

Derivatives by the Chain Rule

4.1 The Chain Rule

You remember that the derivative of $f(x)g(x)$ is not $(df/dx)(dg/dx)$. The derivative of $\sin x$ times x^2 is not $\cos x$ times $2x$. The product rule gave two terms, not one term. But there is another way of combining the sine function f and the squaring function g into a single function. The derivative of that new function does involve the cosine times $2x$ (but with a certain twist). We will first explain the new function, and then find the “*chain rule*” for its derivative.

May I say here that the chain rule is important. It is easy to learn, and you will use it often. I see it as the third basic way to find derivatives of new functions from derivatives of old functions. (So far the old functions are x^n , $\sin x$, and $\cos x$. Still ahead are e^x and $\log x$.) When f and g are added and multiplied, derivatives come from the *sum rule* and *product rule*. This section combines f and g in a third way.

The new function is $\sin(x^2)$ —the sine of x^2 . It is created out of the two original functions: if $x = 3$ then $x^2 = 9$ and $\sin(x^2) = \sin 9$. There is a “chain” of functions, combining $\sin x$ and x^2 into the composite function $\sin(x^2)$. You start with x , *then find $g(x)$, then find $f(g(x))$* :

The squaring function gives $y = x^2$. This is $g(x)$.
The sine function produces $z = \sin y = \sin(x^2)$. This is $f(g(x))$.

The “*inside function*” $g(x)$ gives y . *This is the input to the “outside function” $f(y)$.* That is called *composition*. It starts with x and ends with z . The composite function is sometimes written $f \circ g$ (the circle shows the difference from an ordinary product fg). More often you will see $f(g(x))$:

$$z(x) = f \circ g(x) = f(g(x)). \tag{1}$$

Other examples are $\cos 2x$ and $(2x)^3$, with $g = 2x$. *On a calculator you input x , then push the “g” button, then push the “f” button:*

From x compute $y = g(x)$ From y compute $z = f(y)$.

There is not a button for every function! But the squaring function and sine function are on most calculators, and they are used *in that order*. Figure 4.1a shows how squaring will stretch and squeeze the sine function.

That graph of $\sin x^2$ is a crazy FM signal (the Frequency is Modulated). The wave goes up and down like $\sin x$, but not at the same places. Changing to $\sin g(x)$ moves the peaks left and right. Compare with a product $g(x) \sin x$, which is an AM signal (the Amplitude is Modulated).

Remark $f(g(x))$ is usually different from $g(f(x))$. *The order of f and g is usually important.* For $f(x) = \sin x$ and $g(x) = x^2$, the chain in the opposite order $g(f(x))$ gives something different:

First apply the sine function: $y = \sin x$
 Then apply the squaring function: $z = (\sin x)^2$.

That result is often written $\sin^2 x$, to save on parentheses. It is never written $\sin x^2$, which is totally different. Compare them in Figure 4.1.

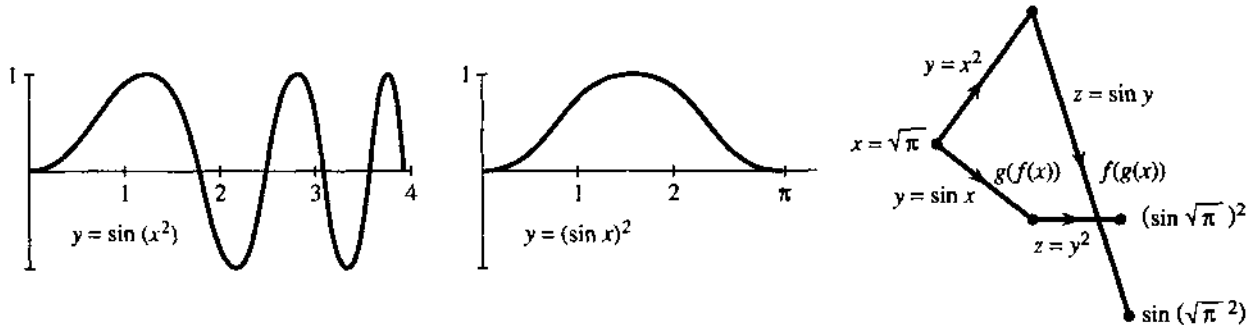


Fig. 4.1 $f(g(x))$ is different from $g(f(x))$. Apply g then f , or f then g .

EXAMPLE 1 The composite function $f \circ g$ can be deceptive. If $g(x) = x^3$ and $f(y) = y^4$, how does $f(g(x))$ differ from the ordinary product $f(x)g(x)$? The ordinary product is x^7 . The chain starts with $y = x^3$, and then $z = y^4 = x^{12}$. The composition of x^3 and y^4 gives $f(g(x)) = x^{12}$.

EXAMPLE 2 In Newton's method, $F(x)$ is composed with itself. This is *iteration*. Every output x_n is fed back as input, to find $x_{n+1} = F(x_n)$. The example $F(x) = \frac{1}{2}x + 4$ has $F(F(x)) = \frac{1}{2}(\frac{1}{2}x + 4) + 4$. That produces $z = \frac{1}{4}x + 6$.

The derivative of $F(x)$ is $\frac{1}{2}$. The derivative of $z = F(F(x))$ is $\frac{1}{4}$, which is $\frac{1}{2}$ times $\frac{1}{2}$. *We multiply derivatives.* This is a special case of the chain rule.

An extremely special case is $f(x) = x$ and $g(x) = x$. The ordinary product is x^2 . The chain $f(g(x))$ produces only x ! The output from the "identity function" is $g(x) = x$.† When the second identity function operates on x it produces x again. The derivative is 1 times 1. I can give more composite functions in a table:

$y = g(x)$	$z = f(y)$	$z = f(g(x))$
$x^2 - 1$	\sqrt{y}	$\sqrt{x^2 - 1}$
$\cos x$	y^3	$(\cos x)^3$
2^x	2^y	2^{2^x}
$x + 5$	$y - 5$	x

The last one adds 5 to get y . Then it subtracts 5 to reach z . So $z = x$. Here output

†A calculator has no button for the identity function. It wouldn't do anything.

equals input: $f(g(x)) = x$. These “*inverse functions*” are in Section 4.3. The other examples create new functions $z(x)$ and we want their derivatives.

THE DERIVATIVE OF $f(g(x))$

What is the derivative of $z = \sin x^2$? It is the limit of $\Delta z/\Delta x$. Therefore we look at a nearby point $x + \Delta x$. That change in x produces a change in $y = x^2$ —which moves to $y + \Delta y = (x + \Delta x)^2$. From this change in y , there is a change in $z = f(y)$. It is a “domino effect,” in which each changed input yields a changed output: Δx produces Δy produces Δz . We have to connect the final Δz to the original Δx .

The key is to write $\Delta z/\Delta x$ as $\Delta z/\Delta y$ times $\Delta y/\Delta x$. Then let Δx approach zero. In the limit, dz/dx is given by the “chain rule”:

$$\frac{\Delta z}{\Delta x} = \frac{\Delta z}{\Delta y} \frac{\Delta y}{\Delta x} \text{ becomes the chain rule } \frac{dz}{dx} = \frac{dz}{dy} \frac{dy}{dx}. \quad (2)$$

As Δx goes to zero, the ratio $\Delta y/\Delta x$ approaches dy/dx . Therefore Δy must be going to zero, and $\Delta z/\Delta y$ approaches dz/dy . The limit of a product is the product of the separate limits (end of quick proof). We multiply derivatives:

4A Chain Rule Suppose $g(x)$ has a derivative at x and $f(y)$ has a derivative at $y = g(x)$. Then the derivative of $z = f(g(x))$ is

$$\frac{dz}{dx} = \frac{dz}{dy} \frac{dy}{dx} = f'(g(x)) g'(x). \quad (3)$$

The slope at x is df/dy (at y) times dg/dx (at x).

Caution The chain rule does *not* say that the derivative of $\sin x^2$ is $(\cos x)(2x)$. True, $\cos y$ is the derivative of $\sin y$. The point is that $\cos y$ must be *evaluated at* y (not at x). We do not want df/dx at x , we want df/dy at $y = x^2$:

$$\text{The derivative of } \sin x^2 \text{ is } (\cos x^2) \text{ times } (2x). \quad (4)$$

EXAMPLE 3 If $z = (\sin x)^2$ then $dz/dx = (2 \sin x)(\cos x)$. Here $y = \sin x$ is *inside*.

In this order, $z = y^2$ leads to $dz/dy = 2y$. It does *not* lead to $2x$. The inside function $\sin x$ produces $dy/dx = \cos x$. The answer is $2y \cos x$. We have not yet found the function whose derivative is $2x \cos x$.

EXAMPLE 4 The derivative of $z = \sin 3x$ is $\frac{dz}{dx} = \frac{dz}{dy} \frac{dy}{dx} = 3 \cos 3x$.

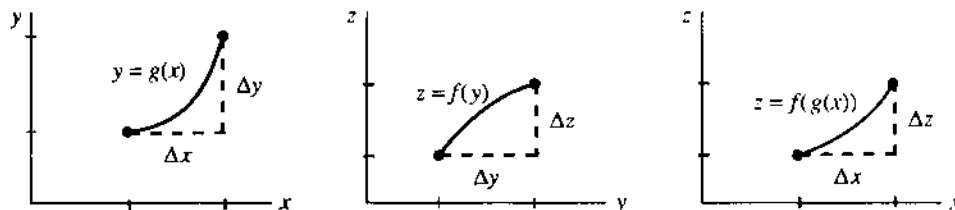


Fig. 4.2 The chain rule: $\frac{\Delta z}{\Delta x} = \frac{\Delta z}{\Delta y} \frac{\Delta y}{\Delta x}$ approaches $\frac{dz}{dx} = \frac{dz}{dy} \frac{dy}{dx}$.

The outside function is $z = \sin y$. The inside function is $y = 3x$. Then $dz/dy = \cos y$ —this is $\cos 3x$, not $\cos x$. Remember the other factor $dy/dx = 3$.

I can explain that factor 3, especially if x is switched to t . The distance is $z = \sin 3t$. That oscillates like $\sin t$ except *three times as fast*. The speeded-up function $\sin 3t$ completes a wave at time $2\pi/3$ (instead of 2π). Naturally the velocity contains the extra factor 3 from the chain rule.

EXAMPLE 5 Let $z = f(y) = y^n$. Find the derivative of $f(g(x)) = [g(x)]^n$.

In this case dz/dy is ny^{n-1} . The chain rule multiplies by dy/dx :

$$\frac{dz}{dx} = ny^{n-1} \frac{dy}{dx} \quad \text{or} \quad \frac{d}{dx} [g(x)]^n = n[g(x)]^{n-1} \frac{dg}{dx}. \quad (5)$$

This is the **power rule!** It was already discovered in Section 2.5. Square roots (when $n = 1/2$) are frequent and important. Suppose $y = x^2 - 1$:

$$\frac{d}{dx} \sqrt{x^2 - 1} = \frac{1}{2} (x^2 - 1)^{-1/2} (2x) = \frac{x}{\sqrt{x^2 - 1}}. \quad (6)$$

Question A Buick uses $1/20$ of a gallon of gas per mile. You drive at 60 miles per hour. How many gallons per hour?

Answer $(\text{Gallons}/\text{hour}) = (\text{gallons}/\text{mile})(\text{miles}/\text{hour})$. The chain rule is $(dy/dt) = (dy/dx)(dx/dt)$. The answer is $(1/20)(60) = 3$ gallons/hour.

Proof of the chain rule The discussion above was correctly based on

$$\frac{\Delta z}{\Delta x} = \frac{\Delta z}{\Delta y} \frac{\Delta y}{\Delta x} \quad \text{and} \quad \frac{dz}{dx} = \frac{dz}{dy} \frac{dy}{dx}. \quad (7)$$

It was here, over the chain rule, that the “battle of notation” was won by Leibniz. His notation practically tells you what to do: Take the limit of each term. (I have to mention that when Δx is approaching zero, it is theoretically possible that Δy might *hit* zero. If that happens, $\Delta z/\Delta y$ becomes $0/0$. We have to assign it the correct meaning, which is dz/dy .) As $\Delta x \rightarrow 0$,

$$\frac{\Delta y}{\Delta x} \rightarrow g'(x) \quad \text{and} \quad \frac{\Delta z}{\Delta y} \rightarrow f'(y) = f'(g(x)).$$

Then $\Delta z/\Delta x$ approaches $f'(y)$ times $g'(x)$, which is the chain rule $(dz/dy)(dy/dx)$. In the table below, the derivative of $(\sin x)^3$ is $3(\sin x)^2 \cos x$. That extra factor $\cos x$ is easy to forget. It is even easier to forget the -1 in the last example.

$$\begin{array}{lll} z = (x^3 + 1)^5 & dz/dx = 5(x^3 + 1)^4 & \text{times } 3x^2 \\ z = (\sin x)^3 & dz/dx = 3 \sin^2 x & \text{times } \cos x \\ z = (1 - x)^2 & dz/dx = 2(1 - x) & \text{times } -1 \end{array}$$

Important All kinds of letters are used for the chain rule. We named the output z . Very often it is called y , and the inside function is called u :

$$\text{The derivative of } y = \sin u(x) \text{ is } \frac{dy}{dx} = \cos u \frac{du}{dx}.$$

Examples with du/dx are extremely common. I have to ask you to accept whatever letters may come. What never changes is the key idea—*derivative of outside function times derivative of inside function*.

EXAMPLE 6 The chain rule is barely needed for $\sin(x-1)$. Strictly speaking the inside function is $u = x-1$. Then du/dx is just 1 (not -1). If $y = \sin(x-1)$ then $dy/dx = \cos(x-1)$. The graph is shifted and the slope shifts too.

Notice especially: The cosine is computed at $x-1$ and not at the unshifted x .

RECOGNIZING $f(y)$ AND $g(x)$

A big part of the chain rule is *recognizing the chain*. The table started with $(x^3+1)^5$. You look at it for a second. Then you see it as u^5 . The inside function is $u = x^3+1$. With practice this decomposition (the opposite of composition) gets easy:

$$\cos(2x+1) \text{ is } \cos u \quad \sqrt{1+\sin t} \text{ is } \sqrt{u} \quad x \sin x \text{ is } \dots \text{ (product rule!)}$$

In calculations, the careful way is to write down all the functions:

$$z = \cos u \quad u = 2x+1 \quad dz/dx = (-\sin u)(2) = -2 \sin(2x+1).$$

The quick way is to keep in your mind "the derivative of what's inside." The slope of $\cos(2x+1)$ is $-\sin(2x+1)$, times 2 from the chain rule. The derivative of $2x+1$ is remembered—without z or u or f or g .

EXAMPLE 7 $\sin \sqrt{1-x}$ is a chain of $z = \sin y$, $y = \sqrt{u}$, $u = 1-x$ (three functions).

With that triple chain you will have the hang of the chain rule:

$$\text{The derivative of } \sin \sqrt{1-x} \text{ is } (\cos \sqrt{1-x}) \left(\frac{1}{2\sqrt{1-x}} \right) (-1).$$

This is $(dz/dy)(dy/du)(du/dx)$. Evaluate them at the right places y , u , x .

Finally there is the question of *second derivatives*. The chain rule gives dz/dx as a product, so d^2z/dx^2 needs the product rule:

$$\frac{dz}{dx} = \frac{dz}{dy} \frac{dy}{dx} \quad \text{leads to} \quad \frac{d^2z}{dx^2} = \frac{dz}{dy} \frac{d^2y}{dx^2} + \frac{d}{dx} \left(\frac{dz}{dy} \right) \frac{dy}{dx} \quad (8)$$

$$u \quad v \qquad \qquad \qquad u \quad v' + \quad u' \quad v$$

That last term needs the chain rule again. It becomes d^2z/dy^2 times $(dy/dx)^2$.

EXAMPLE 8 The derivative of $\sin x^2$ is $2x \cos x^2$. Then the product rule gives $d^2z/dx^2 = 2 \cos x^2 - 4x^2 \sin x^2$. In this case $y'' = 2$ and $(y')^2 = 4x^2$.

4.1 EXERCISES

Read-through questions

$z = f(g(x))$ comes from $z = f(y)$ and $y = \underline{a}$. At $x = 2$, the chain $(x^2-1)^3$ equals \underline{b} . Its inside function is $y = \underline{c}$, its outside function is $z = \underline{d}$. Then dz/dx equals \underline{e} . The first factor is evaluated at $y = \underline{f}$ (not at $y = x$). For $z = \sin(x^4-1)$ the derivative is \underline{g} . The triple chain $z = \cos(x+1)^2$ has a shift and a \underline{h} and a cosine. Then $dz/dx = \underline{i}$.

The proof of the chain rule begins with $\Delta z/\Delta x = (\underline{j})(\underline{k})$ and ends with \underline{l} . Changing letters, $y =$

$\cos u(x)$ has $dy/dx = \underline{m}$. The power rule for $y = [u(x)]^n$ is the chain rule $dy/dx = \underline{n}$. The slope of $5g(x)$ is \underline{o} and the slope of $g(5x)$ is \underline{p} . When $f = \text{cosine}$ and $g = \text{sine}$ and $x = 0$, the numbers $f(g(x))$ and $g(f(x))$ and $f(x)g(x)$ are \underline{q} .

In 1–10 identify $f(y)$ and $g(x)$. From their derivatives find $\frac{dz}{dx}$.

1 $z = (x^2-3)^3$

2 $z = (x^3-3)^2$

3 $z = \cos(x^3)$

4 $z = \tan 2x$

5 $z = \sqrt{\sin x}$

6 $z = \sin \sqrt{x}$

7 $z = \tan(1/x) + 1/\tan x$

8 $z = \sin(\cos x)$

9 $z = \cos(x^2 + x + 1)$

10 $z = \sqrt{x^2}$

In 11–16 write down dz/dx . Don't write down f and g .

11 $z = \sin(17x)$

12 $z = \tan(x + 1)$

13 $z = \cos(\cos x)$

14 $z = (x^2)^{3/2}$

15 $z = x^2 \sin x$

16 $z = (9x + 4)^{3/2}$

Problems 17–22 involve three functions $z(y)$, $y(u)$, and $u(x)$. Find dz/dx from $(dz/dy)(dy/du)(du/dx)$.

17 $z = \sin \sqrt{x+1}$

18 $z = \sqrt{\sin(x+1)}$

19 $z = \sqrt{1 + \sin x}$

20 $z = \sin(\sqrt{x+1})$

21 $z = \sin(1/\sin x)$

22 $z = (\sin x^2)^2$

In 23–26 find dz/dx by the chain rule and also by rewriting z .

23 $z = ((x^2)^2)^2$

24 $z = (3x)^3$

25 $z = (x+1)^2 + \sin(x+\pi)$

26 $z = \sqrt{1 - \cos^2 x}$

27 If $f(x) = x^2 + 1$ what is $f(f(x))$? If $U(x)$ is the unit step function (from 0 to 1 at $x=0$) draw the graphs of $\sin U(x)$ and $U(\sin x)$. If $R(x)$ is the ramp function $\frac{1}{2}(x+|x|)$, draw the graphs of $R(x)$ and $R(\sin x)$.

28 (Recommended) If $g(x) = x^3$ find $f(y)$ so that $f(g(x)) = x^3 + 1$. Then find $h(y)$ so that $h(g(x)) = x$. Then find $k(y)$ so that $k(g(x)) = 1$.

29 If $f(y) = y - 2$ find $g(x)$ so that $f(g(x)) = x$. Then find $h(x)$ so that $f(h(x)) = x^2$. Then find $k(x)$ so that $f(k(x)) = 1$.

30 Find two different pairs $f(y)$, $g(x)$ so that $f(g(x)) = \sqrt{1-x^2}$.

31 The derivative of $f(f(x))$ is _____. Is it $(df/dx)^2$? Test your formula on $f(x) = 1/x$.

32 If $f(3) = 3$ and $g(3) = 5$ and $f'(3) = 2$ and $g'(3) = 4$, find the derivative at $x = 3$ if possible for

(a) $f(x)g(x)$

(b) $f(g(x))$

(c) $g(f(x))$

(d) $f(f(x))$

33 For $F(x) = \frac{1}{2}x + 8$, show how iteration gives $F(F(x)) = \frac{1}{4}x + 12$. Find $F(F(F(x)))$ —also called $F^{(3)}(x)$. The derivative of $F^{(4)}(x)$ is _____.

34 In Problem 33 the limit of $F^{(n)}(x)$ is a constant $C =$ _____. From any start (try $x = 0$) the iterations $x_{n+1} = F(x_n)$ converge to C .

35 Suppose $g(x) = 3x + 1$ and $f(y) = \frac{1}{3}(y - 1)$. Then $f(g(x)) =$ _____ and $g(f(y)) =$ _____. These are *inverse functions*.

36 Suppose $g(x)$ is continuous at $x = 4$, say $g(4) = 7$. Suppose $f(y)$ is continuous at $y = 7$, say $f(7) = 9$. Then $f(g(x))$ is continuous at $x = 4$ and $f(g(4)) = 9$.

Proof ϵ is given. Because _____ is continuous, there is a δ such that $|f(g(x)) - 9| < \epsilon$ whenever $|g(x) - 7| < \delta$. Then

because _____ is continuous, there is a θ such that $|g(x) - 7| < \delta$ whenever $|x - 4| < \theta$. Conclusion: If $|x - 4| < \theta$ then _____. This shows that $f(g(x))$ approaches $f(g(4))$.

37 Only six functions can be constructed by compositions (in any sequence) of $g(x) = 1 - x$ and $f(x) = 1/x$. Starting with g and f , find the other four.

38 If $g(x) = 1 - x$ then $g(g(x)) = 1 - (1 - x) = x$. If $g(x) = 1/x$ then $g(g(x)) = 1/(1/x) = x$. Draw graphs of those g 's and explain from the graphs why $g(g(x)) = x$. Find two more g 's with this special property.

39 Construct functions so that $f(g(x))$ is always zero, but $f(y)$ is not always zero.

40 True or false

(a) If $f(x) = f(-x)$ then $f'(x) = f'(-x)$.

(b) The derivative of the identity function is zero.

(c) The derivative of $f(1/x)$ is $-1/(f(x))^2$.

(d) The derivative of $f(1+x)$ is $f'(1+x)$.

(e) The second derivative of $f(g(x))$ is $f''(g(x))g''(x)$.

41 On the same graph draw the parabola $y = x^2$ and the curve $z = \sin y$ (keep y upwards, with x and z across). Starting at $x = 3$ find your way to $z = \sin 9$.

42 On the same graph draw $y = \sin x$ and $z = y^2$ (y upwards for both). Starting at $x = \pi/4$ find $z = (\sin x)^2$ on the graph.

43 Find the second derivative of

(a) $\sin(x^2 + 1)$

(b) $\sqrt{x^2 - 1}$

(c) $\cos \sqrt{x}$

44 Explain why $\frac{d}{dx} \left(\frac{dz}{dy} \right) = \left(\frac{d^2 z}{dy^2} \right) \left(\frac{dy}{dx} \right)$ in equation (8). Check this when $z = y^2$, $y = x^3$.

Final practice with the chain rule and other rules (and other letters!). Find the x or t derivative of z or y .

45 $z = f(u(t))$

46 $z = u^3$, $u = x^3$

47 $y = \sin u(x) \cos u(x)$

48 $y = \sqrt{u(t)}$

49 $y = x^2 u(x)$

50 $y = f(x^2) + (f(x))^2$

51 $z = \sqrt{1-u}$, $u = \sqrt{1-x}$

52 $z = 1/u^n(t)$

53 $z = f(u)$, $u = v^2$, $v = \sqrt{t}$

54 $y = u$, $u = x$, $x = 1/t$

55 If $f = x^4$ and $g = x^3$ then $f' = 4x^3$ and $g' = 3x^2$. The chain rule multiplies derivatives to get $12x^5$. But $f(g(x)) = x^{12}$ and its derivative is not $12x^5$. Where is the flaw?

56 The derivative of $y = \sin(\sin x)$ is $dy/dx =$

$\cos(\cos x) \sin(\cos x) \cos x \quad \cos(\sin x) \cos x \quad \cos(\cos x) \cos x$

57 (a) A book has 400 words per page. There are 9 pages per section. So there are _____ words per section.

(b) You read 200 words per minute. So you read _____ pages per minute. How many minutes per section?

58 (a) You walk in a train at 3 miles per hour. The train moves at 50 miles per hour. Your ground speed is _____ miles per hour.

(b) You walk in a train at 3 miles per hour. The train is shown on TV (1 mile train = 20 inches on TV screen). Your speed across the screen is _____ inches per hour.

59 Coke costs $1/3$ dollar per bottle. The buyer gets _____ bottles per dollar. If $dy/dx = 1/3$ then $dx/dy =$ _____.

60 (Computer) Graph $F(x) = \sin x$ and $G(x) = \sin(\sin x)$ —not much difference. Do the same for $F'(x)$ and $G'(x)$. Then plot $F''(x)$ and $G''(x)$ to see where the difference shows up.

4.2 Implicit Differentiation and Related Rates

We start with the equations $xy = 2$ and $y^5 + xy = 3$. As x changes, these y 's will change—to keep (x, y) on the curve. *We want to know dy/dx at a typical point.* For $xy = 2$ that is no trouble, but the slope of $y^5 + xy = 3$ requires a new idea.

In the first case, solve for $y = 2/x$ and take its derivative: $dy/dx = -2/x^2$. The curve is a hyperbola. At $x = 2$ the slope is $-2/4 = -1/2$.

The problem with $y^5 + xy = 3$ is that it can't be solved for y . Galois proved that there is no solution formula for fifth-degree equations.† *The function $y(x)$ cannot be given explicitly.* All we have is the *implicit* definition of y , as a solution to $y^5 + xy = 3$. The point $x = 2, y = 1$ satisfies the equation and lies on the curve, but how to find dy/dx ?

This section answers that question. It is a situation that often occurs. Equations like $\sin y + \sin x = 1$ or $y \sin y = x$ (maybe even $\sin y = x$) are difficult or impossible to solve directly for y . Nevertheless we can find dy/dx at any point.

The way out is *implicit differentiation*. Work with the equation as it stands. *Find the x derivative of every term in $y^5 + xy = 3$.* That includes the constant term 3, whose derivative is zero.

EXAMPLE 1 The power rule for y^5 and the product rule for xy yield

$$5y^4 \frac{dy}{dx} + x \frac{dy}{dx} + y = 0. \quad (1)$$

Now substitute the typical point $x = 2$ and $y = 1$, and solve for dy/dx :

$$5 \frac{dy}{dx} + 2 \frac{dy}{dx} + 1 = 0 \quad \text{produces} \quad \frac{dy}{dx} = -\frac{1}{7}. \quad (2)$$

This is implicit differentiation (**ID**), and you see the idea: Include dy/dx from the chain rule, even if y is not known explicitly as a function of x .

EXAMPLE 2 $\sin y + \sin x = 1$ leads to $\cos y \frac{dy}{dx} + \cos x = 0$

EXAMPLE 3 $y \sin y = x$ leads to $y \cos y \frac{dy}{dx} + \sin y \frac{dy}{dx} = 1$

Knowing the slope makes it easier to draw the curve. We still need points (x, y) that satisfy the equation. Sometimes we can solve for x . Dividing $y^5 + xy = 3$ by y

†That was before he went to the famous duel, and met his end. Fourth-degree equations do have a solution formula, but it is practically never used.

gives $x = 3/y - y^4$. Now the derivative (the x derivative!) is

$$1 = \left(-\frac{3}{y^2} - 4y^3 \right) \frac{dy}{dx} = -7 \frac{dy}{dx} \text{ at } y = 1. \quad (3)$$

Again $dy/dx = -1/7$. All these examples confirm the main point of the section:

4B (Implicit differentiation) An equation $F(x, y) = 0$ can be differentiated directly by the chain rule, without solving for y in terms of x .

The example $xy = 2$, done implicitly, gives $x \, dy/dx + y = 0$. The slope dy/dx is $-y/x$. That agrees with the explicit slope $-2/x^2$.

ID is explained better by examples than theory (maybe everything is). The essential theory can be boiled down to one idea: “*Go ahead and differentiate.*”

EXAMPLE 4 Find the tangent direction to the circle $x^2 + y^2 = 25$.

We can solve for $y = \pm \sqrt{25 - x^2}$, or operate directly on $x^2 + y^2 = 25$:

$$2x + 2y \frac{dy}{dx} = 0 \quad \text{or} \quad \frac{dy}{dx} = -\frac{x}{y}. \quad (4)$$

Compare with the radius, which has slope y/x . The radius goes across x and up y . The tangent goes across $-y$ and up x . The slopes multiply to give $(-x/y)(y/x) = -1$.

To emphasize implicit differentiation, go on to the *second derivative*. The top of the circle is concave down, so d^2y/dx^2 is negative. Use the quotient rule on $-x/y$:

$$\frac{dy}{dx} = -\frac{x}{y} \quad \text{so} \quad \frac{d^2y}{dx^2} = -\frac{y \, dx/dx - x \, dy/dx}{y^2} = -\frac{y + (x^2/y)}{y^2} = -\frac{y^2 + x^2}{y^3}. \quad (5)$$

RELATED RATES

There is a group of problems that has never found a perfect place in calculus. They seem to fit here—as applications of the chain rule. The problem is to compute df/dt , but the odd thing is that *we are given another derivative* dg/dt . To find df/dt , we need a relation between f and g .

The chain rule is $df/dt = (df/dg)(dg/dt)$. Here the variable is t because that is typical in applications. From the rate of change of g we find *the rate of change of* f . This is the problem of **related rates**, and examples will make the point.

EXAMPLE 5 The radius of a circle is growing by $dr/dt = 7$. How fast is the circumference growing? Remember that $C = 2\pi r$ (this relates C to r).

Solution
$$\frac{dC}{dt} = \frac{dC}{dr} \frac{dr}{dt} = (2\pi)(7) = 14\pi.$$

That is pretty basic, but its implications are amazing. Suppose you want to put a rope around the earth that any 7-footer can walk under. If the distance is 24,000 miles, what is the additional length of the rope? Answer: Only 14π feet.

More realistically, if two lanes on a circular track are separated by 5 feet, how much head start should the outside runner get? Only 10π feet. If your speed around a turn is 55 and the car in the next lane goes 56, who wins? See Problem 14.

Examples 6–8 are from the 1988 Advanced Placement Exams (copyright 1989 by the College Entrance Examination Board). Their questions are carefully prepared.

4 Derivatives by the Chain Rule

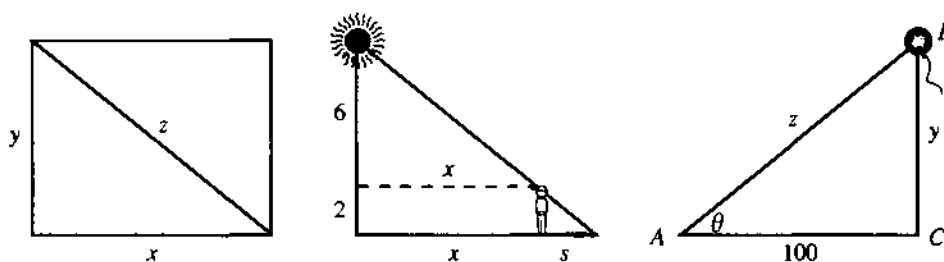


Fig. 4.3 Rectangle for Example 6, shadow for Example 7, balloon for Example 8.

EXAMPLE 6 The sides of the rectangle increase in such a way that $dz/dt = 1$ and $dx/dt = 3dy/dt$. At the instant when $x = 4$ and $y = 3$, what is the value of dx/dt ?

Solution The key relation is $x^2 + y^2 = z^2$. Take its derivative (implicitly):

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 2z \frac{dz}{dt} \quad \text{produces} \quad 8 \frac{dx}{dt} + 6 \frac{dy}{dt} = 10.$$

We used all information, including $z = 5$, except for $dx/dt = 3dy/dt$. The term $6dy/dt$ equals $2dx/dt$, so we have $10dx/dt = 10$. Answer: $dx/dt = 1$.

EXAMPLE 7 A person 2 meters tall walks directly away from a streetlight that is 8 meters above the ground. If the person's shadow is lengthening at the rate of $4/9$ meters per second, at what rate in meters per second is the person walking?

Solution Draw a figure! You must relate the shadow length s to the distance x from the streetlight. The problem gives $ds/dt = 4/9$ and asks for dx/dt :

$$\text{By similar triangles } \frac{x}{6} = \frac{s}{2} \quad \text{so} \quad \frac{dx}{dt} = \frac{6}{2} \frac{ds}{dt} = (3) \left(\frac{4}{9} \right) = \frac{4}{3}.$$

Note This problem was hard. I drew three figures before catching on to x and s . It is interesting that we never knew x or s or the angle.

EXAMPLE 8 An observer at point A is watching balloon B as it rises from point C . (The figure is given.) The balloon is rising at a constant rate of 3 meters per second (this means $dy/dt = 3$) and the observer is 100 meters from point C .

(a) Find the rate of change in z at the instant when $y = 50$. (They want dz/dt .)

$$z^2 = y^2 + 100^2 \Rightarrow 2z \frac{dz}{dt} = 2y \frac{dy}{dt}$$

$$z = \sqrt{50^2 + 100^2} = 50\sqrt{5} \Rightarrow \frac{dz}{dt} = \frac{2 \cdot 50 \cdot 3}{2 \cdot 50\sqrt{5}} = \frac{3\sqrt{5}}{5}.$$

(b) Find the rate of change in the area of right triangle BCA when $y = 50$.

$$A = \frac{1}{2}(100)(y) = 50y \quad \frac{dA}{dt} = 50 \frac{dy}{dt} = 50 \cdot 3 = 150.$$

(c) Find the rate of change in θ when $y = 50$. (They want $d\theta/dt$.)

$$y = 50 \Rightarrow \cos \theta = \frac{100}{50\sqrt{5}} = \frac{2}{\sqrt{5}}$$

$$\tan \theta = \frac{y}{100} \Rightarrow \sec^2 \theta \frac{d\theta}{dt} = \frac{1}{100} \frac{dy}{dt} \Rightarrow \frac{d\theta}{dt} = \left(\frac{2}{\sqrt{5}} \right)^2 \frac{3}{100} = \frac{3}{125}$$

In all problems I first wrote down a relation from the figure. Then I took its derivative. Then I substituted known information. (The substitution is after taking the derivative of $\tan \theta = y/100$. If we substitute $y = 50$ too soon, the derivative of $50/100$ is useless.)

"Candidates are advised to show their work in order to minimize the risk of not receiving credit for it." 50% solved Example 6 and 21% solved Example 7. From 12,000 candidates, the average on Example 8 (free response) was 6.1 out of 9.

EXAMPLE 9 A is a lighthouse and BC is the shoreline (same figure as the balloon). The light at A turns once a second ($d\theta/dt = 2\pi$ radians/second). How quickly does the receiving point B move up the shoreline?

Solution The figure shows $y = 100 \tan \theta$. The speed dy/dt is $100 \sec^2 \theta d\theta/dt$. This is $200\pi \sec^2 \theta$, so B speeds up as $\sec \theta$ increases.

Paradox When θ approaches a right angle, $\sec \theta$ approaches infinity. So does dy/dt . ***B moves faster than light!*** This contradicts Einstein's theory of relativity. The paradox is resolved (I hope) in Problem 18.

If you walk around a light at A , your shadow at B seems to go faster than light. Same problem. This speed is impossible—something has been forgotten.

Smaller paradox (not destroying the theory of relativity). The figure shows $y = z \sin \theta$. Apparently $dy/dt = (dz/dt) \sin \theta$. **This is totally wrong.** Not only is it wrong, the exact opposite is true: $dz/dt = (dy/dt) \sin \theta$. If you can explain that (Problem 15), then ID and related rates hold no terrors.

4.2 EXERCISES

Read-through questions

For $x^3 + y^3 = 2$ the derivative dy/dx comes from a differentiation. We don't have to solve for b. Term by term the derivative is $3x^2 + \underline{c} = 0$. Solving for dy/dx gives d. At $x = y = 1$ this slope is e. The equation of the tangent line is $y - 1 = \underline{f}$.

A second example is $y^2 = x$. The x derivative of this equation is g. Therefore $dy/dx = \underline{h}$. Replacing y by \sqrt{x} , this is $dy/dx = \underline{i}$.

In related rates, we are given dg/dt and we want df/dt . We need a relation between f and j. If $f = g^2$, then $(df/dt) = \underline{k} (dg/dt)$. If $f^2 + g^2 = 1$, then $df/dt = \underline{l}$. If the sides of a cube grow by $ds/dt = 2$, then its volume grows by $dV/dt = \underline{m}$. To find a number (8 is wrong), you also need to know n.

By implicit differentiation find dy/dx in 1–10.

1 $y^n + x^n = 1$

2 $x^2y + y^2x = 1$

3 $(x - y)^2 = 4$

4 $\sqrt{x} + \sqrt{y} = 3$ at $x = 4$

5 $x = F(y)$

6 $f(x) + F(y) = xy$

7 $x^2y = y^2x$

8 $x = \sin y$

9 $x = \tan y$

10 $y^n = x$ at $x = 1$

11 Show that the hyperbolas $xy = C$ are perpendicular to the hyperbolas $x^2 - y^2 = D$. (Perpendicular means that the product of slopes is -1 .)

12 Show that the circles $(x - 2)^2 + y^2 = 2$ and $x^2 + (y - 2)^2 = 2$ are tangent at the point $(1, 1)$.

13 At 25 meters/second, does your car turn faster or slower than a car traveling 5 meters further out at 26 meters/second? Your radius is (a) 50 meters (b) 100 meters.

14 Equation (4) is $2x + 2y dy/dx = 0$ (on a circle). Directly by ID reach d^2y/dx^2 in equation (5).

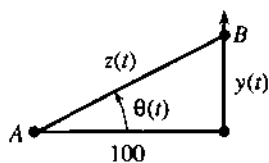
Problems 15–18 resolve the speed of light paradox in Example 9.

15 (Small paradox first) The right triangle has $z^2 = y^2 + 100^2$. Take the t derivative to show that $z' = y' \sin \theta$.

16 (Even smaller paradox) As B moves up the line, why is dy/dt larger than dz/dt ? Certainly z is larger than y . But as θ increases they become _____.

17 (Faster than light) The derivative of $y = 100 \tan \theta$ in Example 9 is $y' = 100 \sec^2 \theta \theta' = 200\pi \sec^2 \theta$. Therefore y'

passes c (the speed of light) when $\sec^2\theta$ passes _____. Such a speed is impossible—we forget that light takes time to reach B .



θ increases by 2π in 1 second

t is arrival time of light

θ is different from $2\pi t$

18 (Explanation by ID) Light travels from A to B in time z/c , distance over speed. Its arrival time is $t = \theta/2\pi + z/c$ so $\theta'/2\pi = 1 - z'/c$. Then $z' = y' \sin \theta$ and $y' = 100 \sec^2\theta \theta'$ (all these are ID) lead to

$$y' = 200\pi c / (c \cos^2\theta + 200\pi \sin \theta)$$

As θ approaches $\pi/2$, this speed approaches _____.

Note: y' still exceeds c for some negative angle. That is for Einstein to explain. See the 1985 *College Math Journal*, page 186, and the 1960 *Scientific American*, "Things that go faster than light."

19 If a plane follows the curve $y = f(x)$, and its ground speed is $dx/dt = 500$ mph, how fast is the plane going up? How fast is the plane going down?

20 Why can't we differentiate $x = 7$ and reach $1 = 0$?

Problems 21–29 are applications of related rates.

21 (Calculus classic) The bottom of a 10-foot ladder is going away from the wall at $dx/dt = 2$ feet per second. How fast is the top going down the wall? Draw the right triangle to find dy/dt when the height y is (a) 6 feet (b) 5 feet (c) zero.

22 The top of the 10-foot ladder can go faster than light. At what height y does $dy/dt = -c$?

23 How fast does the level of a Coke go down if you drink a cubic inch a second? The cup is a cylinder of radius 2 inches—first write down the volume.

24 A jet flies at 8 miles up and 560 miles per hour. How fast is it approaching you when (a) it is 16 miles from you; (b) its

shadow is 8 miles from you (the sun is overhead); (c) the plane is 8 miles from you (exactly above)?

25 Starting from a 3–4–5 right triangle, the short sides increase by 2 meters/second but the angle between them decreases by 1 radian/second. How fast does the area increase or decrease?

26 A pass receiver is at $x = 4$, $y = 8t$. The ball thrown at $t = 3$ is at $x = c(t - 3)$, $y = 10c(t - 3)$.

(a) Choose c so the ball meets the receiver.

*(b) At that instant the distance D between them is changing at what rate?

27 A thief is 10 meters away (8 meters ahead of you, across a street 6 meters wide). The thief runs on that side at 7 meters/second, you run at 9 meters/second. How fast are you approaching if (a) you follow on your side; (b) you run toward the thief; (c) you run away on your side?

28 A spherical raindrop evaporates at a rate equal to twice its surface area. Find dr/dt .

29 Starting from $P = V = 5$ and maintaining $PV = T$, find dV/dt if $dP/dt = 2$ and $dT/dt = 3$.

30 (a) The crankshaft AB turns twice a second so $d\theta/dt =$ _____.

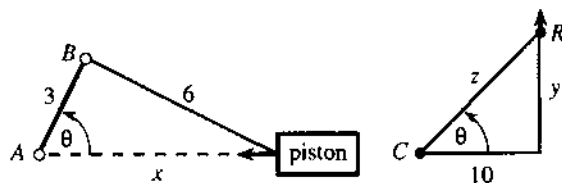
(b) Differentiate the cosine law $6^2 = 3^2 + x^2 - 2(3x \cos \theta)$ to find the piston speed dx/dt when $\theta = \pi/2$ and $\theta = \pi$.

31 A camera turns at C to follow a rocket at R .

(a) Relate dz/dt to dy/dt when $y = 10$.

(b) Relate $d\theta/dt$ to dy/dt based on $y = 10 \tan \theta$.

(c) Relate $d^2\theta/dt^2$ to d^2y/dt^2 and dy/dt .



4.3 Inverse Functions and Their Derivatives

There is a remarkable special case of the chain rule. It occurs when $f(y)$ and $g(x)$ are "inverse functions." That idea is expressed by a very short and powerful equation: $f(g(x)) = x$. Here is what that means.

Inverse functions: Start with any input, say $x = 5$. Compute $y = g(x)$, say $y = 3$. Then compute $f(y)$, and the answer must be 5. What one function does, the inverse function