15.093: Optimization Methods

Lecture 13: Exact Methods for IP

1 Outline

SLIDE 1

- Cutting plane methods
- Branch and bound methods

2 Cutting plane methods

SLIDE 2

$$\begin{aligned} \min \quad & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \\ & \boldsymbol{x} \text{ integer}, \end{aligned}$$

LP relaxation

$$\begin{aligned} \min \quad & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0}. \end{aligned}$$

2.1 Algorithm

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- Solve the LP relaxation. Let $\boldsymbol{x}^*$ be an optimal solution.
- If $\boldsymbol{x}^*$ is integer stop; $\boldsymbol{x}^*$ is an optimal solution to IP.
- If not, add a linear inequality constraint to LP relaxation that all integer solutions satisfy, but $\boldsymbol{x}^*$ does not; go to Step 1.

2.2 Example

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- Let $\boldsymbol{x}^*$ be an optimal BFS to LP ralxation with at least one fractional basic variable.
- N: set of indices of the nonbasic variables.
- Is this a valid cut?

$$\sum_{j \in N} x_j \geq 1.$$

2.3 The Gomory cutting plane algorithm

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- Let $\boldsymbol{x}^*$ be an optimal BFS and $\boldsymbol{B}$ an optimal basis.
- $$\boldsymbol{x}_B + \boldsymbol{B}^{-1}\boldsymbol{A}_N\boldsymbol{x}_N = \boldsymbol{B}^{-1}\boldsymbol{b}.$$
- $\overline{a}_{ij} = \left(\boldsymbol{B}^{-1}\boldsymbol{A}_j\right)_i$, $\overline{a}_{i0} = \left(\boldsymbol{B}^{-1}\boldsymbol{b}\right)_i$.

- $$x_i + \sum_{j \in N} \overline{a}_{ij} x_j = \overline{a}_{i0}.$$
- Since $x_j \geq 0$ for all j,
$$x_i + \sum_{j \in N} \lfloor \overline{a}_{ij} \rfloor x_j \leq x_i + \sum_{j \in N} \overline{a}_{ij} x_j = \overline{a}_{i0}.$$
- Since x_j integer,
$$x_i + \sum_{j \in N} \lfloor \overline{a}_{ij} \rfloor x_j \leq \lfloor \overline{a}_{i0} \rfloor.$$
- Valid cut

2.4 Example

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$$\begin{array}{lrcrcl} \min & x_1 & - & 2x_2 & & \\ \text{s.t.} & -4x_1 & + & 6x_2 & \leq & 9 \\ & x_1 & + & x_2 & \leq & 4 \\ & \multicolumn{5}{l}{x_1, x_2 \geq 0} \\ & \multicolumn{5}{l}{x_1, x_2 \text{ integer.}} \end{array}$$

We transform the problem in standard form

$$\begin{array}{lrcrcrcrcl} \min & x_1 & - & 2x_2 & & & & & & \\ \text{s.t.} & -4x_1 & + & 6x_2 & + & x_3 & & & = & 9 \\ & x_1 & + & x_2 & & & + & x_4 & = & 4 \\ & \multicolumn{9}{l}{x_1, \ldots, x_4 \geq 0} \\ & \multicolumn{9}{l}{x_1, \ldots, x_4 \text{ integer.}} \end{array}$$

LP relaxation: $\boldsymbol{x}^1 = (15/10, 25/10)$. SLIDE 7

- $$x_2 + \frac{1}{10} x_3 + \frac{1}{10} x_4 = \frac{25}{10}.$$
- Gomory cut
$$x_2 \leq 2.$$
- Add constraints $x_2 + x_5 = 2$, $x_5 \geq 0$
- New optimal $\boldsymbol{x}^2 = (3/4, 2)$.
- One of the equations in the optimal tableau is
$$x_1 - \frac{1}{4} x_3 + \frac{6}{4} x_5 = \frac{3}{4}.$$
- New Gomory cut
$$x_1 - x_3 + x_5 \leq 0,$$
- New optimal solution is $\boldsymbol{x}^3 = (1, 2)$.

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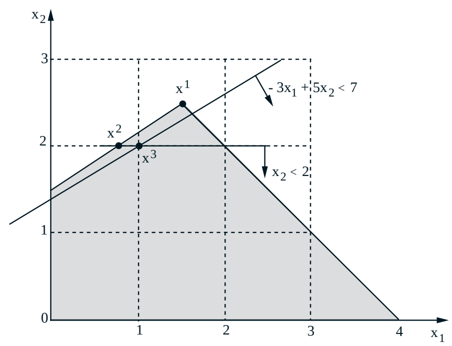

3 Branch and bound

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1. **Branching**: Select an active subproblem F_i
2. **Pruning**: If the subproblem is infeasible, delete it.
3. **Bounding**: Otherwise, compute a lower bound $b(F_i)$ for the subproblem.
4. **Pruning**: If $b(F_i) \geq U$, the current best upperbound, delete the subproblem.
5. **Partitioning**: If $b(F_i) < U$, either obtain an optimal solution to the subproblem (stop), or break the corresponding problem into further subproblems, which are added to the list of active subproblem.

3.1 LP Based

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- Compute the lower bound $b(F)$ by solving the LP relaxation of the discrete optimization problem.
- From the LP solution $\boldsymbol{x}^*$, if there is a component x_i^* which is fractional, we create two subproblems by adding either one of the constraints

$$x_i \leq \lfloor x_i^* \rfloor, \text{ or } x_i \geq \lceil x_i^* \rceil.$$

  Note that both constraints are violated by $\boldsymbol{x}^*$.
- If there are more than 2 fractional components, we use selection rules like maximum infeasibility etc. to determine the inequalities to be added to the problem
- Select the active subproblem using either depth-first or breadth-first search strategies.

3.2 Example

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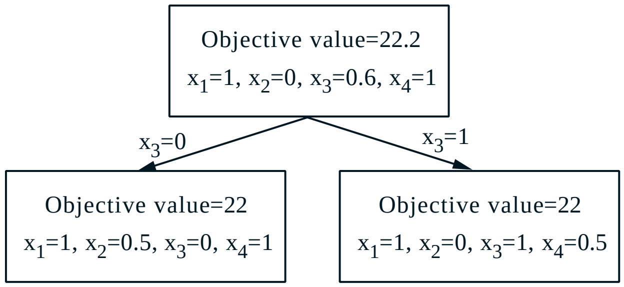
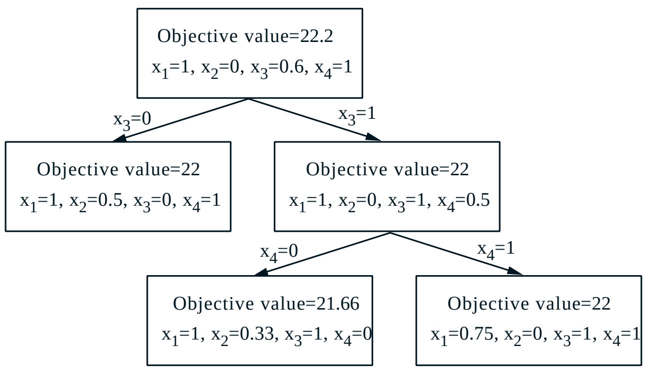
$$\begin{aligned} \max \quad & 12x_1 + 8x_2 + 7x_3 + 6x_4 \\ \text{s.t.} \quad & 8x_1 + 6x_2 + 5x_3 + 4x_4 \leq 15 \\ & x_1, x_2, x_3, x_4 \text{ are binary.} \end{aligned}$$





LP relaxation SLIDE 12

$$\begin{aligned} \max \quad & 12x_1 + 8x_2 + 7x_3 + 6x_4 \\ \text{s.t.} \quad & 8x_1 + 6x_2 + 5x_3 + 4x_4 \leq 15 \\ & x_1 \leq 1, x_2 \leq 1, x_3 \leq 1, x_4 \leq 1 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

LP solution: $x_1 = 1, \; x_2 = 0, \; x_3 = 0.6, \; x_4 = 1$ Profit=22.2

3.2.1 Branch and bound tree

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3.3 Pigeonhole Problem

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- There are $n+1$ pigeons with n holes. We want to place the pigeons in the holes in such a way that no two pigeons go into the same hole.
- Let $x_{ij} = 1$ if pigeon i goes into hole j, 0 otherwise.

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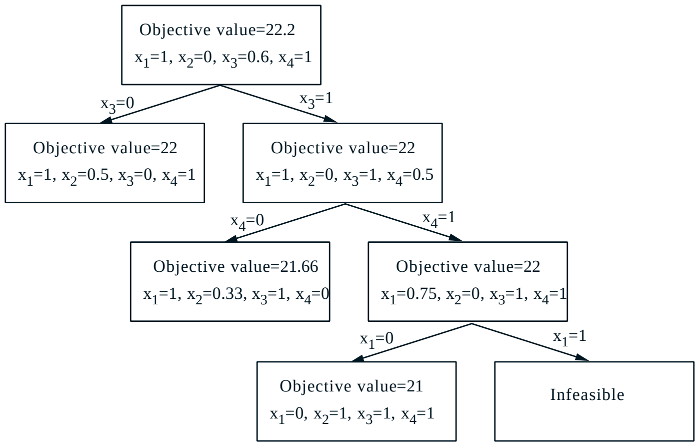
- Formulation 1:

$$\begin{aligned} \textstyle\sum_j x_{ij} = 1, \quad & i = 1, \ldots, n+1 \\ x_{ij} + x_{kj} \leq 1, \quad & \forall j, i \neq k \end{aligned}$$



- Formulation 2:

$$\sum_j x_{ij} = 1, \quad i = 1, \ldots, n+1$$
$$\sum_{i=1}^{n+1} x_{ij} \leq 1, \quad \forall j$$

Which formulation is better for the problem? SLIDE 18

- The pigeonhole problem is infeasible.
- For Formulation 1, feasible solution $x_{ij} = \frac{1}{n}$ for all i, j. $O(n^3)$ constraints. Nearly complete enumeration is needed for LP-based BB, since the problem remains feasible after fixing many variables.
- Formulation 2 Infeasible. $O(n)$ constraints.
- Mesage: Formulation of the problem is important!

3.4 Preprocessing

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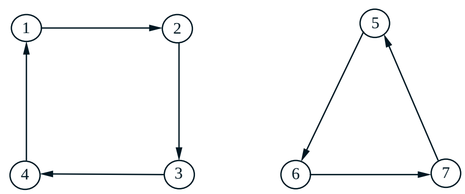
- An effective way of improving integer programming formulations prior to and during branch-and-bound.
- Logical Tests
  - Removal of empty (all zeros) rows and columns;
  - Removal of rows dominated by multiples of other rows;
  - strengthening the bounds within rows by comparing individual variables and coefficients to the right-hand-side.
  - Additional strengthening may be possible for integral variables using rounding.
- Probing : Setting temporarily a 0-1 variable to 0 or 1 and redo the logical tests. Force logical connection between variables. For example, if $5x + 4y + z \leq 8, x, y, z \in \{0, 1\}$, then by setting $x = 1$, we obtain $y = 0$. This leads to an inequality $x + y \leq 1$.



4 Application

4.1 Directed TSP

4.1.1 Assignment Lower Bound

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Given a directed graph $G = (N, A)$ with n nodes, and a cost c_{ij} for every arc, find a tour (a directed cycle that visits all nodes) of minimum cost.

$$\begin{array}{rc} \min & \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} \\ \text{s.t.}: & \sum_{i=1}^{n} x_{ij} = 1, \;\; j = 1, \ldots, n, \\ & \sum_{j=1}^{n} x_{ij} = 1, \;\; i = 1, ..., n, \\ & x_{ij} \in \{0, 1\}. \end{array}$$

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Branching:Set one of the arcs selected in the optimal solution to zero. i.e., add constraints of the type "$x_{ij} = 0$" to exclude the current optimal solution.

4.2 Improving BB

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- Better LP solver
- Use problem structure to derive better branching strategy
- Better choice of lower bound $b(F)$ - better relaxation
- Better choice of upper bound U - heuristic to get good solution
- **KEY: Start pruning the search tree as early as possible**