

15.093 Optimization Methods

Lecture 3: The Simplex Method

1 Outline

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- Reduced Costs
- Optimality conditions
- Improving the cost
- Unboundness
- The Simplex algorithm
- The Simplex algorithm on degenerate problems

2 Matrix View

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$$\begin{aligned} \min \quad & c'x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

$$x = (x_B, x_N) \quad \begin{array}{l} x_B \text{ basic variables} \\ x_N \text{ non-basic variables} \end{array}$$

$$\begin{aligned} A &= [B, N] \\ Ax = b &\Rightarrow B \cdot x_B + N \cdot x_N = b \\ \Rightarrow x_B + B^{-1}Nx_N &= B^{-1}b \\ \Rightarrow x_B &= B^{-1}b - B^{-1}Nx_N \end{aligned}$$

2.1 Reduced Costs

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$$\begin{aligned} z &= c'_B x_B + c'_N x_N \\ &= c'_B (B^{-1}b - B^{-1}Nx_N) + c'_N x_N \\ &= c'_B B^{-1}b + (c'_N - c'_B B^{-1}N) x_N \end{aligned}$$

$$\boxed{\bar{c}_j = c_j - c'_B B^{-1}A_j \quad \text{reduced cost}}$$

2.2 Optimality Conditions

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Theorem:

- x BFS associated with basis B
- \bar{c} reduced costs
Then
- If $\bar{c} \geq 0 \Rightarrow x$ optimal
- x optimal and non-degenerate $\Rightarrow \bar{c} \geq 0$

2.3 Proof

- \mathbf{y} arbitrary feasible solution
- $\mathbf{d} = \mathbf{y} - \mathbf{x} \Rightarrow \mathbf{A}\mathbf{x} = \mathbf{A}\mathbf{y} = \mathbf{b} \Rightarrow \mathbf{A}\mathbf{d} = \mathbf{0}$

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$$\Rightarrow \mathbf{B}\mathbf{d}_B + \sum_{i \in N} \mathbf{A}_i d_i = \mathbf{0}$$

$$\Rightarrow \mathbf{d}_B = - \sum_{i \in N} \mathbf{B}^{-1} \mathbf{A}_i d_i$$

$$\begin{aligned} \Rightarrow \mathbf{c}'\mathbf{d} &= \mathbf{c}'_B \mathbf{d}_B + \sum_{i \in N} c_i d_i \\ &= \sum_{i \in N} (c_i - \mathbf{c}'_B \mathbf{B}^{-1} \mathbf{A}_i) d_i = \sum_{i \in N} \bar{c}_i d_i \end{aligned}$$

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- Since $y_i \geq 0$ and $x_i = 0, i \in N$, then $d_i = y_i - x_i \geq 0, i \in N$
- $\mathbf{c}'\mathbf{d} = \mathbf{c}'(\mathbf{y} - \mathbf{x}) \geq 0 \Rightarrow \mathbf{c}'\mathbf{y} \geq \mathbf{c}'\mathbf{x}$
- $\Rightarrow \mathbf{x}$ optimal

(b) in BT, Theorem 3.1

3 Improving the Cost

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- Suppose $\bar{c}_j = c_j - \mathbf{c}'_B \mathbf{B}^{-1} \mathbf{A}_j < 0$
Can we improve the cost?
- Let $\mathbf{d}_B = -\mathbf{B}^{-1} \mathbf{A}_j$
 $d_j = 1, d_i = 0, i \neq B(1), \dots, B(m), j$.
- Let $\mathbf{y} = \mathbf{x} + \theta \cdot \mathbf{d}, \quad \theta > 0$ scalar

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$$\begin{aligned} \mathbf{c}'\mathbf{y} - \mathbf{c}'\mathbf{x} &= \theta \cdot \mathbf{c}'\mathbf{d} \\ &= \theta \cdot (\mathbf{c}'_B \mathbf{d}_B + c_j d_j) \\ &= \theta \cdot (c_j - \mathbf{c}'_B \mathbf{B}^{-1} \mathbf{A}_j) \\ &= \theta \cdot \bar{c}_j \end{aligned}$$

Thus, if $\bar{c}_j < 0$ cost will decrease.

4 Unboundness

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- Is $\mathbf{y} = \mathbf{x} + \theta \cdot \mathbf{d}$ feasible?
Since $\mathbf{A}\mathbf{d} = \mathbf{0} \Rightarrow \mathbf{A}\mathbf{y} = \mathbf{A}\mathbf{x} = \mathbf{b}$
- $\mathbf{y} \geq \mathbf{0}$?
If $\mathbf{d} \geq \mathbf{0} \Rightarrow \mathbf{x} + \theta \cdot \mathbf{d} \geq \mathbf{0} \quad \forall \theta \geq 0$
 \Rightarrow objective unbounded.

5 Improvement

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If $d_i < 0$, then

$$x_i + \theta d_i \geq 0 \Rightarrow \theta \leq -\frac{x_i}{d_i}$$

$$\Rightarrow \theta^* = \min_{\{i|d_i < 0\}} \left(-\frac{x_i}{d_i} \right)$$

$$\Rightarrow \theta^* = \min_{\{i=1, \dots, m | d_{B(i)} < 0\}} \left(-\frac{x_{B(i)}}{d_{B(i)}} \right)$$

5.1 Example

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$$\begin{array}{llll} \min & x_1 + & 5x_2 & -2x_3 \\ \text{s.t.} & x_1 + & x_2 + & x_3 \leq 4 \\ & x_1 & & \leq 2 \\ & & & x_3 \leq 3 \\ & & 3x_2 + & x_3 \leq 6 \\ & x_1, & x_2, & x_3 \geq 0 \end{array}$$

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$$\left[\begin{array}{ccccccc} \mathbf{A}_1 & \mathbf{A}_2 & \mathbf{A}_3 & \mathbf{A}_4 & \mathbf{A}_5 & \mathbf{A}_6 & \mathbf{A}_7 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 3 & 1 & 0 & 0 & 0 & 1 \end{array} \right] \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 3 \\ 6 \end{pmatrix}$$

$$B = [\mathbf{A}_1, \mathbf{A}_3, \mathbf{A}_6, \mathbf{A}_7]$$

$$\text{BFS: } \mathbf{x} = (2, 0, 2, 0, 0, 1, 4)'$$

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$$\mathbf{B} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \quad \mathbf{B}^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{bmatrix} \quad \bar{\mathbf{c}}' = (0, 7, 0, 2, -3, 0, 0)$$

$$d_5 = 1, d_2 = d_4 = 0, \quad \begin{pmatrix} d_1 \\ d_3 \\ d_6 \\ d_7 \end{pmatrix} = -\mathbf{B}^{-1} \mathbf{A}_5 = \begin{pmatrix} -1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$$

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$$\mathbf{y}' = \mathbf{x}' + \theta \mathbf{d}' = (2 - \theta, 0, 2 + \theta, 0, \theta, 1 - \theta, 4 - \theta)$$

What happens as θ increases?

$$\theta^* = \min_{\{i=1, \dots, m \mid d_{B(i)} < 0\}} \left(-\frac{x_{B(i)}}{d_i} \right) = \min \left(-\frac{2}{(-1)}, -\frac{1}{(-1)}, -\frac{4}{(-1)} \right) = 1.$$

$l = 6$ (\mathbf{A}_6 exits the basis).

New solution

$$\mathbf{y} = (1, 0, 3, 0, 1, 0, 3)'$$

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New basis $\bar{\mathbf{B}} = (\mathbf{A}_1, \mathbf{A}_3, \mathbf{A}_5, \mathbf{A}_7)$

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$$\bar{\mathbf{B}} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \quad \bar{\mathbf{B}}^{-1} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$$\bar{\mathbf{c}}' = \mathbf{c}' - \mathbf{c}'_{\bar{\mathbf{B}}} \bar{\mathbf{B}}^{-1} \mathbf{A} = (0, 4, 0, -1, 0, 3, 0)$$

Need to continue, column \mathbf{A}_4 enters the basis.

6 Correctness

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$$-\frac{x_{B(l)}}{d_{B(l)}} = \min_{i=1, \dots, m, d_{B(i)} < 0} \left(-\frac{x_{B(i)}}{d_{B(i)}} \right) = \theta^*$$

Theorem

- $\bar{\mathbf{B}} = \{\mathbf{A}_{B(i)}, i \neq l, \mathbf{A}_j\}$ basis
- $\mathbf{y} = \mathbf{x} + \theta^* \mathbf{d}$ is a BFS associated with basis $\bar{\mathbf{B}}$.

7 The Simplex Algorithm

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1. Start with basis $\mathbf{B} = [\mathbf{A}_{B(1)}, \dots, \mathbf{A}_{B(m)}]$ and a BFS \mathbf{x} .
2. Compute $\bar{c}_j = c_j - \mathbf{c}'_B \mathbf{B}^{-1} \mathbf{A}_j$
 - If $\bar{c}_j \geq 0$; \mathbf{x} optimal; stop.
 - Else select $j : \bar{c}_j < 0$.

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3. Compute $\mathbf{u} = -\mathbf{d} = \mathbf{B}^{-1} \mathbf{A}_j$.
 - If $\mathbf{u} \leq \mathbf{0} \Rightarrow$ cost unbounded; stop
 - Else
4. $\theta^* = \min_{1 \leq i \leq m, u_i > 0} \frac{x_{B(i)}}{u_i} = \frac{u_{B(l)}}{u_l}$
5. Form a new basis by replacing $\mathbf{A}_{B(l)}$ with \mathbf{A}_j .
6. $y_j = \theta^*$
 $y_{B(i)} = x_{B(i)} - \theta^* u_i$

7.1 Finite Convergence

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Theorem:

- $P = \{\mathbf{x} \mid \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\} \neq \emptyset$
- Every BFS non-degenerate
Then
- Simplex method terminates after a finite number of iterations
- At termination, we have optimal basis \mathbf{B} or we have a direction $\mathbf{d} : \mathbf{A}\mathbf{d} = \mathbf{0}, \mathbf{d} \geq \mathbf{0}, \mathbf{c}'\mathbf{d} < \mathbf{0}$ and optimal cost is $-\infty$.

7.2 Degenerate problems

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- θ^* can equal zero (why?) $\Rightarrow \mathbf{y} = \mathbf{x}$, although $\overline{\mathbf{B}} \neq \mathbf{B}$.
- Even if $\theta^* > 0$, there might be a tie

$$\min_{1 \leq i \leq m, u_i > 0} \frac{x_{B(i)}}{u_i} \Rightarrow$$

next BFS degenerate.

- Finite termination not guaranteed; cycling is possible.

7.3 Avoiding Cycling

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- Cycling can be avoided by carefully selecting which variables enter and exit the basis.
- Example: among all variables $\bar{c}_j < 0$, pick the smallest subscript; among all variables eligible to exit the basis, pick the one with the smallest subscript.