -4x_1 + x_2 - x_3 + x_4 \\
\text{subject to} \quad & -7x_1 + 5x_2 + x_3 + x_4 = 8 \\
& -2x_1 + 4x_2 + 2x_3 - x_4 = 10 \\
& x_1, \; x_2, \; x_3, \; x_4 \geq 0
\end{aligned}
$$

We will describe LPs that start with the following form:

- equality constraints
- non-negative RHS
- nonnegative variables

Basic Feasible Solutions

maximize $z = -4x_1 + x_2 - x_3 + x_4$

subject to

$$-7x_1 + 5x_2 + x_3 + x_4 = 8$$

$$-2x_1 + 4x_2 + 2x_3 - x_4 = 10$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Suppose there are m constraints, n variables

A basic solution is found by setting n-m variables to 0 and solving the remaining system with n variables and n constraints.

- The n – m variables are called non-basic variables
- The m variables are called basic variables

Basic Feasible Solutions

maximize $z = -4x_1 + x_2 - x_3 + x_4$

subject to
$$-7x_1 + 5x_2 + x_3 + x_4 = 8$$
$$-2x_1 + 4x_2 + 2x_3 - x_4 = 10$$
$$x_1, x_2, x_3, x_4 \geq 0$$

-z	x_1	x_2	x_3	x_4		
1	-4	1	-1	1	=	0
0	-7	5	1	1	=	8
0	-2	4	2	-1	=	10

Basic Feasible Solutions

-z	x_1	x_2	x_3	x_4		
1	-11	6	0	2	=	8
0	-7	5	1	1	=	8
0	12	6	0	-3	=	-6

Example: Suppose we want the solution with basic variables x_3 and x_4, and thus x_1 and x_2 are non-basic.

We then perform pivot operations.

Basic Feasible Solutions

-z	x_1	x_2	x_3	x_4		
1	-3	2	0	0	=	4
0	-3	3	1	0	=	6
0	-4	2	0	1	=	2

Next pivot on the –3.

Basic Feasible Solutions

Canonical form: basic variables have a single one in the column.

reduced costs →

-z	x_1	x_2	x_3	x_4		
1	-3	2	0	0	=	4
0	-3	3	1	0	=	6
0	-4	2	0	1	=	2

The basic solution is found by setting non-basic variables to 0. We get x_1=0, x_2=0, x_3=6, x_4=2.

This solution also satisfies $x \geq 0$. It is called a **basic feasible solution**.

More on Basic Feasible Solutions

Each corner point solution of the polyhedron is a basic feasible solution.

The simplex method is a systematic way of moving from one basic feasible solution to another, always improving the solution, until the optimum solution is obtained.

The network simplex algorithm is a special case of the simplex algorithm.

The Simplex Algorithm

	-z	x_1	x_2	x_3	x_4		
$z = -3x_1 + 2x_2 - 4$	1	-3	2	0	0	=	4
	0	-3	3	1	0	=	6
	0	-4	2	0	1	=	2

The entering variable for a max problem is a variable with positive reduced cost.

The pivot element is chosen uniquely in the column of the entering variable so that the next basis is feasible.

The Simplex Algorithm

The pivot element is chosen according to a min ratio rule.

-z	x_1	x_2	x_3	x_4		
1	1	0	0	-1	=	-2
0	3	0	1	-1.5	=	3
0	-2	1	0	.5	=	1

A pivot is carried out, leading to the next bfs.

Variable x_4 has left the basis.
The new basis consists of x_2 and x_3.

The Simplex Algorithm

Pivots are carried out until the bfs is optimal.

-z	x_1	x_2	x_3	x_4		
1	0	0	-1/3	-1/2	=	-3
0	1	0	1/3	-1/2	=	1
0	0	1	2/3	-1/2	=	3

$z = -x_3/3 - x_4/2 + 3$

This new bfs is optimal. Increasing x_3 or x_4 makes the solution worse.

The Simplex Algorithm

-z	x_1	x_2	x_3	x_4		
1	0	0	-1/3	-1/2	=	-3
0	1	0	1/3	-1/2	=	1
0	0	1	2/3	-1/2	=	3

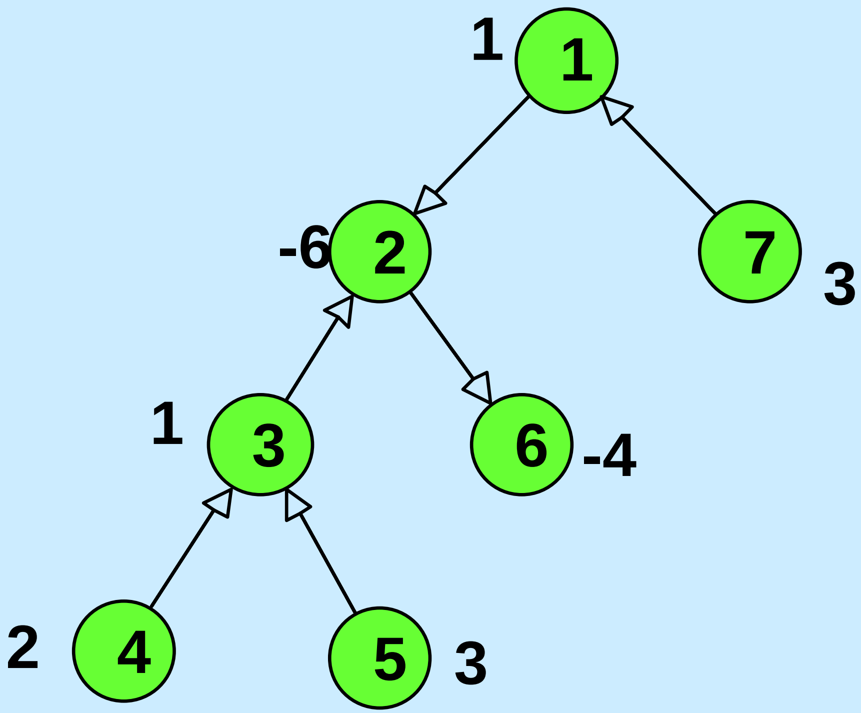
Optimality conditions for a maximization problem: all reduced costs are non-positive.

Network Simplex

We have already seen network simplex.

In the following, we show a connection between network simplex and ordinary simplex.

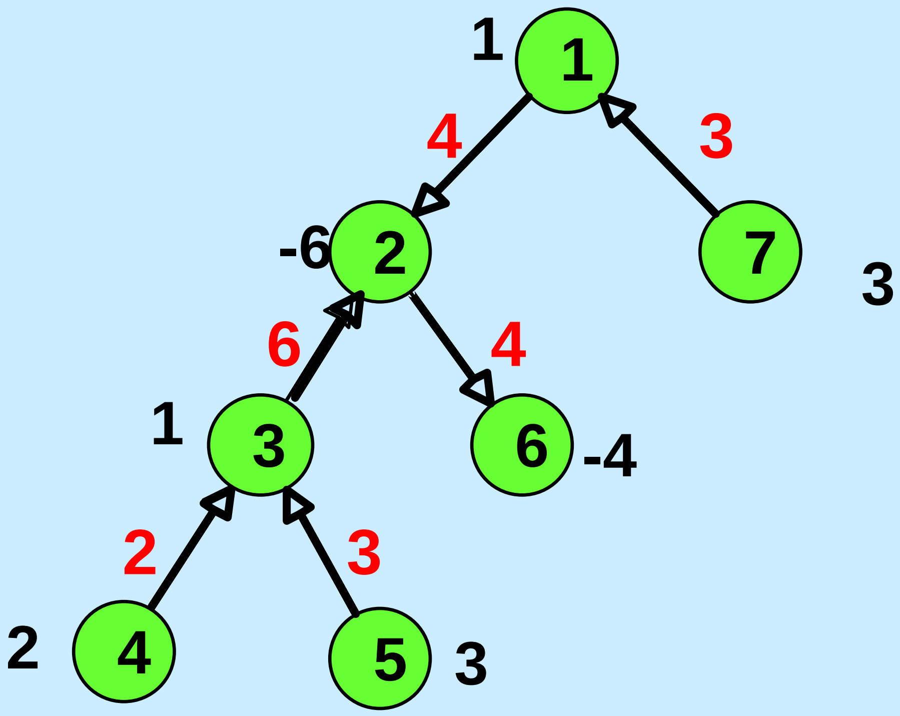
Network Simplex Algorithm



In the min cost flow problem, there are n constraints, one for each node, but one constraint is redundant.

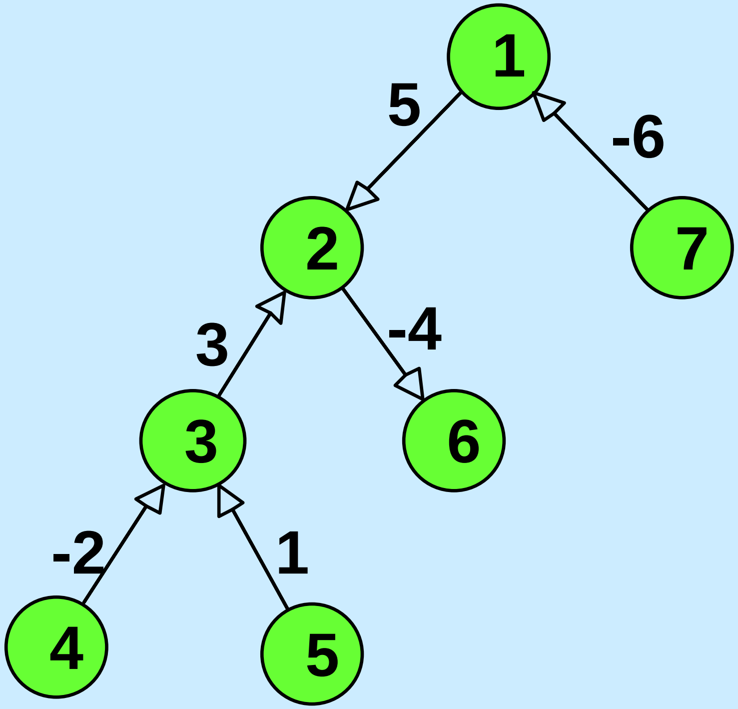
Basic solutions correspond to spanning trees.

We can calculate the bfs directly

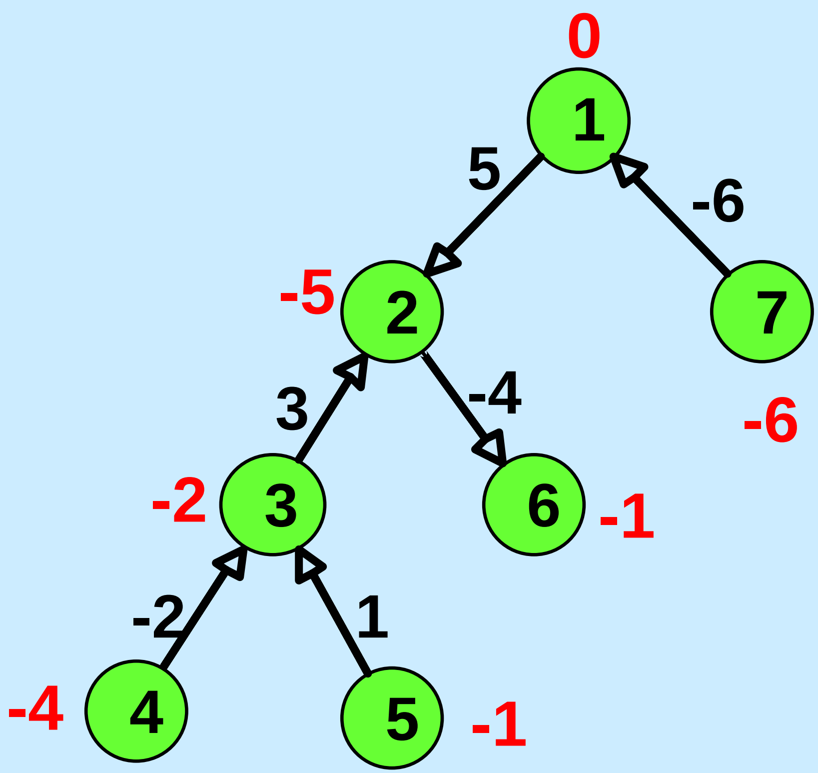


This corresponds to solving the system of equations.

We calculate node potentials directly



We calculate node potentials directly



Node potentials are chosen so that reduced costs of tree arcs are 0.

We can then determine the reduced costs of non-basic arcs directly.

$c^{\pi}_{ij} = c_{ij} - \pi_i + \pi_j$.

Entering and Leaving variables

The entering variable is chosen so that it violates its optimality condition. (same as for simplex)

The exiting variable is chosen so that the next spanning tree solution is feasible.

Determined by sending flow around a cycle.

So, operations on the trees avoid all of the matrix manipulations.

Simplex Method in Matrix Form

x^B: basic variables of final tableau

x^L: non-basic variables of final tableau

Rearranging columns, we can express the LP as:

$A = [B \;\; L]$: the original constraint matrix

$c = [c^B \; c^L]$: the original objective coefficient vector

Initial Problem:

$$\text{Maximize} \quad z = c^B x^B + c^L x^L$$

$$B x^B + L x^L = b$$

$$x^B, x^L \geq 0$$

Basic Feasible Solutions in Matrix Form

Initial Problem:

minimize $z = c_B x_B + c_L x_L$

$$B x_B + L x_L = b$$

$$x_B , x_L \geq 0$$

Let the bar above a vector or matrix represent the updated values in the tableau after pivoting.

Final System in canonical form (after pivoting):

minimize $-z + \bar{c}_L x_L = -z_0$

$$x_B + \bar{L} x_L = \bar{b}$$

$$x_B, x_L \geq 0$$

$\bar{L}$ and $\bar{b}$

Start with: $B\, x_B + L\, x_L = b$

$$B^{-1}B\, x_B + B^{-1} L\, x_L = B^{-1} b$$

$$x_B + B^{-1} L\, x_L = B^{-1} b$$

$$\text{-->} \quad \bar{L} = B^{-1} L, \text{ and } \quad \bar{b} = B^{-1} b,$$

The basic feasible solution is obtained by setting x_L to 0.

Therefore, $x_B = B^{-1} b$.

$\bar{c}^L$

Start with: $z = c_B\, x_B + c_L\, x_L$

$$z = c_B\,(B^{-1} b - B^{-1} L\, x_L) + c_L\, x_L$$

$$z = c_B\, B^{-1} b + (c_L - c_B\, B^{-1} L)\, x_L$$

$$\Rightarrow \quad \bar{c}_L = c_L - c_B\, B^{-1} L \quad \text{and} \quad \bar{z} = c_B\, B^{-1} b,$$

There is no coefficient for x_B in the final system.

So, we substitute for x_B.

Notation: let $\pi = c^B B^{-1}$

Start with: $z = c_B x_B + c_L x_L$

$\bar{c}_L = c_L - c_B B^{-1} L$ and $\bar{z} = c_B B^{-1} b$,

$\bar{c}_L = c_L - \pi L$ and $\bar{z} = \pi b$,

π is called the vector of ***simplex multipliers.***

Duality via the Lagrangian

Primal Problem P

minimize $z = cx$

subject to $Ax = b$

$x \geq 0$

optimum value is z*

Lagrangian Problem $D(\pi)$

minimize $v(\pi) = cx - \pi(Ax - b)$

subject to $x \geq 0$

optimum value is $v^*(\pi)$

Note: if x* is feasible for P, then it is also feasible for $D(\pi)$.

Thus $v^*(\pi) \leq z^*$, and $D(\pi)$ provides a lower bound for P.

Computing the Highest Lower Bound

Lagrangian Problem $D(\pi)$

minimize $v(\pi) = cx - \pi(Ax - b) = (c - \pi A)x + \pi b$

subject to $x \geq 0$

$v(\pi) = -\infty$ **unless** $(c - \pi A) \geq 0.$

$v(\pi) = \pi b$ **if** $(c - \pi A) \geq 0$

The highest lower bound is found by solving

maximize $v = \pi b$

subject to $c - \pi A \geq 0$

Primal Problem P

minimize $z = cx$

subject to $Ax = b$

$x \geq 0$

optimum value is z*

Dual Problem D

maximize $v = \pi b$

subject to $\pi A \leq c$

optimum value is v*

Theorem. (Strong Duality) If both P and D are feasible, then $z^* = v^*$.

PRIMAL PROBLEM:

maximize $z = 3x_1 + 4x_2 + 6x_3 + 8x_4$

subject to

$$x_1 + x_2 + x_3 + x_4 = 1$$
$$2x_1 + 3x_2 + 4x_3 + 5x_4 = 3$$
$$x_1, x_2, x_3, x_4 \geq 0$$

DUAL PROBLEM:

minimize $y_1 + 3y_2$

Subject to

$$y_1 + 2y_2 \geq 3$$
$$y_1 + 3y_2 \geq 4$$
$$y_1 + 4y_2 \geq 6$$
$$y_1 + 5y_2 \geq 8$$

Observation 1.

The constraint matrix in the primal is the transpose of the constraint matrix in the dual.

Observation 2.

The RHS coefficients in the primal become the cost coefficients in the dual.

PRIMAL PROBLEM:

maximize $z = 3x_1 + 4x_2 + 6x_3 + 8x_4$

subject to

$$x_1 + x_2 + x_3 + x_4 = 1$$
$$2x_1 + 3x_2 + 4x_3 + 5x_4 = 3$$
$$x_1, x_2, x_3, x_4 \geq 0$$

DUAL PROBLEM:

minimize $y_1 + 3y_2$

Subject to

$$y_1 + 2y_2 \geq 3$$
$$y_1 + 3y_2 \geq 4$$
$$y_1 + 4y_2 \geq 6$$
$$y_1 + 5y_2 \geq 8$$

Observation 3. The cost coefficients in the primal become the RHS coefficients in the dual.

Observation 4. The primal (in this case) is a max problem with equality constraints and non-negative variables

The dual (in this case) is a minimization problem with ≥ constraints and variables unconstrained in sign.

The min cost flow problem and its dual

Minimize $\sum_{(i,j)\in A} c_{ij}x_{ij}$

$\sum_j x_{ij} - \sum_k x_{ki} = b_i$ for all $i \in N$.

and $x_{ij} \geq 0$ for all $(i,j) \in A$.

Primal

Minimize $\sum_{i=1}^{n} \pi_i b_i$

Dual

subject to $\pi_i - \pi_j \leq c_{ij}$ for all $(i, j) \in A$

Summary

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