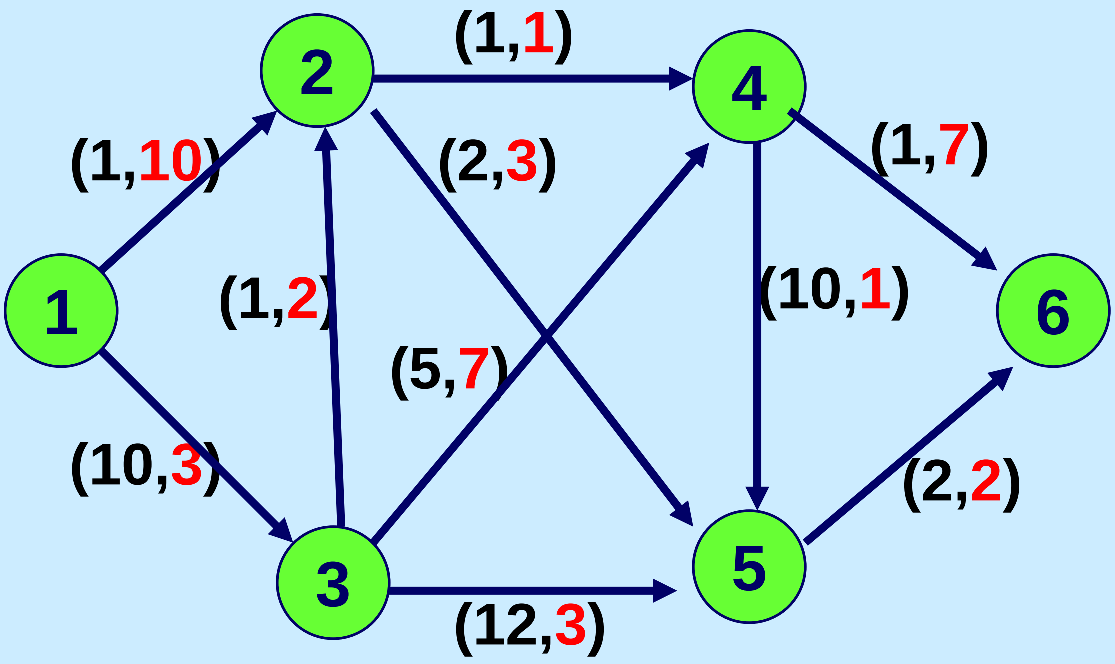
15.082J and 6.855J

Lagrangian Relaxation 2

- Algorithms
- Application to LPs

The Constrained Shortest Path Problem



Find the shortest path from node 1 to node 6 with a transit time at most 10

Constrained Shortest Paths: LP Formulation

Given: a network G = (N,A)

c_{ij} cost for arc (i,j)

t_{ij} traversal time for arc (i,j)

Z* = Min $\sum_{(i,j)\in A} c_{ij} x_{ij}$

s. t. $\sum_j x_{ij} - \sum_j x_{ji} = \begin{cases} 1 & \text{if i = s} \\ -1 & \text{if i = t} \\ 0 & \text{otherwise} \end{cases}$

$\sum_{(i,j)\in A} t_{ij} x_{ij} \le T$ Complicating constraint

$x_{ij} = 0 \text{ or } 1 \quad \text{for all } (i,j) \in A$

Lagrangian Relaxation and Inequality Constraints

$$
\begin{array}{llll}
z^* = & \min & cx & \\
 & \text{subject to} & Ax \le b, & \text{(P)} \\
 & & x \in X. &
\end{array}
$$

$$
\begin{array}{llll}
L(\mu) = & \min & cx + \mu(Ax - b) & \text{(P}(\mu)\text{)} \\
 & \text{subject to} & x \in X. &
\end{array}
$$

$L^* = \max\ (L(\mu) : \mu \ge 0)$.

So we want to maximize over μ, while we are minimizing over x.

An alternative representation

Suppose that $X = \{x^1, x^2, x^3, \ldots, x^K\}$. Possibly K is exponentially large; e.g., X is the set of paths from node s to node t.

$$L(\mu) \quad = \quad \min \quad cx + \mu(Ax - b) = (c + \mu A)x - \mu b$$

$$\text{subject to} \quad x \in X.$$

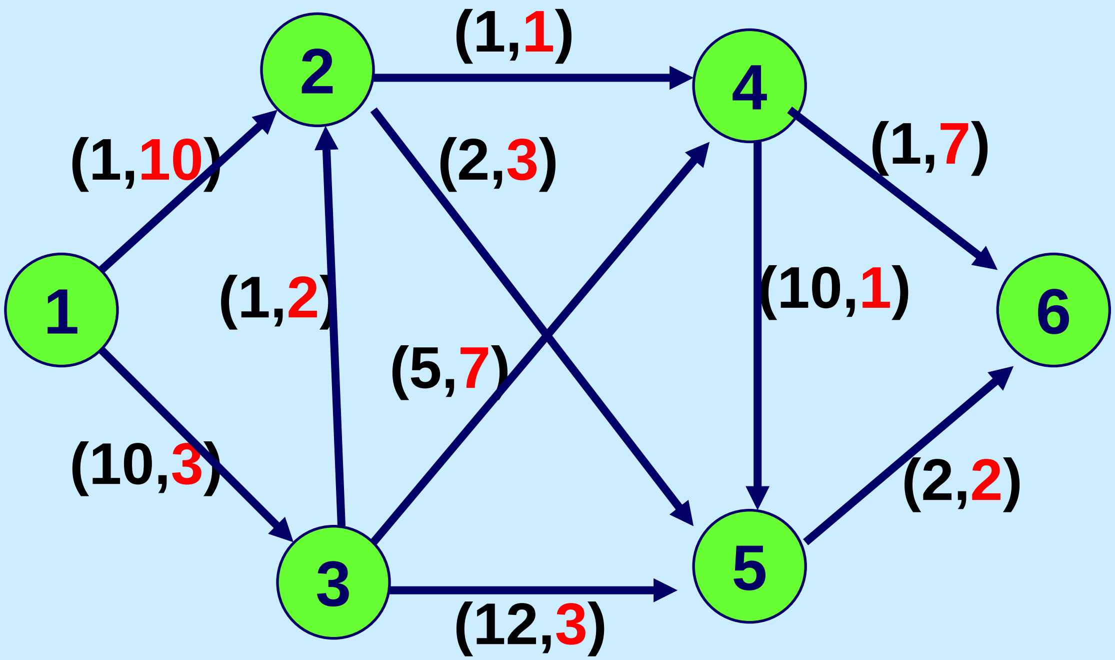
$$L(\mu) \quad = \quad \min \; \{(c + \mu A)x^k - \mu b : k = 1 \text{ to } K\}$$

Solving the Lagrange Multiplier Problem

$L(\mu) \quad = \quad \min \ \{(c + \mu A)x^k - \mu b : k = 1 \text{ to } K\}$

Determine $L^* = \max\ (L(\mu) : \mu \geq 0\}$

Suppose we want the min cost path from 1 to 6 with transit time at most 14.



We now list all K paths from 1 to 6.

P	c_P	t_P	$c_P + \mu\,(t_P - 14)$
1-2-4-6	3	18	$3 + 4\,\mu$
1-2-5-6	5	15	$5 + \mu$
1-2-4-5-6	14	14	14
etc.			

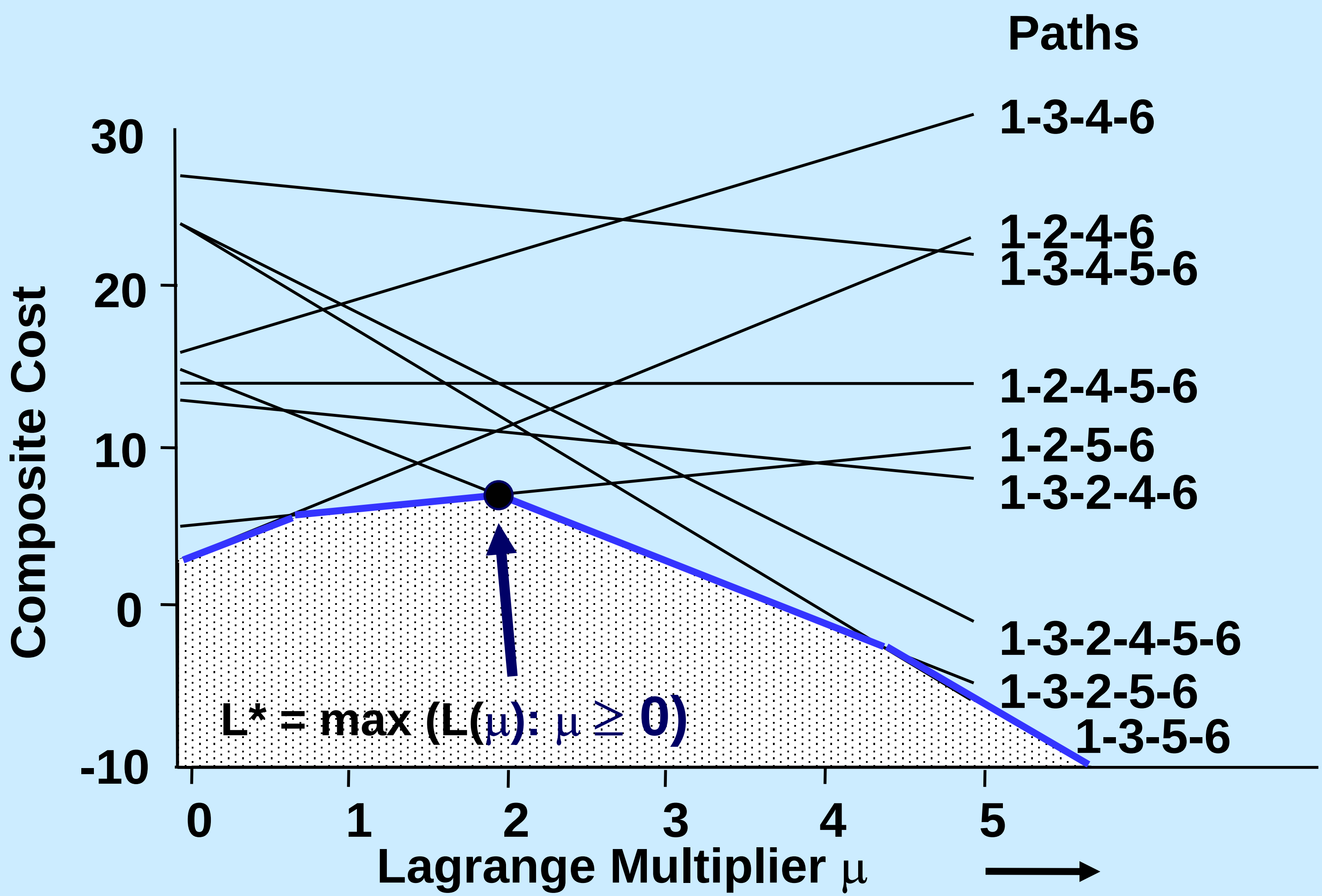


Figure 16.3 The Lagrangian function for T = 14.

Solving the Lagrange Multiplier Problem

$$L(\mu) = \min \{(c + \mu A)x^k - \mu b : k = 1 \text{ to } K\}$$

In the algorithm S will be a subset of {1, 2, …, K}

Let $L_S(\mu) = \min \{(c + \mu A)x^k - \mu b : k \in S\}$

$L_S(\mu) \geq L(\mu)$.

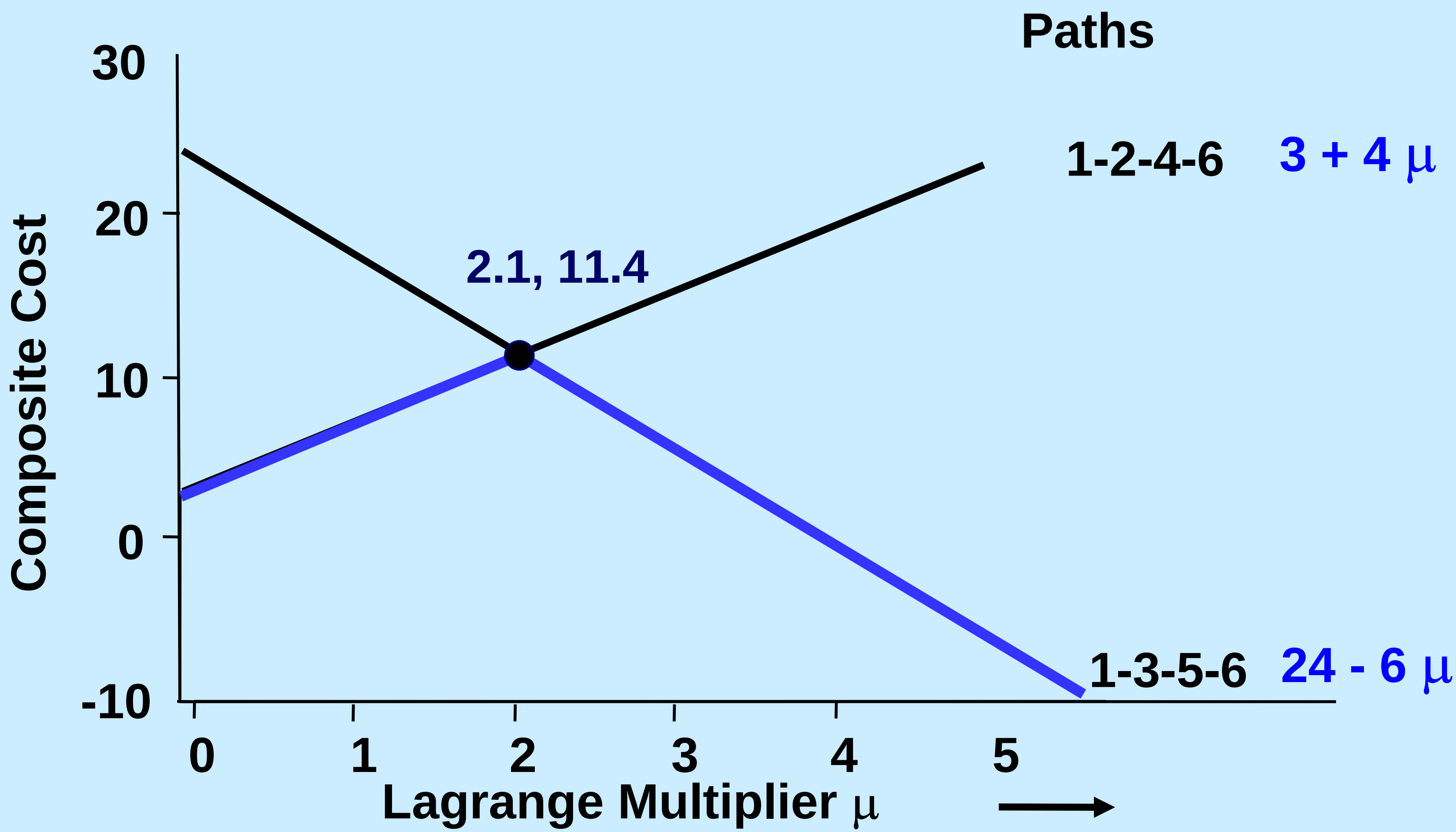
So, $L^*_S \geq L^*$.

1. Initialize S
2. Find μ^* that maximizes $L^*_S = L_S(\mu)$.
3. Find $x^k \in X$ that minimizes $(c + \mu^* A)x$
4. If $k \in S$, quit with the optimal solution
5. Else, add k to S, and return to step 2.

At end,

$$\begin{aligned} L^*_S &= L_S(\mu^*) \\ &= L(\mu^*) \\ &\leq L^*. \end{aligned}$$

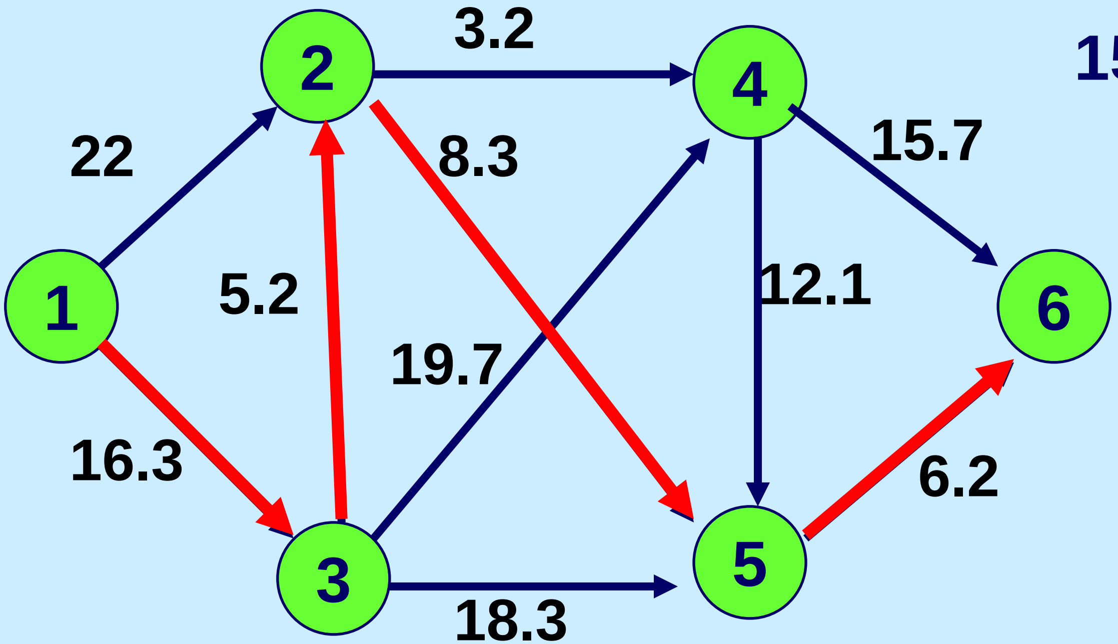
We start with the paths 1-2-4-6, and 1-3-5-6 which are optimal for L(0) and L(∞).



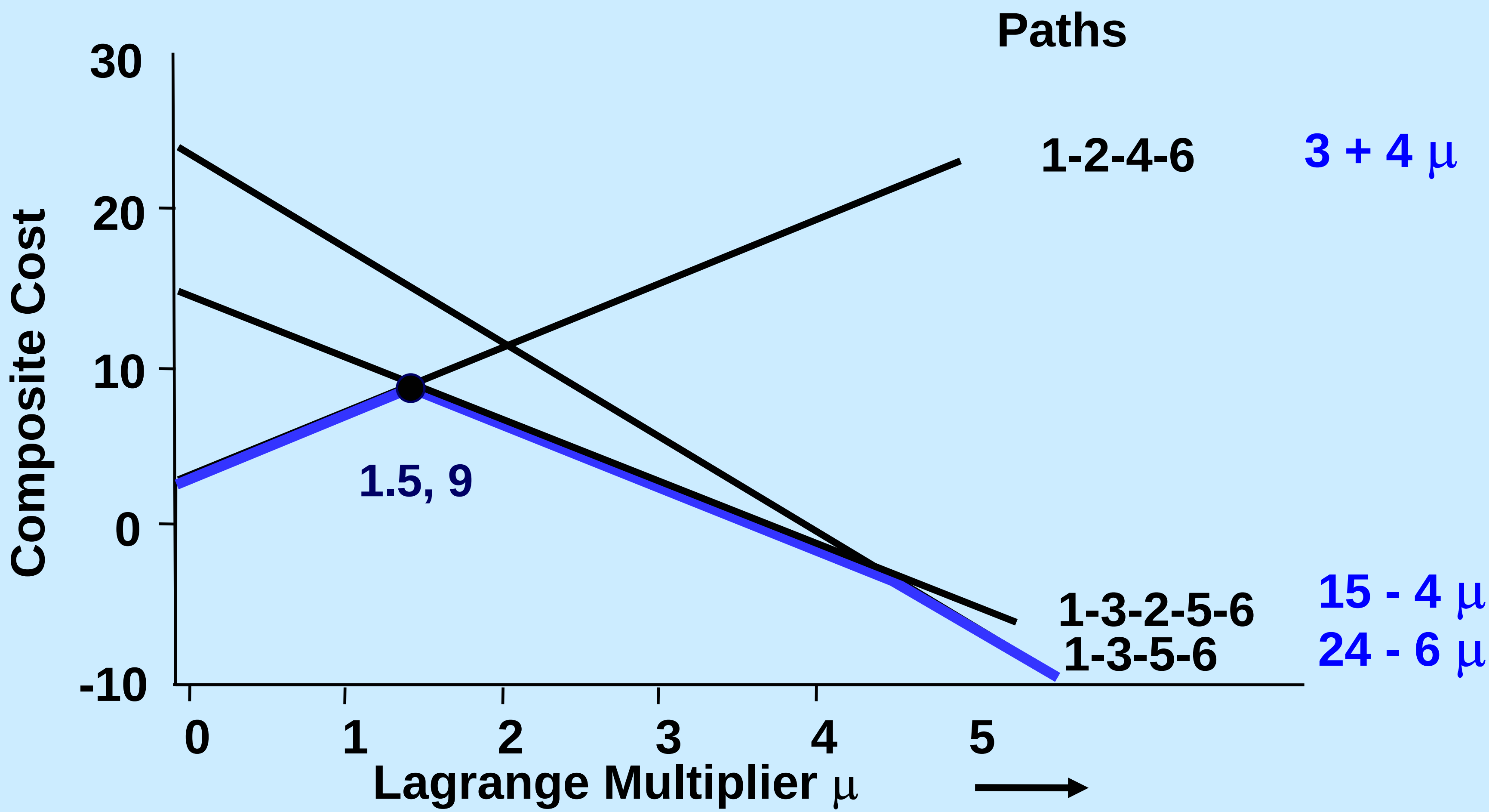
Set μ = 2.1 and solve the constrained shortest path problem

The optimum path is 1-3-2-5-6

15 + 10 μ

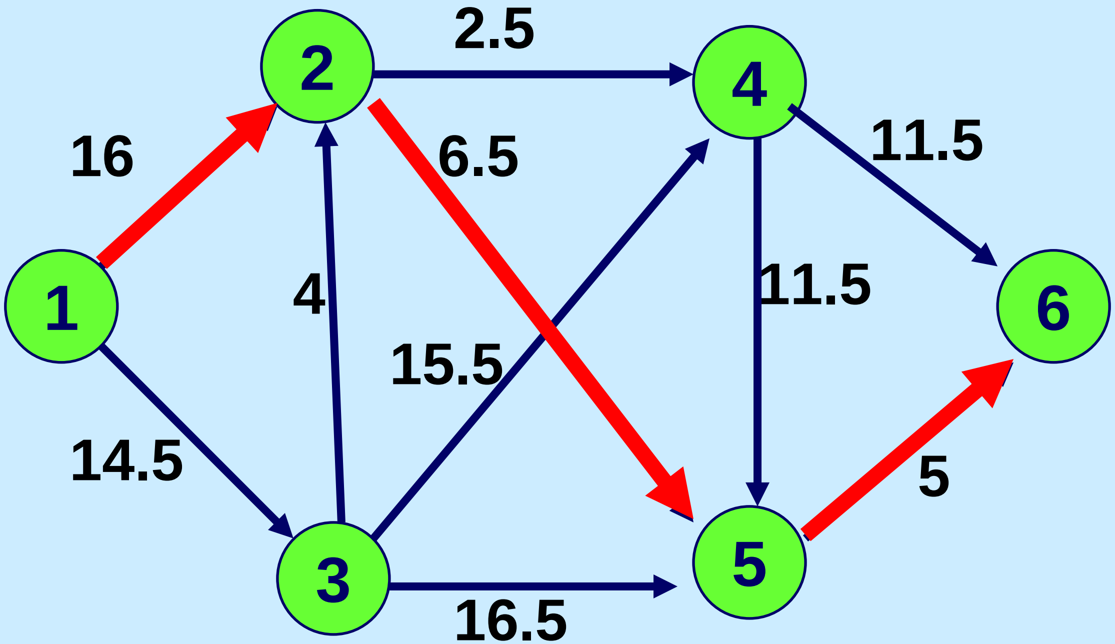


Add Path 1-3-2-5-6 and reoptimize

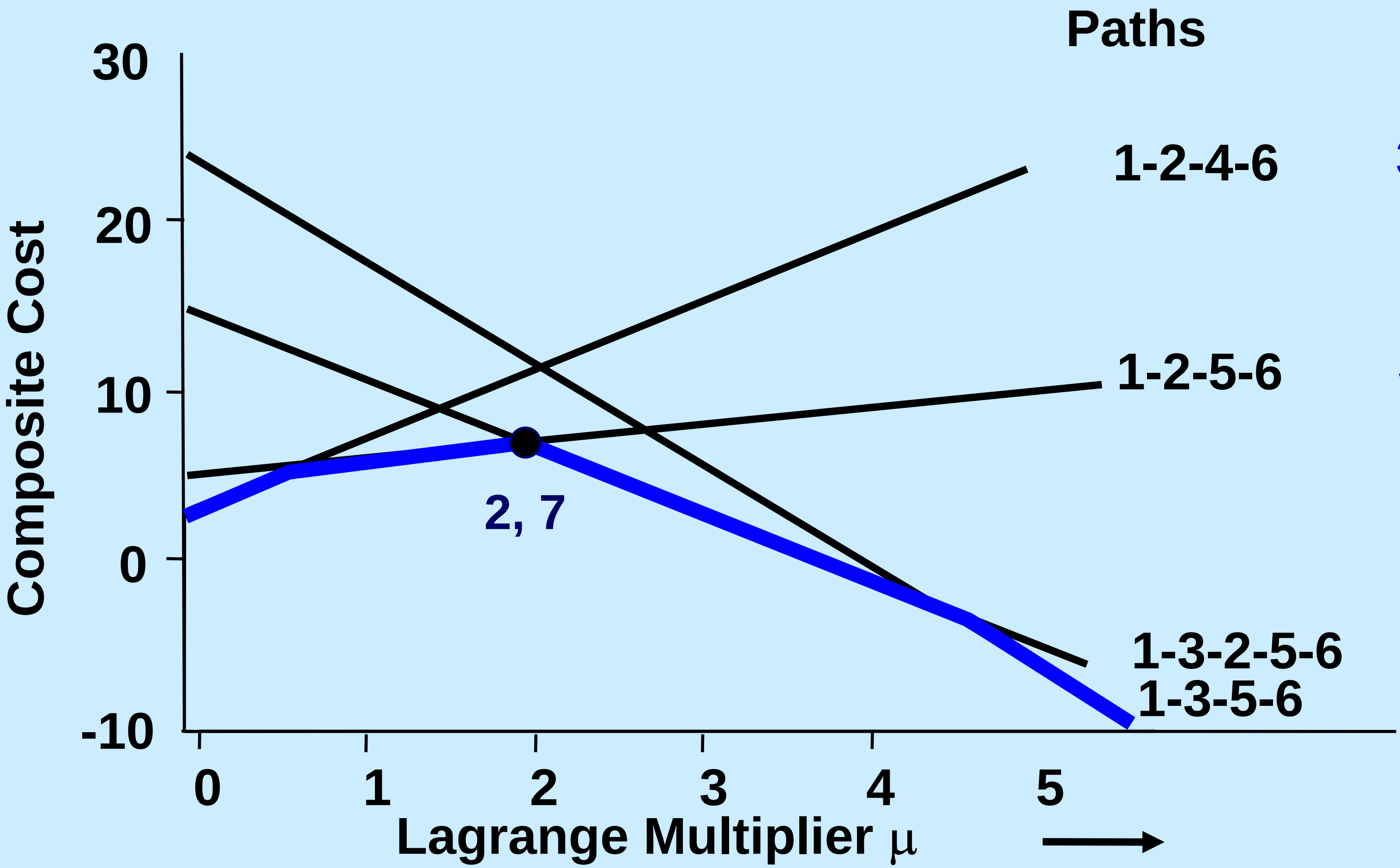


Set $\mu = 1.5$ and solve the constrained shortest path problem

The optimum path is 1-2-5-6.



Add Path 1-2-5-6 and reoptimize



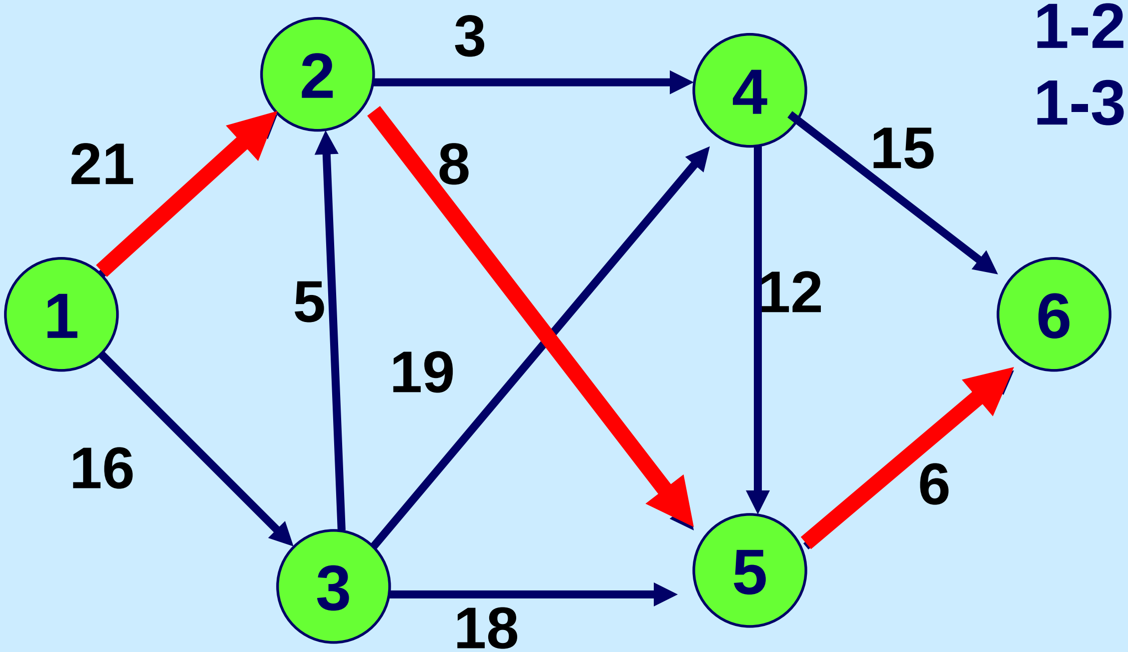
3 + 4 μ

5 + μ

15 - 4 μ

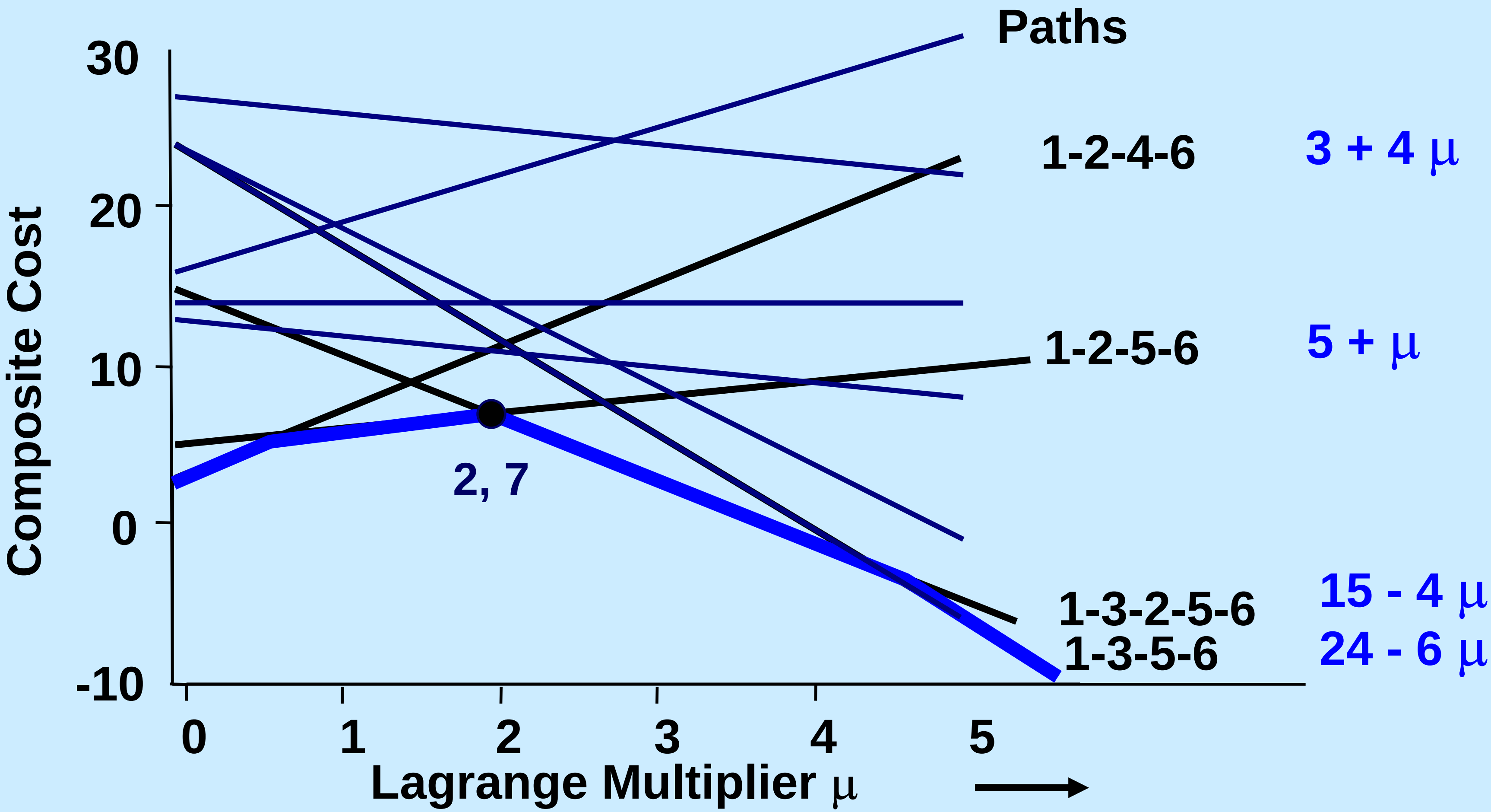
24 - 6 μ

Set μ = 2 and solve the constrained shortest path problem



The optimum paths are 1-2-5-6 and 1-3-2-5-6

There are no new paths to add.
μ* is optimal for the multiplier problem



Solving the Lagrange Multiplier Problem

$L(\mu) = \min \{(c + \mu A)x^k - \mu b : k = 1 \text{ to } K\}$

Let S be a subset of {1, 2, …, K}

Let $L_S(\mu) = \min \{(c + \mu A)x^k - \mu b : k \in S\}$

$L_S(\mu) \geq L(\mu)$.

So, $L^*_S \geq L^*$.

1. Initialize S
2. Find μ^* that maximizes $L^*_S = L_S(\mu)$.
3. Find $x^k \in X$ that minimizes $(c + \mu^* A)x$
4. If $k \in S$, quit with the optimal solution
5. Else, add k to S, and return to step 2.

At end,

$$L^*_S = L_S(\mu^*) = L(\mu^*) \leq L^*.$$

Solving the Lagrange Multiplier Problem

Let $L_S(\mu) = \min \{(c + \mu A)x^k - \mu b : k \in S\}$

Find μ^* that maximizes $L_S(\mu)$.

- **This is a linear program (see next slide)**

Find $x^k \in X$ that minimizes $(c + \mu^* A)x$

- **This is optimizing over X**

Usually X is chosen so that optimizing over X is something that "easy" to do

e.g., finding a minimum cost spanning tree or finding a minimum cost flow

Determining L^*_S by solving an LP

$$\min(a_1, a_2, \ldots, a_n) = \max\{ w : w \le a_j \text{ for } j = 1 \text{ to } n \}$$

$$\min(4, 9, 3) = \max\{ w : w \le 4;\ z \le 9;\ z \le 3 \}.$$

$$L_S(\mu) = \min \{(c + \mu A)x^k - \mu b : k \in S\}$$

$$L_S(\mu) = \max \{w : w \le (c + \mu A)x^k - \mu b \text{ for } k \in S\}$$

$$
\begin{aligned}
L^*_S = \max \quad & w \\
& w \le (c + \mu A)x^k - \mu b \quad \text{for } k \in S \\
& \mu \ge 0
\end{aligned}
$$

Theorem 16.5

The Lagrangian multiplier problem $L^* = \max_{\mu} L(\mu)$ with

$L(\mu) = \{\min cx + \mu(Ax - b) : x \in X\}$ is equivalent to the linear programming problem:

$$
\begin{aligned}
L^* \quad = \quad \max \quad & w \\
& w \leq (c + \mu A)x^k - \mu b \quad \text{for } k = 1 \text{ to } K \\
& \mu \geq 0
\end{aligned}
$$

Subgradient optimization

Another major solution technique for solving the Lagrange Multiplier Problem is subgradient optimization.

Based on ideas from non-linear programming.

It converges (often slowly) to the optimum.

See the textbook for more information.

Application	*Embedded Network Structure*
Networks with side constraints	**minimum cost flows** **shortest paths**
Traveling Salesman Problem	**assignment problem** **minimum cost spanning tree**
Vehicle routing	**assignment problem** **variant of min cost spanning tree**
Network design	**shortest paths**
Two-duty operator scheduling	**shortest paths** **minimum cost flows**
Multi-time production planning	**shortest paths / DPs** **minimum cost flows**

Lagrangian Relaxation and LPs

Theorem 16.6. Suppose that we apply the Lagrangian relaxation technique to solving a linear program P' defined as

$$\begin{aligned} \text{minimize} \quad & cx \\ \text{subject to} \quad & Ax = b \\ & Dx \le q \\ & x \ge 0 \end{aligned} \qquad \text{P'}$$

By relaxing the constraint $Ax = b$. Then the optimal value L* of the Lagrangian multiplier problem equals the optimal objective value of P'.

Proof. See AMO.

On Lagrangian Relaxation and LPs

Advantage of solving the Lagrangian Relaxation

- The relaxed problem may be much easier to solve
- The relaxed problem may decompose into smaller problems

Comment on Theorem 16.6

- We will use it to develop a stronger result on Lagrangian Relaxations based on convex hulls.

Convex Hulls

Suppose that $X = \{x^1, x^2, \ldots, x^K\}$ is a finite set.

Vector x is a convex combination of $X = \{x^1, x^2, \ldots, x^K\}$ if there is a feasible solution to

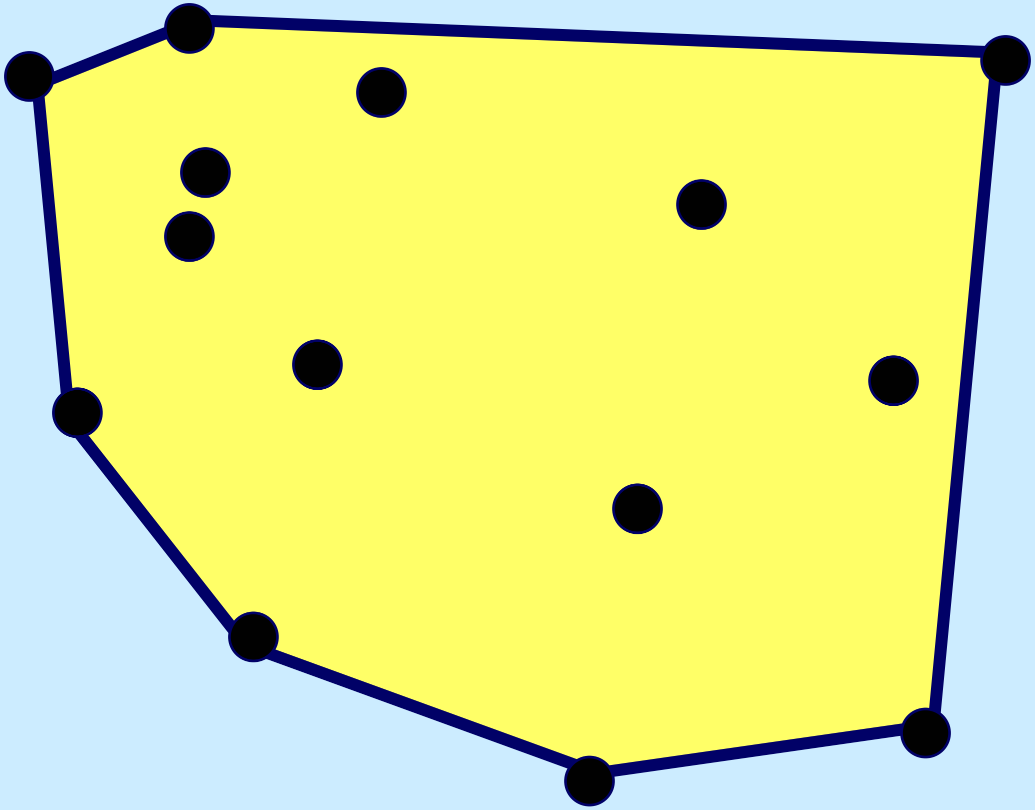
$$x = \sum_{k=1}^{K} \lambda_k x^k$$

$$\sum_{k=1}^{K} \lambda_k = 1$$

$$\lambda_k \geq 0 \;\; \text{for } k = 1 \text{ to K}$$

The ***convex hull*** of X is H(X) = { x : x can be expressed as a convex combination of points in X.}

Convex Hulls in two dimensions



The convex Hull H(X) is the smallest LP feasible region that contains all points of X.

Property 16.7

1. The set H(X) is a polyhedron, that is, it can be expressed as $H(X) = \{x : Ax \le b\}$ for some matrix A and vector b.
2. Each extreme point of H(X) is in X. If we minimize $\{cx : x \in H(X)\}$, the optimum solution lies in X.
3. Suppose $X \subseteq Y = \{x : Dx \le c \text{ and } x \ge 0\}$.
   Then $H(X) \subseteq Y$.

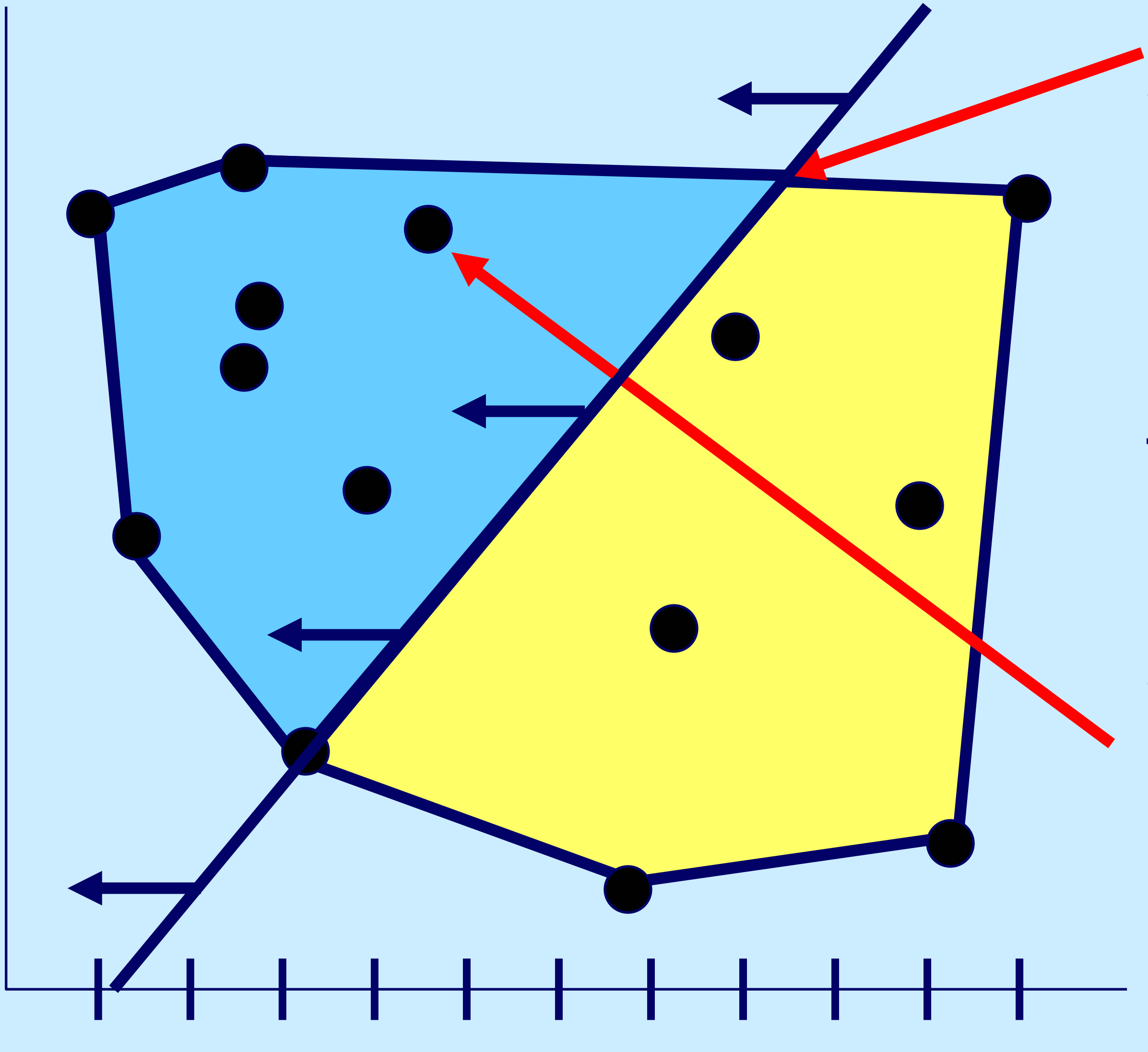
Theorem 16.8

The optimal objective function value L* of the Lagrangian multiplier problem equals the optimal objective function value of the linear program

$$\begin{array}{ll} \textbf{minimize} & cx \\ \textbf{subject to} & Ax = b \\ & x \in H(X) \end{array} \qquad \text{P*}$$

$$\begin{array}{ll} \textbf{minimize} & cx \\ \textbf{subject to} & Ax = b \\ & x \in X \end{array} \qquad \text{P}$$

First, note that P* $\neq$ P



max x_1

s.t.

$x_1 - x_2 \leq 1$

$x \in H(X)$

is different from

max x_1

s.t.

$x_1 - x_2 \leq 1$

$x \in X$

Theorem 16.8. The optimal objective function value L* of the Lagrangian multiplier problem equals the optimal objective function value of the linear program P*.

$$\begin{aligned} \textbf{minimize} \quad & cx \\ \textbf{subject to} \quad & Ax = b \\ & x \in H(X) \end{aligned} \qquad \text{P*}$$

Proof. For multiplier μ, the Lagrangian Relaxation is:

$$\begin{aligned} L(\mu) = \min \quad & cx + \mu(Ax - b) \\ \text{subject to} \quad & x \in X \end{aligned}$$

which by Property 16.7 is equivalent to:

$$\begin{aligned} L(\mu) = \min \quad & cx + \mu(Ax - b) \\ \text{subject to} \quad & x \in H(X) \end{aligned} \qquad (16.2)$$

$$L(\mu) = \min \quad cx + \mu(Ax - b)$$
$$\text{subject to} \quad x \in H(X)$$

(16.2)

(16.2) can be expressed as a linear program.

Moreover, (16.2) is a Lagrangian relaxation for:

$$\begin{aligned} \text{minimize} \quad & cx \\ \text{subject to} \quad & Ax = b \\ & x \in H(X) \end{aligned}$$

P*

By Theorem 16.6, the optimal solution for the Lagrangian multiplier problem for P is equal to the optimal solution value for P*.

Theorem 16.9

Suppose we solve the Lagrangian multiplier problem for

$$\begin{aligned} \min \quad & cx \\ \text{s.t.} \quad & Ax = b \\ & Dx \le q \\ & x \ge 0 \quad x \text{ integer} \end{aligned}$$

Then $z^0 \le L^*$, where

$$z^0 = \begin{aligned} \min \quad & cx \\ \text{s.t} \quad & Ax = b \\ & Dx \le q \\ & x \ge 0 \end{aligned}$$

Proof.

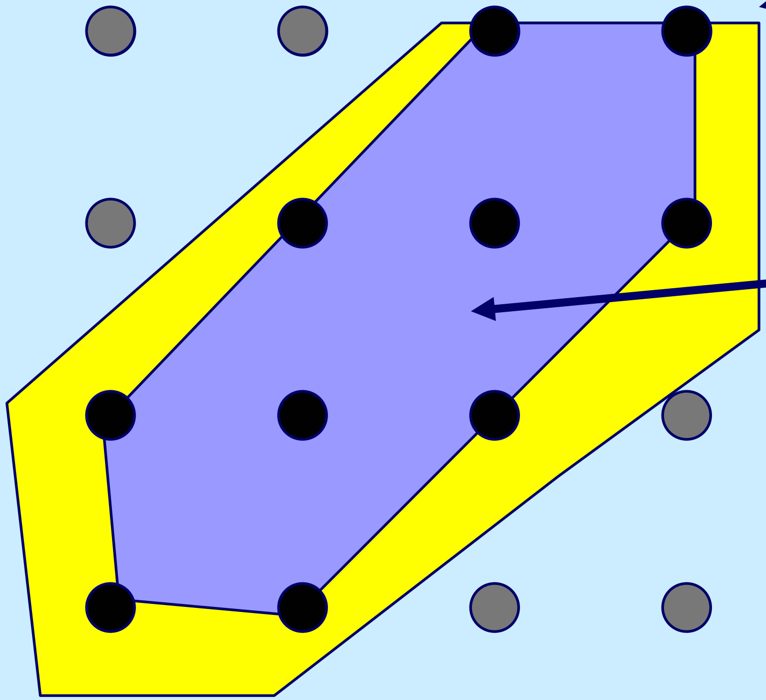
$$L^* = \begin{aligned} \min \quad & cx \\ \text{s.t} \quad & Ax = b \\ & x \in H(X) \end{aligned}$$

$$H(X) =$$

$$H(\{x : Dx \le q,\ x \ge 0,\ x \text{ integer}\})$$

$$\subseteq \{x : Dx \le q,\ \ x \ge 0\}$$

An Example of Theorem 16.9



The feasible region for $Dx \leq q$, $x \geq 0$

The feasible region for $x \in H(X)$

The feasible points for

$Dx \leq q$, $x \geq 0$,
x integer

are in black.

Integrality Property

Suppose X = {x : Dx $\leq$ q , x $\geq$ 0, x integer}.

We say that X satisfies if the ***integrality property*** if the following LP has integer solutions for all d

$$\begin{array}{ll} \textbf{minimize} & dx \\ \textbf{subject to} & Dx \leq q \\ & x \geq 0 \end{array}$$

Fact: if X satisfies the integrality property then

min (dx: Dx $\leq$ q , x $\geq$ 0)

= min (dx : x $\in$ X)

= min (dx : x $\in$ H(x))

Theorem 16.10

If the Lagrangian subproblem of the optimization problem P satisfies the integrality property then $L^* = z^0$, the solution to the LP relaxation of P.

Example: Generalized Assignment

Minimize $\sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij}$ (16.10a)

$\sum_{j \in J} x_{ij} = 1$ for each $i \in I$ (16.10b)

$\sum_{i \in I} a_{ij} x_{ij} \le d_j$ for each $j \in J$ (16.10c)

$x_{ij} \ge 0$ and integer for all $(i, j) \in A$ (16.10d)

If we relax (16.10c), the bound for the Lagrangian multiplier problem is the same as the bound for the LP relaxation.

If we relax (16.10b), the LP does not satisfy the integrality property, and we should get a better bound than z^0.

Summary

- A decomposition approach for Lagrangian Relaxations
- Relating Lagrangian Relaxations to LPs