

15.093: Optimization Methods

Lecture 9: Large Scale Optimization

1 Outline

SLIDE 1

1. The idea of column generation
2. The cutting stock problem
3. Stochastic programming

2 Column Generation

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- For $\mathbf{x} \in \mathbb{R}^n$ and n large consider the LOP:

$$\begin{aligned} \min \quad & \mathbf{c}'\mathbf{x} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

- Restricted problem

$$\begin{aligned} \min \quad & \sum_{i \in I} c_i x_i \\ \text{s.t.} \quad & \sum_{i \in I} \mathbf{A}_i x_i = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

2.1 Two Key Ideas

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- Generate columns \mathbf{A}_j only as needed.
- Calculate $\min_i \bar{c}_i$ efficiently without enumerating all columns.

3 The Cutting Stock Problem

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- Company has a supply of large rolls of paper of width W .
- b_i rolls of width w_i , $i = 1, \dots, m$ need to be produced.
- Example: $w = 70$ inches, can be cut in 3 rolls of width $w_1 = 17$ and 1 roll of width $w_2 = 15$, waste:

$$70 - (3 \times 17 + 1 \times 15) = 4$$

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- Given w_1, \dots, w_m and W there are many cutting patterns: $(3, 1)$ and $(2, 2)$ for example

$$\begin{aligned} 3 \times 17 + 1 \times 15 &\leq 70 \\ 2 \times 17 + 2 \times 15 &\leq 70 \end{aligned}$$

- Pattern: (a_1, \dots, a_m) integers:

$$\sum_{i=1}^m a_i w_i \leq W$$

3.1 Problem

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- Given $w_i, b_i, i = 1, \dots, m$ (b_i : number of rolls of width w_i demanded, and W (width of large rolls):
- Find how to cut the large rolls in order to minimize the number of rolls used.

3.2 Concrete Example

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- What is the solution for $W = 70, w_1 = 21, w_2 = 11, b_1 = 40, b_2 = 40$?
- feasible patterns: $(2, 2), (3, 0), (0, 6)$
- Solution 1: $(2, 2)$: 20 patterns; 20 rolls used
- Solution 2: $(3, 0)$: 12, $(0, 6)$: 9, $(2, 2)$: 2 patterns: 23 rolls used

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- $W = 70, w_1 = 20, w_2 = 11, b_1 = 12, b_2 = 17$
- Feasible patterns: $\binom{1}{0}, \binom{2}{0}, \binom{3}{0}, \binom{0}{1}, \binom{1}{1}, \binom{2}{1}, \binom{0}{2}, \binom{1}{2}, \binom{2}{2}, \binom{0}{3}, \binom{1}{3}, \binom{0}{4}, \binom{1}{4}, \binom{0}{5}, \binom{0}{6}$
- $x_1, \dots, x_{15} = \#$ of feasible patterns of the type $\binom{1}{0}, \dots, \binom{0}{6}$ respectively

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$$\begin{aligned} \min \quad & x_1 + \dots + x_{15} \\ \text{s.t.} \quad & x_1 \binom{1}{0} + x_2 \binom{2}{0} + \dots + x_{15} \binom{0}{6} = \binom{12}{17} \\ & x_1, \dots, x_{15} \geq 0 \end{aligned}$$

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- Example: $2 \binom{0}{6} + 1 \binom{0}{5} + 4 \binom{3}{0} = \binom{12}{17}$ 7 rolls used

$$4 \binom{0}{4} + \binom{0}{1} + 4 \binom{3}{0} = \binom{12}{17} \quad 9 \text{ rolls used}$$

- Any ideas?

3.3 Formulation

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Decision variables: x_j = number of rolls cut by pattern j characterized by vector \mathbf{A}_j :

$$\begin{aligned} \min \quad & \sum_{j=1}^n x_j \\ \sum_{j=1}^n \mathbf{A}_j \cdot x_j = \quad & \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \\ x_j \geq 0 \quad & (\text{integer}) \end{aligned}$$

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- Huge number of variables.
- Can we apply column generation, that is generate the patterns \mathbf{A}_j on the fly?

3.4 Algorithm

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Idea: Generate feasible patterns as needed.

1) Start with initial patterns: $\begin{pmatrix} \lfloor \frac{W}{w_1} \rfloor \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \lfloor \frac{W}{w_2} \rfloor \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \lfloor \frac{W}{w_3} \rfloor \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \lfloor \frac{W}{w_4} \rfloor \end{pmatrix}$

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2) Solve:

$$\begin{aligned} \min \quad & x_1 + \cdots + x_m \\ x_1 \mathbf{A}_1 + \cdots + x_m \mathbf{A}_m = \quad & \mathbf{b} \\ x_i \geq 0 \end{aligned}$$

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3) Compute reduced costs

$$\bar{c}_j = 1 - \mathbf{p}' \mathbf{A}_j \text{ for all patterns } j$$

If $\bar{c}_j \geq 0$ current set of patterns optimal

If $\bar{c}_s < 0 \Rightarrow x_s$ needs to enter basis

How are we going to compute reduced costs $\bar{c}_j = 1 - \mathbf{p}' \mathbf{A}_j$ for all j ? (huge number)

3.4.1 Key Idea

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4) Solve

$$z^* = \max \sum_{i=1}^m p_i a_i$$
$$\text{s.t. } \sum_{i=1}^m w_i a_i \leq W$$
$$a_i \geq 0, \text{ integer}$$

This is the integer knapsack problem

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- If $z^* \leq 1 \Rightarrow 1 - \mathbf{p}'\mathbf{A}_j > 0 \forall j \Rightarrow$ current solution optimal
- If $z^* > 1 \Rightarrow \exists s: 1 - \mathbf{p}'\mathbf{A}_s < 0 \Rightarrow$ Variable x_s becomes basic, i.e., a new pattern \mathbf{A}_s will enter the basis.
- Perform min-ratio test and update the basis.

3.5 Dynamic Programming

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$$F(u) = \max p_1 a_1 + \dots + p_m a_m$$
$$\text{s.t. } w_1 a_1 + \dots + w_m a_m \leq u$$
$$a_i \geq 0, \text{ integer}$$

- For $u \leq w_{min}$, $F(u) = 0$.
- For $u \geq w_{min}$

$$F(u) = \max_{i=1, \dots, m} \{p_i + F(u - w_i)\}$$

Why ?

3.6 Example

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$$\max 11x_1 + 7x_2 + 5x_3 + x_4$$
$$\text{s.t. } 6x_1 + 4x_2 + 3x_3 + x_4 \leq 25$$
$$x_i \geq 0, x_i \text{ integer}$$

$$F(0) = 0$$
$$F(1) = 1$$
$$F(2) = 1 + F(1) = 2$$
$$F(3) = \max(5 + F(0)^*, 1 + F(2)) = 5$$
$$F(4) = \max(7 + F(0)^*, 5 + F(1), 1 + F(3)) = 7$$
$$F(5) = \max(7 + F(1)^*, 5 + F(2), 1 + F(4)) = 8$$
$$F(6) = \max(11 + F(0)^*, 7 + F(2), 5 + F(3), 1 + F(5)) = 11$$
$$F(7) = \max(11 + F(1)^*, 7 + F(2), 5 + F(3), 1 + F(4)) = 12$$
$$F(8) = \max(11 + F(2)^*, 7 + F(4)^*, 5 + F(5), 1 + F(7)) = 14$$
$$F(9) = 11 + F(3) = 16$$
$$F(10) = 11 + F(4) = 18$$
$$F(u) = 11 + F(u - 6) = 16 \quad u \geq 11$$

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$$\Rightarrow F(25) = 11 + F(19) = 11 + 11 + F(13) = 11 + 11 + 11 + F(7) = 33 + 12 = 45$$

$$x^* = (4, 0, 0, 1)$$

4 Stochastic Programming

4.1 Example

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	Wrenches	Pliers	Cap.
Steel (lbs)	1.5	1.0	27,000
Molding machine (hrs)	1.0	1.0	21,000
Assembly machine (hrs)	0.3	0.5	9,000*
Demand limit (tools/day)	15,000	16,000	
Contribution to earnings (\$/1000 units)	\$130*	\$100	

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$$\begin{aligned} \max \quad & 130W + 100P \\ \text{s.t.} \quad & W \leq 15 \\ & P \leq 16 \\ & 1.5W + P \leq 27 \\ & W + P \leq 21 \\ & 0.3W + 0.5P \leq 9 \\ & W, P \geq 0 \end{aligned}$$

4.1.1 Random data

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- Assembly capacity is random: $\begin{cases} 8000 & \text{with probability } \frac{1}{2} \\ 10,000 & \text{with probability } \frac{1}{2} \end{cases}$
- Contribution from wrenches: $\begin{cases} 160 & \text{with probability } \frac{1}{2} \\ 90 & \text{with probability } \frac{1}{2} \end{cases}$

4.1.2 Decisions

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- Need to decide steel capacity in the current quarter. Cost 58\$/1000lbs.
- Soon after, uncertainty will be resolved.
- Next quarter, company will decide production quantities.

4.1.3 Formulation

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State	Cap.	W. contr.	Prob.
1	8,000	160	0.25
2	10,000	160	0.25
3	8,000	90	0.25
4	10,000	90	0.25

Decision Variables: S : steel capacity,
 $P_i, W_i : i = 1, \dots, 4$ production plan under state i .

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$$\begin{aligned}
 & \max && -58S + 0.25Z_1 + 0.25Z_2 + 0.25Z_3 + 0.25Z_4 \\
 & \text{s.t.} && \\
 \text{Ass. 1} & && 0.3W_1 + 0.5P_1 \leq 8 \\
 \text{Mol. 1} & && W_1 + P_1 \leq 21 \\
 \text{Ste. 1} & && -S + 1.5W_1 + P_1 \leq 0 \\
 \text{W.d. 1} & && W_1 \leq 15 \\
 \text{P.d. 1} & && P_1 \leq 16 \\
 \text{Obj. 1} & && -Z_1 + 160W_1 + 100P_1 = 0
 \end{aligned}$$

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$$\begin{aligned}
 \text{Ass. 2} & && 0.3W_2 + 0.5P_2 \leq 10 \\
 \text{Mol. 2} & && W_2 + P_2 \leq 21 \\
 \text{Ste. 2} & && -S + 1.5W_2 + P_2 \leq 0 \\
 \text{W.d. 2} & && W_2 \leq 15 \\
 \text{P.d. 2} & && P_2 \leq 16 \\
 \text{Obj. 2} & && -Z_2 + 160W_2 + 100P_2 = 0
 \end{aligned}$$

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$$\begin{aligned}
 \text{Ass. 3} & && 0.3W_3 + 0.5P_3 \leq 8 \\
 \text{Mol. 3} & && W_3 + P_3 \leq 21 \\
 \text{Ste. 3} & && -S + 1.5W_3 + P_3 \leq 0 \\
 \text{W.d. 3} & && W_3 \leq 15 \\
 \text{P.d. 3} & && P_3 \leq 16 \\
 \text{Obj. 3} & && -Z_3 + 90W_3 + 100P_3 = 0
 \end{aligned}$$

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$$\begin{aligned}
 \text{Ass. 4} & && 0.3W_4 + 0.5P_4 \leq 10 \\
 \text{Mol. 4} & && W_4 + P_4 \leq 21 \\
 \text{Ste. 4} & && -S + 1.5W_4 + P_4 \leq 0 \\
 \text{W.d. 4} & && W_4 \leq 15 \\
 \text{P.d. 4} & && P_4 \leq 16 \\
 \text{Obj. 4} & && -Z_4 + 90W_4 + 100P_4 = 0 \\
 & && S, W_i, P_i \geq 0
 \end{aligned}$$

4.1.4 Solution

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Solution: $S = 27, 2501b$.

	W_i	P_i
1	15,000	4,750
2	15,000	4,750
3	12,500	8,500
4	5,000	16,000

4.2 Two-stage problems

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- Random scenarios indexed by $w = 1, \dots, k$. Scenario w has probability α_w .
- First stage decisions: x : $Ax = b, x \geq 0$.
- Second stage decisions: y_w : $w = 1, \dots, k$.
- Constraints: $B_w x + D_w y_w = d_w, y_w \geq 0$.

4.2.1 Formulation

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$$\begin{array}{llllll}
 \min & c'x + & \alpha_1 f'_1 y_1 + & \dots + & \alpha_k f'_k y_k & \\
 & Ax & & & & = b \\
 & B_1 x + & D_1 y_1 & & & = d_1 \\
 & B_2 x + & & D_2 y_2 & & = d_2 \\
 & \vdots & & \ddots & & \vdots \\
 & B_k x + & & & D_k y_k & = d_k \\
 & x, & y_1, & y_2, \dots, & y_k & \geq 0.
 \end{array}$$

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Structure: $x \quad y_1 \quad y_2 \quad y_3 \quad y_4$

Objective

