15.093 Optimization Methods

Lecture 23: Semidefinite Optimization

1 Outline

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1. Preliminaries
2. SDO
3. Duality
4. SDO Modeling Power
5. Barrier Algorithm for SDO

2 Preliminaries

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- A symmetric matrix $\boldsymbol{A}$ is positive semidefinite ($\boldsymbol{A} \succeq \boldsymbol{0}$) if and only if

$$\boldsymbol{u}'\boldsymbol{A}\boldsymbol{u} \geq 0 \qquad \forall\, \boldsymbol{u} \in \mathcal{R}^n$$

- $\boldsymbol{A} \succeq \boldsymbol{0}$ if and only if all eigenvalues of $\boldsymbol{A}$ are nonnegative
- Inner product $\boldsymbol{A} \bullet \boldsymbol{B} = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} B_{ij}$

2.1 The trace

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- The *trace* of a matrix $\boldsymbol{A}$ is defined

$$\text{trace}(A) = \sum_{j=1}^{n} A_{jj}$$

- $\text{trace}(\boldsymbol{AB}) = \text{trace}(\boldsymbol{BA})$
- $\boldsymbol{A} \bullet \boldsymbol{B} = \text{trace}(\boldsymbol{A}'\boldsymbol{B}) = \text{trace}(\boldsymbol{B}'\boldsymbol{A})$

3 SDO

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- $\boldsymbol{C}$ symmetric $n \times n$ matrix
- $\boldsymbol{A}_i, i = 1, \ldots, m$ symmetric $n \times n$ matrices
- $b_i,\ i = 1, \ldots, m$ scalars
- Semidefinite optimization problem (SDO)

$$\begin{array}{lll} (P): & \min & \boldsymbol{C} \bullet \boldsymbol{X} \\ & \text{s.t.} & \boldsymbol{A}_i \bullet \boldsymbol{X} = b_i \quad i = 1, \ldots, m \\ & & \boldsymbol{X} \succeq \boldsymbol{0} \end{array}$$

3.1 Example

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$n = 3$ and $m = 2$

$$\boldsymbol{A}_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 3 & 7 \\ 1 & 7 & 5 \end{pmatrix}, \quad \boldsymbol{A}_2 = \begin{pmatrix} 0 & 2 & 8 \\ 2 & 6 & 0 \\ 8 & 0 & 4 \end{pmatrix}, \quad \boldsymbol{C} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 9 & 0 \\ 3 & 0 & 7 \end{pmatrix}$$

$$b_1 = 11, \quad b_2 = 19$$

$$\boldsymbol{X} = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$$

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$$\begin{aligned} (P): \quad \min \quad & x_{11} + 4x_{12} + 6x_{13} + 9x_{22} + 7x_{33} \\ \text{s.t.} \quad & x_{11} + 2x_{13} + 3x_{22} + 14x_{23} + 5x_{33} = 11 \\ & 4x_{12} + 16x_{13} + 6x_{22} + 4x_{33} = 19 \\ & \boldsymbol{X} = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} \succeq \boldsymbol{0} \end{aligned}$$

3.2 Convexity

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$$\begin{aligned} (P): \quad \min \quad & \boldsymbol{C} \bullet \boldsymbol{X} \\ \text{s.t.} \quad & \boldsymbol{A}_i \bullet \boldsymbol{X} = b_i \quad i = 1, \ldots, m \\ & \boldsymbol{X} \succeq \boldsymbol{0} \end{aligned}$$

The feasible set is **convex**:

$$\boldsymbol{X_1}, \boldsymbol{X_2} \text{ feasible} \Longrightarrow \lambda \boldsymbol{X_1} + (1-\lambda)\boldsymbol{X_2} \text{ feasible}, \quad 0 \le \lambda \le 1$$

.

$$\boldsymbol{A_i} \bullet (\lambda \boldsymbol{X_1} + (1-\lambda)\boldsymbol{X_2}) = \lambda \underbrace{\boldsymbol{A_i} \bullet \boldsymbol{X_1}}_{b_i} + (1-\lambda) \underbrace{\boldsymbol{A_i} \bullet \boldsymbol{X_2}}_{b_i} = b_i$$

.

$$\boldsymbol{u}'(\lambda \boldsymbol{X_1} + (1-\lambda)\boldsymbol{X_2})\boldsymbol{u} = \lambda \underbrace{\boldsymbol{u}'\boldsymbol{X_1}\boldsymbol{u}}_{\ge 0} + (1-\lambda) \underbrace{\boldsymbol{u}'\boldsymbol{X_2}\boldsymbol{u}}_{\ge 0} \ge 0$$

3.3 LO as SDO

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$$LO: \quad \min \quad \boldsymbol{c}'\boldsymbol{x}$$
$$\text{s.t.} \quad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$$
$$\boldsymbol{x} \geq \boldsymbol{0}$$

$$\boldsymbol{A}_i = \begin{pmatrix} a_{i1} & 0 & \dots & 0 \\ 0 & a_{i2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{in} \end{pmatrix}, \quad \boldsymbol{C} = \begin{pmatrix} c_1 & 0 & \dots & 0 \\ 0 & c_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & c_n \end{pmatrix}$$

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$$\begin{aligned} (P): \quad \min \quad & \boldsymbol{C} \bullet \boldsymbol{X} \\ \text{s.t.} \quad & \boldsymbol{A}_i \bullet \boldsymbol{X} = b_i, \quad i = 1, \dots, m \\ & X_{ij} = 0, \ i = 1, \dots, n, \ j = i+1, \dots, n \\ & \boldsymbol{X} \succeq \boldsymbol{0} \end{aligned}$$

$$\boldsymbol{X} = \begin{pmatrix} x_1 & 0 & \dots & 0 \\ 0 & x_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & x_n \end{pmatrix}$$

4 Duality

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$$\begin{aligned} (D): \quad \max \quad & \sum_{i=1}^m y_i b_i \\ \text{s.t.} \quad & \sum_{i=1}^m y_i \boldsymbol{A}_i + \boldsymbol{S} = \boldsymbol{C} \\ & \boldsymbol{S} \succeq \boldsymbol{0} \end{aligned}$$

Equivalently,

$$\begin{aligned} (D): \quad \max \quad & \sum_{i=1}^m y_i b_i \\ \text{s.t.} \quad & \boldsymbol{C} - \sum_{i=1}^m y_i \boldsymbol{A}_i \succeq \boldsymbol{0} \end{aligned}$$

4.1 Example

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$$(D) \quad \max \quad 11y_1 + 19y_2$$

$$\text{s.t.} \quad y_1 \begin{pmatrix} 1 & 0 & 1 \\ 0 & 3 & 7 \\ 1 & 7 & 5 \end{pmatrix} + y_2 \begin{pmatrix} 0 & 2 & 8 \\ 2 & 6 & 0 \\ 8 & 0 & 4 \end{pmatrix} + \boldsymbol{S} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 9 & 0 \\ 3 & 0 & 7 \end{pmatrix}$$

$$\boldsymbol{S} \succeq \mathbf{0}$$

$$(D) \quad \max \quad 11y_1 + 19y_2$$

$$\text{s.t.} \quad \begin{pmatrix} 1 - 1y_1 - 0y_2 & 2 - 0y_1 - 2y_2 & 3 - 1y_1 - 8y_2 \\ 2 - 0y_1 - 2y_2 & 9 - 3y_1 - 6y_2 & 0 - 7y_1 - 0y_2 \\ 3 - 1y_1 - 8y_2 & 0 - 7y_1 - 0y_2 & 7 - 5y_1 - 4y_2 \end{pmatrix} \succeq \mathbf{0}$$

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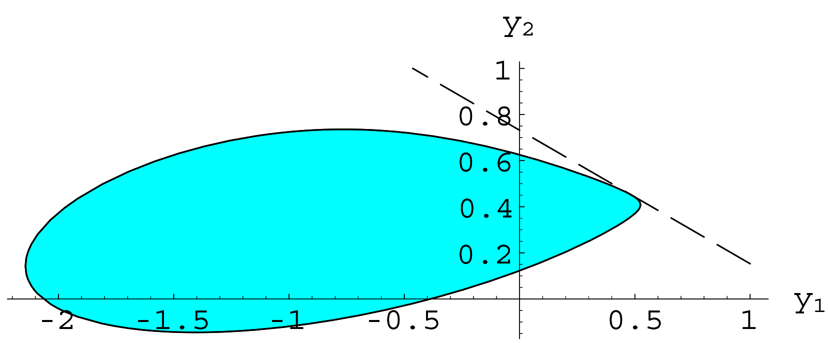

Optimal value $\approxeq 13.9022$

$$y_1^* \approxeq 0.4847, \quad y_2^* \approxeq 0.4511$$

4.2 Weak Duality

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<u>Theorem</u> Given a feasible solution $\boldsymbol{X}$ of (P) and a feasible solution $(\boldsymbol{y}, \boldsymbol{S})$ of (D),

$$\boldsymbol{C} \bullet \boldsymbol{X} - \sum_{i=1}^{m} y_i b_i = \boldsymbol{S} \bullet \boldsymbol{X} \geq 0$$

If $\boldsymbol{C} \bullet \boldsymbol{X} - \sum_{i=1}^{m} y_i b_i = 0$, then $\boldsymbol{X}$ and $(\boldsymbol{y}, \boldsymbol{S})$ are each optimal solutions to (P) and (D) and $\boldsymbol{S}\boldsymbol{X} = \mathbf{0}$

4.3 Proof

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- We must show that if $\boldsymbol{S} \succeq \mathbf{0}$ and $\boldsymbol{X} \succeq \mathbf{0}$, then $\boldsymbol{S} \bullet \boldsymbol{X} \geq 0$
- Let $\boldsymbol{S} = \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}'$ and $\boldsymbol{X} = \boldsymbol{Q}\boldsymbol{E}\boldsymbol{Q}'$ where $\boldsymbol{P}, \boldsymbol{Q}$ are orthonormal matrices and $\boldsymbol{D}, \boldsymbol{E}$ are nonnegative diagonal matrices
- $$\begin{aligned} \boldsymbol{S} \bullet \boldsymbol{X} &= \operatorname{trace}(\boldsymbol{S}'\boldsymbol{X}) = \operatorname{trace}(\boldsymbol{S}\boldsymbol{X}) \\ &= \operatorname{trace}(\boldsymbol{P}\boldsymbol{D}\boldsymbol{P}'\boldsymbol{Q}\boldsymbol{E}\boldsymbol{Q}') \\ = \operatorname{trace}(\boldsymbol{D}\boldsymbol{P}'\boldsymbol{Q}\boldsymbol{E}\boldsymbol{Q}'\boldsymbol{P}) &= \sum_{j=1}^{n} D_{jj}(\boldsymbol{P}'\boldsymbol{Q}\boldsymbol{E}\boldsymbol{Q}'\boldsymbol{P})_{jj} \geq 0, \end{aligned}$$

  since $D_{jj} \geq 0$ and the diagonal of $\boldsymbol{P}'\boldsymbol{Q}\boldsymbol{E}\boldsymbol{Q}'\boldsymbol{P}$ must be nonnegative.
- Suppose that $\operatorname{trace}(\boldsymbol{S}\boldsymbol{X}) = 0$. Then

  $$\sum_{j=1}^{n} D_{jj}(\boldsymbol{P}'\boldsymbol{Q}\boldsymbol{E}\boldsymbol{Q}'\boldsymbol{P})_{jj} = 0$$
- Then, for each $j = 1, \ldots, n$, $D_{jj} = 0$ or $(\boldsymbol{P}'\boldsymbol{Q}\boldsymbol{E}\boldsymbol{Q}'\boldsymbol{P})_{jj} = 0$.
- The latter case implies that the j^{th} row of $\boldsymbol{P}'\boldsymbol{Q}\boldsymbol{E}\boldsymbol{Q}'\boldsymbol{P}$ is all zeros. Therefore, $\boldsymbol{D}\boldsymbol{P}'\boldsymbol{Q}\boldsymbol{E}\boldsymbol{Q}'\boldsymbol{P} = 0$, and so $\boldsymbol{S}\boldsymbol{X} = \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}'\boldsymbol{Q}\boldsymbol{E}\boldsymbol{Q}' = \mathbf{0}$.

4.4 Strong Duality

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- (P) or (D) might not attain their respective optima
- There might be a duality gap, unless certain regularity conditions hold

<u>Theorem</u>

- If there exist feasible solutions $\hat{\boldsymbol{X}}$ for (P) and $(\hat{\boldsymbol{y}}, \hat{\boldsymbol{S}})$ for (D) such that $\hat{\boldsymbol{X}} \succ \mathbf{0}$, $\hat{\boldsymbol{S}} \succ \mathbf{0}$
- Then, both (P) and (D) attain their optimal values z_P^* and z_D^*
- Furthermore, $z_P^* = z_D^*$

5 SDO Modeling Power

5.1 Quadratically Constrained Problems

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$$\begin{aligned} \min \quad & (\boldsymbol{A}_0\boldsymbol{x} + \boldsymbol{b}_0)'(\boldsymbol{A}_0\boldsymbol{x} + \boldsymbol{b}_0) - \boldsymbol{c}_0'\boldsymbol{x} - d_0 \\ \text{s.t.} \quad & (\boldsymbol{A}_i\boldsymbol{x} + \boldsymbol{b}_i)'(\boldsymbol{A}_i\boldsymbol{x} + \boldsymbol{b}_i) - \boldsymbol{c}_i'\boldsymbol{x} - d_i \ \leq \ 0 \ , \end{aligned}$$

$$i = 1, \ldots, m$$

$$(\boldsymbol{A}\boldsymbol{x}+\boldsymbol{b})'(\boldsymbol{A}\boldsymbol{x}+\boldsymbol{b}) - \boldsymbol{c}'\boldsymbol{x} - d \leq 0 \qquad \Leftrightarrow$$

$$\begin{bmatrix} \boldsymbol{I} & \boldsymbol{A}\boldsymbol{x}+\boldsymbol{b} \\ (\boldsymbol{A}\boldsymbol{x}+\boldsymbol{b})' & \boldsymbol{c}'\boldsymbol{x}+d \end{bmatrix} \succeq \boldsymbol{0}$$

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$$\begin{aligned} \min \quad & t \\ \text{s.t.} \quad & (\boldsymbol{A}_0\boldsymbol{x}+\boldsymbol{b}_0)'(\boldsymbol{A}_0\boldsymbol{x}+\boldsymbol{b}_0) - \boldsymbol{c}_0'\boldsymbol{x} - d_0 - t \leq 0 \\ & (\boldsymbol{A}_i\boldsymbol{x}+\boldsymbol{b}_i)'(\boldsymbol{A}_i\boldsymbol{x}+\boldsymbol{b}_i) - \boldsymbol{c}_i'\boldsymbol{x} - d_i \leq 0, \quad \forall\, i \end{aligned}$$

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$\Leftrightarrow$

$$\begin{aligned} \min \quad & t \\ \text{s.t.} \quad & \begin{bmatrix} \boldsymbol{I} & \boldsymbol{A}_0\boldsymbol{x}+\boldsymbol{b}_0 \\ (\boldsymbol{A}_0\boldsymbol{x}+\boldsymbol{b}_0)' & \boldsymbol{c}_0'\boldsymbol{x}+d_0+t \end{bmatrix} \succeq \boldsymbol{0} \\ & \begin{bmatrix} \boldsymbol{I} & \boldsymbol{A}_i\boldsymbol{x}+\boldsymbol{b}_i \\ (\boldsymbol{A}_i\boldsymbol{x}+\boldsymbol{b}_i)' & \boldsymbol{c}_i'\boldsymbol{x}+d_i \end{bmatrix} \succeq \boldsymbol{0} \quad \forall\, i \end{aligned}$$

5.2 Eigenvalue Problems

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- $\boldsymbol{X}$: symmetric $n \times n$ matrix
- $\lambda_{\max}(\boldsymbol{X})$ = largest eigenvalue of $\boldsymbol{X}$
- $\lambda_1(\boldsymbol{X}) \geq \lambda_2(\boldsymbol{X}) \geq \cdots \geq \lambda_m(\boldsymbol{X})$ eigenvalues of $\boldsymbol{X}$
- $\lambda_i(\boldsymbol{X} + t \cdot \boldsymbol{I}) = \lambda_i(\boldsymbol{X}) + t$
- Theorem: $\lambda_{\max}(\boldsymbol{X}) \leq t \;\Leftrightarrow\; t \cdot \boldsymbol{I} - \boldsymbol{X} \succeq \boldsymbol{0}$
- Sum of k largest eigenvalues:

$$\sum_{i=1}^{k} \lambda_i(\boldsymbol{X}) \leq t \qquad \Leftrightarrow \qquad \begin{aligned} t - k \cdot s - \text{trace}(\boldsymbol{Z}) &\geq 0 \\ \boldsymbol{Z} &\succeq \boldsymbol{0} \\ \boldsymbol{Z} - \boldsymbol{X} + s\,\boldsymbol{I} &\succeq \boldsymbol{0} \end{aligned}$$

- Follows from the characterization:

$$\sum_{i=1}^{k} \lambda_i(\boldsymbol{X}) = \max\{X \bullet V : \text{trace}(V) = k, \boldsymbol{0} \preceq \boldsymbol{V} \preceq \boldsymbol{I}\}$$

5.3 Optimizing Structural Dynamics

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- Select x_i, cross-sectional area of structure i, $i = 1, \ldots, n$
- $\boldsymbol{M}(\boldsymbol{x}) = \boldsymbol{M}_0 + \sum_i x_i \boldsymbol{M}_i$, mass matrix
- $\boldsymbol{K}(\boldsymbol{x}) = \boldsymbol{K}_0 + \sum_i x_i \boldsymbol{K}_i$, stiffness matrix
- Structure weight $w = w_0 + \sum_i x_i w_i$
- Dynamics

$$\boldsymbol{M}(\boldsymbol{x})\ddot{\boldsymbol{d}} + \boldsymbol{K}(\boldsymbol{x})\boldsymbol{d} = \boldsymbol{0}$$

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- $\boldsymbol{d}(t)$ vector of displacements
- $d_i(t) = \sum_{j=1}^{n} \alpha_{ij} \cos(\omega_j t - \phi_j)$
- $\det(\boldsymbol{K}(\boldsymbol{x}) - \boldsymbol{M}(\boldsymbol{x})\omega^2) = 0$; $\omega_1 \le \omega_2 \le \cdots \le \omega_n$
- Fundamental frequency: $\omega_1 = \lambda_{\min}^{1/2}(\boldsymbol{M}(\boldsymbol{x}), \boldsymbol{K}(\boldsymbol{x}))$
- We want to bound the fundamental frequency

$$\omega_1 \ge \Omega \Longleftrightarrow \boldsymbol{M}(\boldsymbol{x})\Omega^2 - \boldsymbol{K}(\boldsymbol{x}) \preceq \boldsymbol{0}$$

- Minimize weight

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Problem: Minimize weight subject to
Fundamental frequency $\omega_1 \ge \Omega$
Limits on cross-sectional areas
Formulation

$$\begin{aligned} \min \quad & w_0 + \textstyle\sum_i x_i w_i \\ \text{s.t.} \quad & \boldsymbol{M}(\boldsymbol{x})\,\Omega^2 - \boldsymbol{K}(\boldsymbol{x}) \preceq \boldsymbol{0} \\ & l_i \le x_i \le u_i \end{aligned}$$

5.4 Measurements with Noise

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- $\boldsymbol{x}$: ability of a random student on k tests
  $\boldsymbol{E}[\boldsymbol{x}] = \bar{\boldsymbol{x}}$, $\boldsymbol{E}[(\boldsymbol{x} - \bar{\boldsymbol{x}})(\boldsymbol{x} - \bar{\boldsymbol{x}})'] = \boldsymbol{\Sigma}$
- $\boldsymbol{y}$: score of a random student on k tests
- $\boldsymbol{v}$: testing error of k tests, independent of $\boldsymbol{x}$
  $\boldsymbol{E}[\boldsymbol{v}] = \boldsymbol{0}$, $\boldsymbol{E}[\boldsymbol{v}\boldsymbol{v}'] = \boldsymbol{D}$, diagonal (unknown)

- $\boldsymbol{y} = \boldsymbol{x} + \boldsymbol{v}; \quad \boldsymbol{E}[\boldsymbol{y}] = \bar{\boldsymbol{x}}$

$$\boldsymbol{E}[(\boldsymbol{y} - \bar{\boldsymbol{x}})(\boldsymbol{y} - \bar{\boldsymbol{x}})'] = \widehat{\boldsymbol{\Sigma}} = \boldsymbol{\Sigma} + \boldsymbol{D}$$

- Objective: Estimate reliably $\bar{\boldsymbol{x}}$ and $\boldsymbol{\Sigma}$

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- Take samples of $\boldsymbol{y}$ from which we can estimate $\bar{\boldsymbol{x}}$, $\widehat{\Sigma}$
- $\boldsymbol{e}'\boldsymbol{x}$: total ability on tests
- $\boldsymbol{e}'\boldsymbol{y}$: total test score
- Reliability of test:=

$$\frac{\text{Var}[\boldsymbol{e}'\boldsymbol{x}]}{\text{Var}[\boldsymbol{e}'\boldsymbol{y}]} = \frac{\boldsymbol{e}'\boldsymbol{\Sigma}\boldsymbol{e}}{\boldsymbol{e}'\widehat{\boldsymbol{\Sigma}}\boldsymbol{e}} = 1 - \frac{\boldsymbol{e}'\boldsymbol{D}\boldsymbol{e}}{\boldsymbol{e}'\widehat{\boldsymbol{\Sigma}}\boldsymbol{e}}$$

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We can find a lower bound on the reliability of the test

$$\begin{aligned} \min \quad & \boldsymbol{e}'\boldsymbol{\Sigma}\boldsymbol{e} \\ \text{s.t.} \quad & \boldsymbol{\Sigma} + \boldsymbol{D} = \widehat{\boldsymbol{\Sigma}} \\ & \boldsymbol{\Sigma}, \boldsymbol{D} \succeq \boldsymbol{0} \\ & \boldsymbol{D} \ \text{ diagonal} \end{aligned}$$

Equivalently,

$$\begin{aligned} \max \quad & \boldsymbol{e}'\boldsymbol{D}\boldsymbol{e} \\ \text{s.t.} \quad & \boldsymbol{0} \preceq \boldsymbol{D} \preceq \widehat{\boldsymbol{\Sigma}} \\ & \boldsymbol{D} \ \text{ diagonal} \end{aligned}$$

5.5 Further Tricks

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- If $B \succ 0$,

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{B} & \boldsymbol{C}' \\ \boldsymbol{C} & \boldsymbol{D} \end{bmatrix} \succeq \boldsymbol{0} \iff \boldsymbol{D} - \boldsymbol{C}\boldsymbol{B}^{-1}\boldsymbol{C}' \succeq \boldsymbol{0}$$

- $$\boldsymbol{x}'\boldsymbol{A}\boldsymbol{x} + 2\boldsymbol{b}'\boldsymbol{x} + c \geq 0, \ \forall\, \boldsymbol{x} \iff \begin{bmatrix} c & \boldsymbol{b}' \\ \boldsymbol{b} & \boldsymbol{A} \end{bmatrix} \succeq \boldsymbol{0}$$

5.6 MAXCUT

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- Given $G = (N, E)$ undirected graph, weights $w_{ij} \geq 0$ on edge $(i, j) \in E$
- Find a subset $S \subseteq N$: $\sum_{i \in S, j \in \bar{S}} w_{ij}$ is maximized
- $x_j = 1$ for $j \in S$ and $x_j = -1$ for $j \in \bar{S}$

$$\begin{aligned} MAXCUT: \quad & \max \quad \frac{1}{4}\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}(1 - x_i x_j) \\ & \text{s.t.} \quad x_j \in \{-1, 1\}, \quad j = 1, \ldots, n \end{aligned}$$

5.6.1 Reformulation

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- Let $\boldsymbol{Y} = \boldsymbol{x}\boldsymbol{x}'$, i.e., $Y_{ij} = x_i x_j$
- Let $\boldsymbol{W} = [w_{ij}]$
- Equivalent Formulation

$$\begin{aligned} MAXCUT: \quad & \max \quad \frac{1}{4}\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij} - \boldsymbol{W} \bullet \boldsymbol{Y} \\ & \text{s.t.} \quad x_j \in \{-1, 1\}, \quad j = 1, \ldots, n \\ & \qquad Y_{jj} = 1, \quad j = 1, \ldots, n \\ & \qquad \boldsymbol{Y} = \boldsymbol{x}\boldsymbol{x}' \end{aligned}$$

5.6.2 Relaxation

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- $\boldsymbol{Y} = \boldsymbol{x}\boldsymbol{x}' \succeq \boldsymbol{0}$
- Relaxation

$$\begin{aligned} RELAX: \quad & \max \quad \frac{1}{4}\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij} - \boldsymbol{W} \bullet \boldsymbol{Y} \\ & \text{s.t.} \quad Y_{jj} = 1, \quad j = 1, \ldots, n \\ & \qquad \boldsymbol{Y} \succeq \boldsymbol{0} \end{aligned}$$

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- $$MAXCUT \leq RELAX$$
- It turns out that:

$$0.87856 \, RELAX \leq MAXCUT \leq RELAX$$

- The value of the SDO relaxation is guaranteed to be no more than 12% higher than the value of the very difficult to solve problem MAXCUT

6 Barrier Algorithm for SDO

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- $\boldsymbol{X} \succeq \mathbf{0} \Leftrightarrow \lambda_1(\boldsymbol{X}) \geq 0, \ldots, \lambda_n(\boldsymbol{X}) \geq 0$
- Natural barrier to repel $\boldsymbol{X}$ from the boundary $\lambda_1(\boldsymbol{X}) > 0, \ldots, \lambda_n(\boldsymbol{X}) > 0$:

$$-\sum_{j=1}^{n} \log(\lambda_i(\boldsymbol{X})) =$$

$$-\log(\prod_{j=1}^{n} \lambda_i(\boldsymbol{X})) = -\log(\det(\boldsymbol{X}))$$

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- Logarithmic barrier problem

$$\begin{array}{ll} \min & B_\mu(\boldsymbol{X}) = \boldsymbol{C} \bullet \boldsymbol{X} - \mu \log(\det(\boldsymbol{X})) \\ \text{s.t.} & \boldsymbol{A}_i \bullet \boldsymbol{X} = b_i \quad , i = 1, \ldots, m, \\ & \boldsymbol{X} \succ \mathbf{0} \end{array}$$

- Barrier algorithm needs $O\left(\sqrt{n} \log \frac{\epsilon_0}{\epsilon}\right)$ iterations to reduce duality gap from ϵ_0 to ϵ

7 Conclusions

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- SDO is a very powerful modeling tool
- SDO represents the present and future in continuous optimization
- Barrier Algorithm is very powerful
- Many good solvers available: SeDuMi, SDPT3, SDPA, etc.
- Pointers to literature and solvers:
`www-user.tu-chemnitz.de/~helmberg/semidef.html`