15.082 and 6.855J

Recitation about LP

Linear Program

Minimize $c'x$

Subject to

$Ax \geq b,$

$Bx \geq d,$

$Cx \leq f,$

The variables can be non negative, non positive or free.

An example of Linear Program

Minimize $-x_1 - x_2$

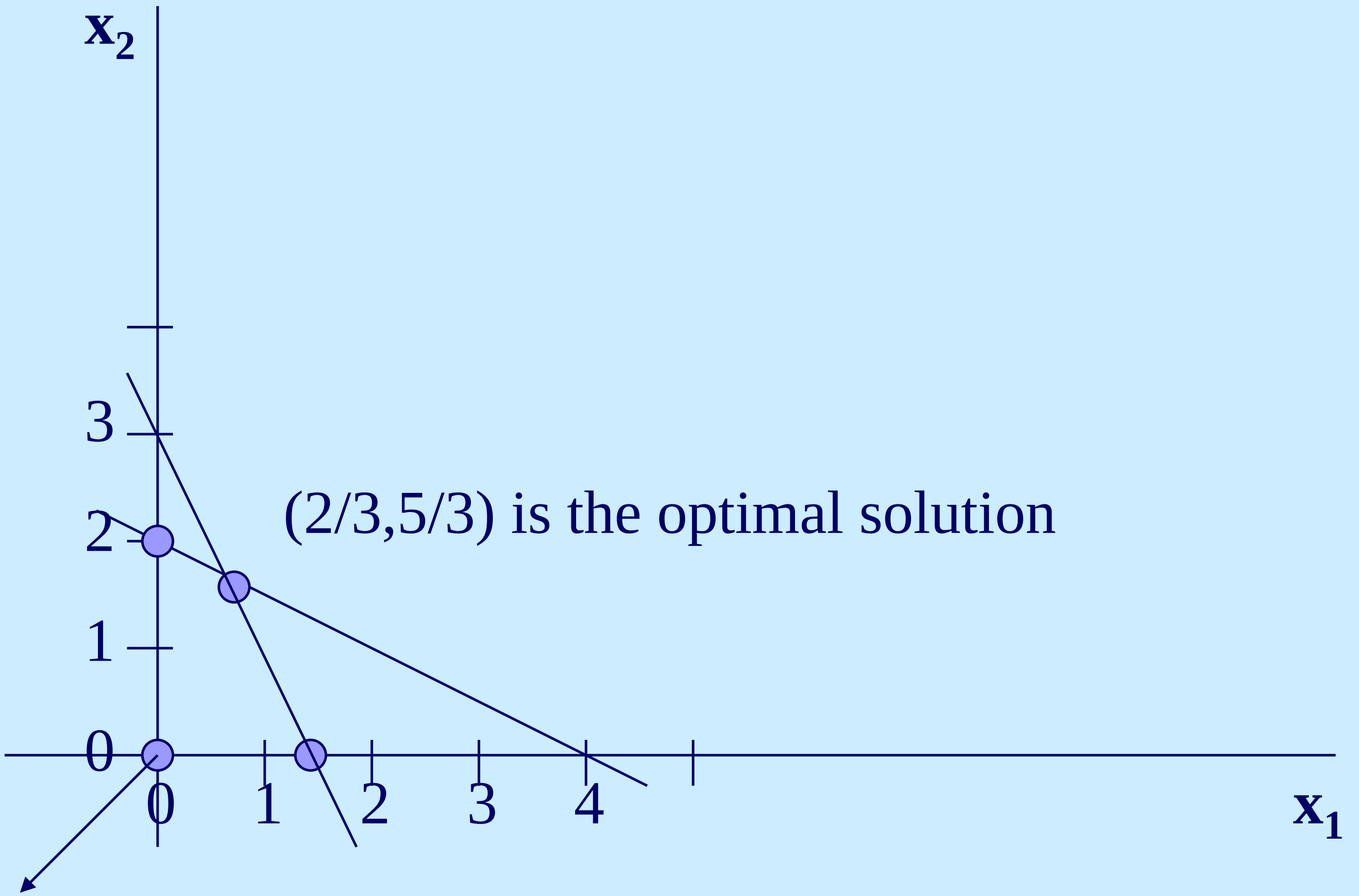
Subject to

$x_1 + 2x_2 \leq 4$,

$2x_1 + x_2 \leq 3$,

And $0 \leq x_1, x_2$

Example (cont'd)



Simplex method

We need the LP to be in standard form in order to run simplex method.

This means that the LP has:

-Only equality constraints

-Non negative RHS

-Non negative variables

We need an initial *basic feasible solution* to start the method.

Standard form of the example

We add extra non negative variables s_1, s_2 to transform the inequality constraints into equality constraints.

The standard form:

Minimize $-x_1 - x_2$

Subject to

$$x_1 + 2x_2 + s_1 = 4,$$

$$2x_1 + x_2 + s_2 = 3,$$

And $0 \leq x_1, x_2, s_1, s_2$

Basic Feasible Solutions

Suppose there are m constraints, n variables

A basic solution is found by setting n - m variables to 0 and solving the remaining system with n variables and n constraints.

- The n – m variables are called non-basic variables
- The m variables are called basic variables

In our example, n = 4 and m = 2.

A basic feasible solution is $(x_1, x_2, s_1, s_2) = (0, 0, 4, 3)$

Simplex Method in our example

Initial Basic Feasible solution: (0, 0, 4, 3),

Cost: 0

-z	x_1	x_2	s_1	s_2		
1	-1	-1	0	0	=	0
0	1	2	1	0	=	4
0	2	1	0	1	=	3

x_1 will enter to the basis because it has negative reduced cost

s_2 will leave the basis.

Next iteration

Basic Feasible solution: (3/2, 0, 5/2, 0),

Cost: -3/2

-z	x_1	x_2	s_1	s_2		
1	0	-1/2	0	1/2	=	3/2
0	0	3/2	1	-1/2	=	5/2
0	1	1/2	0	1/2	=	3/2

x_2 will enter to the basis because it has negative reduced cost

s_1 will leave the basis.

Last iteration

Basic Feasible solution: (5/3, 2/3, 0, 0),

Cost: -7/3

-z	x_1	x_2	s_1	s_2		
1	0	0	1/3	1/3	=	7/3
0	0	1	2/3	-1/3	=	5/3
0	1	0	-1/3	1/3	=	2/3

This solution is optimal because all the reduced costs are non negative.

Duality

Primal Problem P

minimize $z = cx$

subject to $Ax = b$

$x \geq 0$

optimum value is z*

Dual Problem D

maximize $v = \pi b$

subject to $\pi A \leq c$

optimum value is v*

Theorem. (Strong Duality) If both P and D are feasible, then $z^* = v^*$.

Duality

Complementary Slackness:

Let x, π be feasible solutions to the primal and dual problem, respectively. Then they are optimal solutions for the two respective problems if and only if:

$\pi_i\ (a_i'x - b_i) = 0$ for all i,

$(c_j - \pi' A_j)x_j = 0$ for all j.