

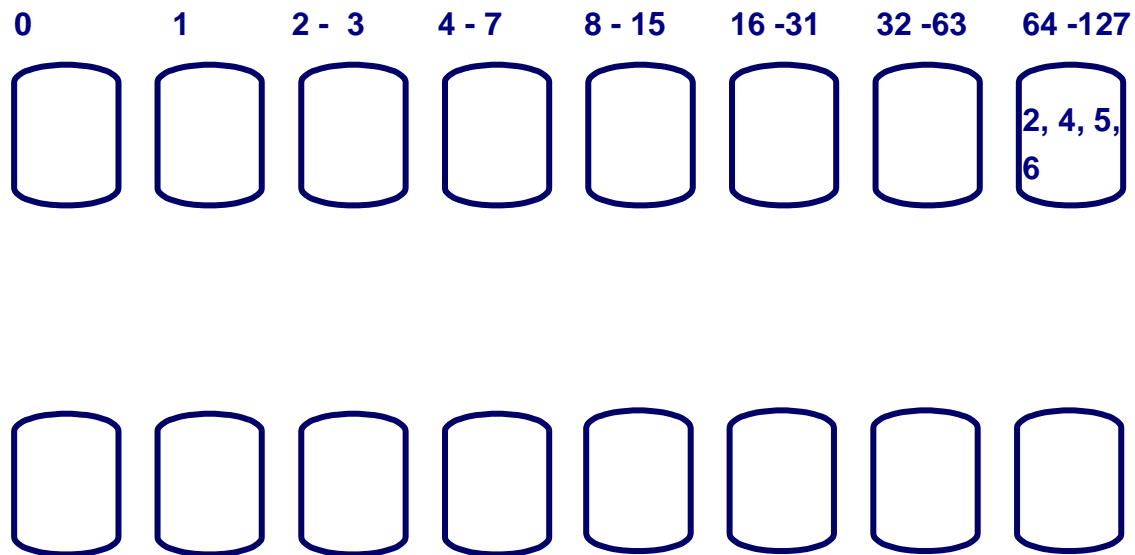
## **15.082 Midterm**

**(closed book, except for  
2 sides of one sheet of paper )**

1. Answer all questions in exam books provided, except for number 3, which part of the answer is on the exam book.
2. Budget your time. If a problem is taking too long, you may want to go on to the next one. You may want to consider doing problems out of sequence and leave more time-consuming problems for the end of the exam.
3. The scores add up to 105.

1. (10 points) Here are the buckets for some iteration of the R-heap algorithm. The algorithm is called again as part of the find-min operation". What is the state of the buckets after the find-min computation is completed, that is after node 2 is deleted from the buckets and made permanent. (What is the range of each bucket, and what nodes are in which buckets?) You can write the ranges and put elements on the buckets below.

node	1	2	3	4	5	6
d(node)	0	71	1	93	78	120



**The buckets for your answer for question 1.**

2. (10 points) Give a flow decomposition of the flow in Figure 2. The numbers on the arcs represent the flows. As usual, any path in the flow decomposition should be directed from a supply node to a demand node.

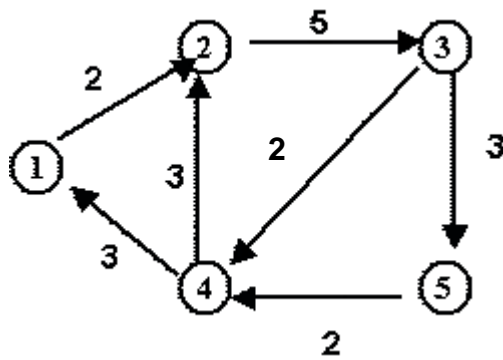


Figure 2. The network for problem 2.

3. (10 points). Consider the max flow problem in the diagram Figure 3.

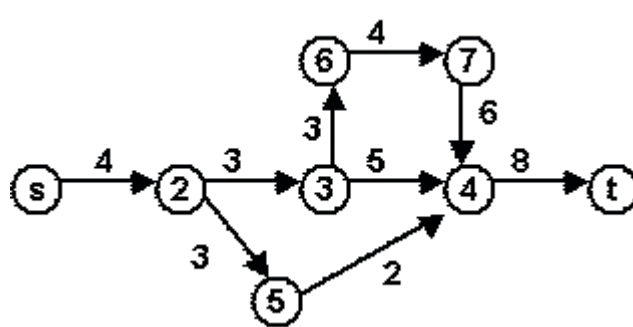
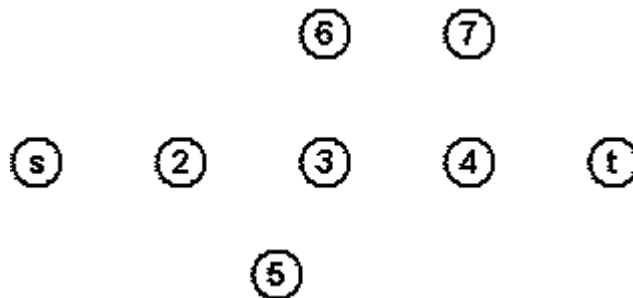


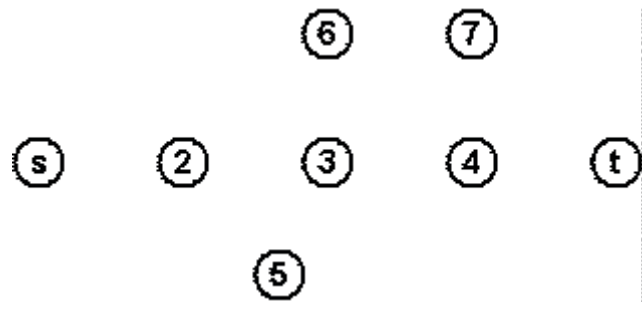
Figure 3. The network for Problem 3

Use the Ford-Fulkerson algorithm, and use breadth first search to find the augmenting paths.

a. (10 points) What are the augmenting paths? Show the residual network after each augmentation. You may draw the residual network on the paper below.

b. (5 points) What is the minimum cut?





4. (15 points, 3 points per part). Let  $G = (N, A)$  be a network with  $n$  nodes and  $m$  arcs. Assume that  $m > n$ . What is the running time for the algorithms presented for the following problems? Express the running time in terms of  $n$  and  $m$ .

- a. Carrying out a depth first search.
- b. Identifying all of the undirected components of a graph.
- c. Determining a flow decomposition.
- d. Finding the shortest path from one node to all others in network in which arc lengths may be non-positive, but there are no negative length cycles.
- e. Find the maximum cardinality matching on a network with  $2n$  nodes ( $n$  on each side) by first converting it to a max flow problem, and then using the Ford Fulkerson augmenting path algorithm. (Also, give an upper bound on the number of augmentations.)

5. (15 points) Several families go out to dinner together. To increase their social interaction, they would like to sit at tables so that no two members of the same family are at the same table. Show how to formulate the problem of finding a feasible seating arrangement (one that meets the above conditions) as a max flow problem. Assume that the dinner contingent has  $p$  families and that the  $i$ th family has  $q(i)$  members. Assume also that there are  $t$  tables, and that the  $j$ th table has a seating capacity of  $b(j)$ .

6. (20 points, 5 points apiece.) For each of the following questions, answer True or False, and briefly justify your answer. Each of the problems is defined on a network  $G = (N, A)$ . Except for part b, you may assume that the network is connected. In general,  $c_{ij}$  denotes the arc length of arc  $(i, j)$ .

- a. The running time for determining a topological sort is  $O(nm)$ .
- b. Suppose that we consider a max flow problem on an undirected network  $G = (N, A)$ . Suppose we replace each arc  $(i, j)$  with capacity  $u_{ij}$  with two directed arcs  $(i, j)$  and  $(j, i)$  each with the same capacity  $u_{ij}$ . Then the flow value of this transformed problem is the same as the flow value of the original problem. Assume that all lower bounds are 0.
- c. Suppose that all paths in a network  $G = (N, A)$  have at most  $K$  arcs. Then the running time for the FIFO label correcting algorithm is  $O(mK)$ .
- d. If the arc costs in a network are all relatively prime, then the network has a unique shortest path tree. (Two integers are relatively prime if their greatest common divisor is 1. For example, 10 and 21 are relatively prime, but 6 and 9 are not.)

7. (10 points, 5 points per part). Multiple Choice. Give the best answer to each question.
- a. Consider the problem of determining the shortest path from node 1 to each other node. Let  $d^*(j)$  denote the shortest path distance to node  $j$ . Suppose that the shortest path from node 1 to node  $k$  has exactly three arcs on it. Let  $c'$  be obtained from  $c$  as follows:  $c'_{ij} = c_{ij} + 10$  for each  $(i,j) \in A$ , and let  $d'(j)$  denote the shortest path distance to node  $j$  using costs  $c'$ . Which of the following is the best answer:
- $d'(k) = d^*(k) + 30$ .
  - $d'(k) \leq d^*(k) + 30$
  - $d'(k) \geq d^*(k) + 30$
  - there is not enough information
- b. Suppose that  $D$  is a directed cycle, and let  $|D|$  denote the number of arcs on the cycle  $D$ . Let  $c(D)$  denote the cost of the cycle  $D$ . Let the mean cost of cycle  $D$  be defined as  $\mu(D) = c(D)/|D|$ . Suppose now that each arc length in the network increases by exactly 10 units, and let  $\mu'(D)$  be the mean of the cycle  $D$  after the change. Which of the following is the best answer.
- $\mu'(D) = \mu(D) + 10$ .
  - $\mu'(D) \leq \mu(D) + 10$
  - $\mu'(D) \geq \mu(D) + 10$
  - none of the above
8. (10 points, 5 points per part) Consider a shortest path problem on a network  $G = (N, A)$  in which negative arc lengths are permitted, but in which all of the negative arc lengths are on arcs directed into node 4. Show how to find the shortest paths from node 1 to all other nodes by solving two shortest path problems, each on networks with non-negative costs. We do this in parts a and b.
- Show how to determine whether there is a negative cost cycle in the original graph by solving just one shortest path problem on a network with non-negative arc lengths.
  - Suppose that there is no negative cost cycle. Use the results of part a and one additional shortest path computation on a network with non-negative arc lengths in order to find the shortest paths from node 1 to all other nodes. HINT: "reduced costs"