1.054 Mechanics and Design of Concrete Structures
Term Project

Analysis and Form Finding Methods of Proper Free-Edged Concrete Shell

Katherina Santoso
May 6, 2004

Introduction

Thin shell concrete structures were developed in the nineteen-sixties in response to the need for economy in large-span structures and the design and aesthetic program of the modern movement in architecture. There are many characteristics of reinforced concrete shell that made it a popular system for large area enclosures – high strength, cheap, easy to construct and durable. It was however the one characteristic – its ability to be easily shaped into any form, that makes it the system of choice among many architects up till today. The concrete shell system has resulted in many interesting architectural forms such as the restaurant in Xochimilco, Mexico, by Felix Candela and the Algeciras Market Hall and Zarzuela Hippodrome in Spain by Eduardo Torroja (Fig 1.).

In designing a concrete shell structure, the shape of the shell can be selected purely arbitrarily, or thought of with care to minimize the weight corresponding to a given loading and boundary condition. With progress made in the development of finite element analysis and the construction of shell structure, shell of almost any shape can now be analyzed and constructed. However, shell of arbitrary forms may not be self-stabilizing and need the help of stiffening elements to restrain its boundaries. An example of such form is the spherical shell roof of Kresge Auditorium whose sides are heavily restrained by edge beams. An alternative and more elegant way of designing the shell would be to analyze the key geometrical parameters (like span and height), loading and desired stress state, and then design a natural self-supporting shell with free edges. This inverse formulation method of designing shell structure will be investigated in this paper. To use this inverse formulation method, an objective of the shape formulation needs to be established. For this paper, the objective shall be to obtain the most effective shape for free-edged shell, in which all forces are carried in the shell by membrane action.

To aid in the understanding of form finding methods, this paper will first introduce the structural behavior of proper shell and the characteristics needed for a shell to have free edges. The behavior of proper shell will be illustrated with specific reference to the dome, the earliest shell form constructed. Investigations on the characteristics of free-edged shells will be supplemented with an in depth study of the hyperbolic paraboloid, a mathematically prescribed form which can be numerically proven to support free edges. Finally, two form finding methods of free-form shells would be explored – the more primitive experimental procedures made famous by Frei Otto, and the more recently developed and widely used computer shape-finding programs.

Shell

A shell, or a laminar shell, is defined loosely as a structure whose thickness is very much smaller than its two other dimensions. Shells may be distinguished into proper shells and improper shells. A proper shell may be described as a laminar structure which acts mostly by membrane or direct stresses, those being equally distributed in the slab’s thickness and parallel to the tangential plane of the surface. These stresses may be tensile or compressive, but has to be compressive in our case of concrete shells. To be considered a proper shell, it must have a doubly-curved surface. Following this definition, an improper shell is a laminar structure in which a crucial part of the structural action is performed by bending of the slab.[1] The limiting case of the improper shells is the ordinary flat slab, which works exclusively through bending. Another common example of improper shells is the cylindrical vault. Although cylindrical vaults work exclusively with direct stresses under uniformly distributed vertical load condition, under non-uniform accidental live load, the vault develops bending. Improper shells usually have simple or singly-curved surfaces. Proper shells are more efficient in its load-carrying capacity, resisting a greater stress value per unit mass. Unlike improper shells, proper shells need minimum stiffening members to resist bending. The most commonly found proper shell is the perfect hemispherical dome.

Dome vs Arch

A dome is essentially an arch rotated 360 degrees about a point. Since the form of a dome is obtained from the rotation of an arch, the force carrying mechanism of a dome can be rationalized from the analysis of the way an arch supports its loads. The simplest way of visualizing how an arch works can be obtained from the hanging string experiment. The hanging string, which is an inversed arch, has negligible bending strength under its own dead weight, a uniform vertical load. All of the loads are carried by direct tensile stress. In an arch, load is thus carried in the opposite direction, by pure compression with no bending, if the footings do not permit horizontal displacement. In the case of the dome, the uniform load is carried by forces in the plane of the shell that are similar to the forces which carry the uniform load in the arch.

A key difference between a dome and an arch is that under non-uniform vertical loading, bending would be developed in an arch, while the dome is able to carry the non-uniform loading with non-significant bending. An arch is also very sensitive to edge forces. A horizontal load exerted on an arch due to boundary conditions may result in a large bending moment. A dome is not affected by horizontal force at its circular edge and edge restraints will not affect the performance of the dome.

There are two sets of forces acting in a dome, the meridional forces and hoop forces. The meridional forces acts along the same line as the axial forces in an arch. The hoop forces act in the horizontal direction, at right angle to the meridional forces (Fig. 2a). Since the hoop forces do not enter into the equations of vertical equilibrium and all vertical loads are transmitted by the meridional forces.

Under non-uniform vertical load with smooth transition, that is to say there is no discontinuity in the load curve at the point where the load goes to zero, it is observed that the shell carries this load almost entirely by the same arching forces as before. This action is unlike the action of the arch under the same non-uniform load (Fig. 2b). The reason for this difference is the existence of the hoop forces. Physically, it can be visualized as when an arch segment of the dome attempts to bend under the partial load in the same way the arch bends, the hoop forces restrain it as if stiff concentric rings are wrapped around the structure.

---

[1] Faber, Colin. Candela/The Shell Builder (London: The Architectural Press, 1963) pp225-226.

Under a concentrated horizontal load at the foot of the arch, which simulates the force exerted by an edge restraint, a large bending moment is formed throughout the entire arch system. The maximum bending moment occurs at the crown. This horizontal force at the foot of an arch may be considered as uniform horizontal thrust acting around the circular edge of the dome in the direction of the center of the dome. As this horizontal force tries to bend the shell, the hoop forces cause a rapid damping of the bending so that at a relatively short distance from the edge the bending effect is no longer observable. Thus edge forces in equilibrium applied to an arch propagate throughout the entire structural system and create large bending moments while similar forces acting on a dome create bending moments in a narrow region near the edge and have almost no effect throughout a large portion of the structure (Fig 2b).

Design of Reinforced Concrete Dome

The invention of reinforced concrete in the early 1900s affected the design of concrete domes dramatically. Reinforced concrete, which has a higher strength per unit volume than pure concrete, allows for a much thinner shell structure to be formed. To illustrate the magnitude of the effect that the reinforced concrete brought to the design of shell, a comparison between a pure concrete shell, the Pantheon in Rome and a reinforced concrete dome, the Kingdome in Washington State can be made. The unreinforced hemispherical dome roof of the Pantheon has a height and radius of 21.6 meter and a thickness of 1.2 m at the top while getting progressively thicker towards the bottom. This gives a radius to thickness ratio of 18. The Kingdome which has a clear span diameter of 201.6 meters has a shell thickness of only 120 mm(Fig.3), giving a radius to thickness ratio of 1750. This radius to thickness ratio of the Kingdome shell is smaller than that of a hen’s egg.

This small thickness is made possible through the use of reinforcement bar integrated in the concrete. There are three main sets of reinforcement bars which are usually used in a concrete dome – the dome hoop reinforcement, the meridional bending reinforcement and the ring reinforcement. The dome hoop reinforcements are concentric steel rings which are arranged parallel to the hoop forces to resist all the hoop tension. In a homogeneous concrete structure, the hoop stress is resisted by the compressive strength of the concrete, thus requiring a considerable thickness of the dome. In a reinforced concrete dome, the hoop reinforcement bars takes this duty of resisting the hoop stress through tension. Since steel has a much greater strength than concrete, a smaller cross sectional area is needed, reducing the thickness of the dome. The meridional bending reinforcement runs along the meridional forces, to resist moment at the bottom of the shell caused by edge restraint or horizontal loading. The third reinforcement, ring reinforcement, is only necessary for a shallow dome. In a pure hemispherical shell, the meridional forces at the edge of the shell run vertically (since the tangent of the hemisphere is vertical at this point) and are met by the abutting structure in vertical support. No horizontal forces are generated at the edge and a tension ring is unnecessary (Fig. 4). However, pure hemispherical shape is rarely used in architecture. Very often, segments of the dome surfaces are cut, generating a new shape, such as the flat dome (Fig 5). In a shallow dome, the tangent of the meridional force at the edge occurs at an angle and a horizontal thrust is experienced at the bottom of the dome. This horizontal force may be contained by having ring reinforcement at this edge, arresting the outward forces by tensile stresses. The use of ring reinforcement is often accompanied by an edge beam.

Besides having pre-tensioned ring reinforcement, the containment of the horizontal thrust at the bottom of a shallow dome could be solved through other designs of the edge zone. The most commonly used solution is the ring wall system, as used in the reservoir in Ohio (Fig. 6). The ring wall is a pre-stressed concrete system, which work similarly to the ring reinforcement, arresting the horizontal thrust at the bottom of the dome through tensile forces.

Another solution, an innovative one, is developed by Eduardo Torroja for the Algericas Market hall. To arrest the horizontal thrust at the bottom of the shallow dome in the market hall, Torroja designed a special edge in the form of barrel shells to divert the forces into the six columns. The diagonal meridional forces from the dome are guided along the barrel shells to the abutting structure, on to which they converge as an oblique force (Fig. 7). The vertical action of this force is sustained by the columns, while the horizontal element is absorbed by a pre-stressed tension ring linking the six columns (Fig. 8).

Free Edge Shell

An important lesson learnt from the shallow dome is that while it is convenient to develop an enclosure shape from taking segments from the dome and being able to analyze the structure with a closed form solution, out of plane stresses and bending moments may exist in the edges and reinforcements of the regions are necessary. In short, a pre-determined forms often do not allow for free-edges in the structure. Before I proceed to investigate on how to form structural shapes with free edges, I would present the behavior of free-edged shells and the theory behind the hyperbolic paraboloid (HP) shape, a geometry which allow for free-edges under a determined boundary condition.

Behavior of Free Edge Shell

It has been suggested that the free edge is the ultimate refinement of shell design. Free-edges have always been the dream of architects, where the architecture stands elegantly without any visible supporting structures. For a long time, engineers have always thought that the free edge was a practical idea. However no one understood how it worked or dared to build it until Felix Candela designed the world’s first free edge concrete shell, the roofs of San Antonio de las Huertas, in 1956. Even this discovery of the free edge form was almost accidental.

Candela has been designing concrete shells since 1941. He has often thought of the possibility of building a shell with no edge beams or edge reinforcements. From his studies of shell structures, he knew that there exists a form which has a stress condition producing no normal stresses along the edge. He envisions that at some line along a surface, internal stresses counter balance each other, and this line would provide an ideal section which may be cut to form a free edge. An example of this stress-free line is the circular line that divides a hollow sphere into two perfect hemispheres. In the spherical hemisphere supported at the horizontal circle, all forces are transmitted by normal forces along the edge with no existing tangential forces. Candela, however, hopes to find this line in a vertical plane. Candela also discovered that the crux of this free-edge problem lies in the boundary condition of the shell. However, he tries to no avail in developing a form which would make this free edge possible. He once related, “The real significance of the boundary conditions, never clearly explained in most texts, was at last clear to me. I tried then, without success, to equalize normal forces to the groins, but the shell plan was not quite symmetrical. Despite all the time I was given to make the calculations, they were wrong in many details.”[2] Then, in June 1956, he was watching architects Enrique de la Mora and Fernando Lopez Carmona sketch the roofs of San Antonio de las Huertas, when suddenly the fundamentals of free-edge shells dawned on him. Then and there, he was confident enough to suggest, given the symmetry of the vaults, that their edges be left free of arches.

During his work on the church, Candela discovered the possibility of having free-edge shell for the hyperbolic paraboloid (HP) form and proved this mathematically.

Hyperbolic Paraboloid (HP)

Candela did not discover the hyperbolic paraboloid shape. The HP was an established geometrical form, prescribed by mathematical equations $x\ /a\ - y\ /b\ = z$ for a negative Gaussian curvature and $x\ /a\ + y\ /b\ = z$ for a positive Gaussian curvature. A surface with a negative Gaussian curvature can be generated by translating a convex parabola over a concave parabolic curve. A surface with positive Gaussian curvature is formed by translating a convex parabola over a convex parabola (Fig. 9).

Like the dome, the HP is a doubly-curved surface, permitting it to work with only membrane stresses. The HP also has a very important asset -- the simplicity of the surface equation makes it possible for its membrane stresses to be analyzed by simple algebraic expressions for any boundary condition. In the following example, the absence of normal stresses along the edge of a hyperbolic paraboloid is proven by the general membrane theory.

General Membrane Theory for Hyperbolic Paraboloid[3]

For translation shells of double survature, the general equilibrium equations of the membrane theory reduces to the following equations,

[2] Faber, Colin. Candela/The Shell Builder (London: The Architectural Press, 1963) p177.

[3] Billington, David P. Thin Shell Concrete Structures (New York: Mc Graw-Hill Book Company, 1965) p245-246

$$N_x = \frac{\partial^2 F}{\partial y^2} - \int p_z dx$$
$$N_y = \frac{\partial^2 F}{\partial x^2} - \int p_y dy \qquad (1)$$
$$N_{xy} = -\frac{\partial^2 F}{\partial x \partial y}$$

where Nx is the axial force acting in the x-direction, Ny is the axial force in the y direction and Nxy the shear force (Fig. 10).
The stress function F is governed by the following differential equation

$$\frac{\partial^2 F}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - 2\frac{\partial^2 F}{\partial x \partial y}\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 F}{\partial y^2}\frac{\partial^2 z}{\partial x^2} = q \qquad (2a)$$

In which

$$q = -p_z + p_x \frac{\partial z}{\partial x} + p_y \frac{\partial z}{\partial y} + \frac{\partial^2 z}{\partial x^2}\int p_x dx + \frac{\partial^2 z}{\partial y^2}\int p_y dy \qquad (2b)$$
$$p_x = p_y = 0 \quad q = -p_Z$$

where px, py and pz are loads acting in the x, y and z direction.

Fig. 11 shows a hyperbolic paraboloid surface with a negative Gaussian curvature. The equation for this surface is

$$z = \frac{y^2}{h_2} - \frac{x^2}{h_1}$$
where $h_1 = \frac{a^2}{c_1}$ (3)
$$h_2 = \frac{b^2}{c_2}$$

When Eq.(3)is introduced into Eq. (2a), we get

$$\frac{\partial^2 F}{\partial x^2}\frac{2}{h_2} - \frac{\partial^2 F}{\partial y^2}\frac{2}{h_1} = q \qquad (4)$$

Consider the simple case of uniform vertical loading over the horizontal projection of the surface, ie q = -pz. Under this loading, there are three solutions for the value of F, depending on the boundary condition of the edge supports[4].

Where there is a full support along y =   b and no appreciable stiffening along the edge parabolas at x =   a, the solution for F is as follow.

[4] Flugge, W. Stresses in Shells (Berlin: Springer-Verlag OHG, 1960) p 184

$$F = -\frac{1}{4} p_z h_2 x^2 \tag{5a}$$

Substituting this value of F into Eq. (1), the following values of forces are obtained.

$$N_x = 0$$

$$N_y = \frac{-p_z h_2}{2} \tag{5b}$$

$$N_{xy} = 0$$

Where arches are provided at x =   a and no support along y =   b, the value of F and the corresponding forces are

$$F = +\frac{1}{4} p_z h_1 y^2$$

$$N_x = \frac{p_z h_1}{2}$$

$$N_y = 0 \tag{6}$$

$$N_{xy} = 0$$

Where both pairs of edges are equally stiff, the third solution of F and its corresponding forces are obtained.

$$F = -\frac{1}{8} p_z \left(h_2 x^2 - h_1 y^2 \right)$$

$$N_x = \frac{p_z h_1}{4}$$

$$N_y = -\frac{p_z h_2}{4} \tag{7}$$

$$N_{xy} = 0$$

A special attention should be paid to equation 5(a) and 5(b). Since 5(a) and 5(b) are similar, for the sake of discussion, equation 5(a) will be quoted. In equation 5(a), it is shown that when only the parabola along y =   b are supported, there is no normal forces acting on the parabolic edge along x =   a. This is as proven by the equation Nx = 0. Under this support condition, Nz = 0 too. This indicates that in this structure, the entire vertical load is carried by the shell along the parabolas parallel to the yz plane and the load is transferred down to the support at the base where y =   b. The application of this theory is illustrated in the Lomas de Cuernavaca Chapel in Mexico (Fig. 12). The HP form of the roof is supported at the longitudinal parabolic side, which is the y =   b curve described in the theory. And as shown in the diagram, the structure support phenomenal free parabolic edges, which is that described as x =   a in the theory.

Another point to note is that the different boundary conditions determine how the load is transferred to the supports and the stresses experienced by the shell surfaces. The same structure, loaded in the same way, but supported differently will have a different load path pattern. This is illustrated by the instance where the shells are supported at all four parabolic edges (at y =   b and x =   a). In this situation, load is transferred to all four directions (as opposed to just two directions when supports are provided at only two edges).

Form finding method

Form finding, as the term itself suggests, refers to a process of determining a geometrical form, in most cases of a surface, which is in force equilibrium and satisfies additional design constraints. These design constraints may include key geometrical parameters (like span and height), loading combinations, and boundary conditions. In this paper, the objective of the form finding method is to find a shape which minimizes the bending stresses in the structure. An ideal form would be a form that functions as a membrane oriented shell, supporting forces through pure axial stresses. And since all form finding methods give structures which are in force equilibrium, free-edges come naturally as part of the final forms.

Two form-finding methods would be explored in this paper, a more traditional experimental form finding technique and form finding by computer analysis.

Experimental Form Finding Method

Experimental form finding methods provide an easy yet clear visualization of how a structure works. They are suitable for determining the optimum shell form only when specific loading conditions are given. The most commonly used experimental method for concrete shell form finding is the inverse principles of hanging models. This method involves hanging a membrane clamped at its boundary conditions as prescribed by the architectural constraints, and placed under a defined loading condition. This hanging model renders pure tension, when if stiffened by some means, turns into pure compression after inversion (Fig. 13).

This procedure not only simplifies the development of favorable shell forms, but also achieves the attainment of ideal edge conditions. A previous illustration of the general membrane theory for hyperbolic paraboloid has shown that the effectiveness of a shell is decisively influenced by the type of support. A shell form favorable in itself may still be exposed to appreciable bending moments if the edge design is structurally unfavorable. The shape of the edge seen in the models is at once clear. By referring to the edge condition of the model, architects could then design the edge of the actual structure, thus ensuring that bending moments are completely excluded in the final product.

At the time before computer was easily available, physical modeling techniques were widely used in the design of free-form shells. After obtaining a shape from the experiments, the stresses in the structure is analyzed by finite element methods to determine the need for reinforcements. In an instance where the form is too complex to be examined analytically, a live scale model is built and tested under the expected loading conditions (Fig. 14). From this test, locations of possible failures and the amount of deformation could be observed and measured. Design decisions, with regards to reinforcement placements and changing the stiffness of the shell to obtain ideal deformed shape, are then made in response to the observed test results.

Computer Shape Finding Method

The concept behind computer shape finding methods is similar to that of the experimental method. For a given arbitrary plan, a flat membrane with a given boundary condition is subjected to a uniform pressure, line or concentrated loads, and the desired shape can be generated. A very useful feature about the computer method, which is not found in the experimental method, is that the computer aided form-finding process yields directly numerical data on the generated shape, which can be further processed in structural analysis software. In this paper, I will not describe the different computer software available to do form finding methods. Instead, the paper will illustrate the outlying theory for the most widely used programming approach for shape-finding in shell structures.

The computer form finding method is in fact an optimization technique developed as a synthesis of geometrical design, structural analysis, and mathematical optimization. Geometrical design is done using the Computer Aided Geometric Design (CAGD) concept, where surfaces are created using a few natural variables, i.e. coordinates of selected nodes. With CAGD, nodes could be conveniently varied, either manually or automatically using a computer program, for future iteration. In addition to shape, thickness variation can also be made. After a geometrical form is developed, boundaries of the structure are defined, loading is applied, and this structural model is fed into an analysis program. Results from the analysis are then supplied into a mathematical optimization program, which will alter the geometric design to achieve a desired objective. The objective of an optimization varies, such as minimum surface or weight. In our case, the intention is to get rid of any bending moment, leading to the formation of a member oriented shell. In finite element, this can be done by strain energy minimization. Different shapes for different loading conditions are obtained at the end of the process, and it is up to the human designer to make the final decision.

Examples

For a better understanding of the computer shape finding methods, two examples of shape finding process are presented here. The first example involves the formation of a form from a flat sheet that covers the intended floor area. The second case illustrates an optimization of a pre-designed canopy form to obtain a more efficient shape.

Kassel Model Shell[5]

In this start-from-nothing form-optimization procedure, only the floor area, height of enclosure, boundary conditions and a rough idea of how the enclosure shape (curved edged or straight edged) need to be established. The formulation of the form is left entirely to the computer.

This reinforced concrete model shell was built at the University of Kassel. The shape was determined by a large displacement finite element analysis starting from a plane system. The finite element (FE) plane model as shown in Fig. 15 is first made using nine-nodes degenerated shell elements. The model covers a plan area of 28 m . The boundaries of

[5] Kollegger, J. and Mehlhornm G. 'Traglastversuch an einer freigeformten Stahlbetonschale'. Res Rep. no. 7, Div. of Reinforced Concrete, University of Kassel, 1989.

the mesh were defined by quadratic parabola, an aesthetic decision based on how the designer wanted the final design to look. A linear elastic material with modulus of elasticity, E=30 000 N/mm and Poisson ratio, ν=0.2 was assumed. A uniform slab thickness of 16 mm is set. (Thickness optimization was not done for this model). The slab was then stepwise subjected to a uniform vertical load of increasing intensity, until the desired displacement (height) of the canopy is reached at 1000mm. The form finding process, from the initial plane system to the final desired shape, is shown in Fig. 16.

Lecture Hall of University of Mainz[6]

This project involves an investigation whether an existing structure, a lecture hall at the University of Mainz (Fig. 17), could be made more efficient by varying its shape and the shell thickness. Although this is a re-design project of an existing building, a similar approach can be undertaken for the form-finding of a new building. An established canopy form of a desired shape can be first modeled, and optimization is then based on the model.

The lecture hall is a three-point supported shallow reinforced concrete shell with a uniform thickness of 0.2m. It has a span of 55.7m. The shell was designed by A. Mehmel in 1967 on an intuitive basis. Mehmel rationed that the form is designed so that forces reaches the ground by the shortest possible distance, through the triangular rib originating from the midpoint. In this project, the shell is qualitatively investigated using the shape optimization concepts by applying strain energy minimization. Due to symmetry, only one sixth of the shell is modeled using 8-node finite element model. An optimization program is then run and the result is shown in Fig. 18. The darker lines denote the original shape while the lighter line represents the optimal shape. It can be seen that the curvature of the free edge is increased. This move reduces the bending stress state in the area.

A second optimization process is run with thickness variation (Fig. 19). The optimal shape for optimization with thickness variation is shown in Fig. 20. Comparing the results from Fig. 18 and Fig. 20, it can be noted that the optimum shape changes very slightly. This implies that thickness variation has little effect on the optimum shape. The practical implication to thickness variation is a little tricky. Although a varied thickness ensures that material is only used in regions where it is needed which ensures a minimization of material volume, construction of varied thickness shell tends to be more complex and may not translate in cost reduction.

Conclusion

Proper shell, which carries load by membrane action, is indeed a very efficient load bearing system. This system evolves as a solution for wide-span problems which the slab-beam-column system is inefficient of doing. For the beam-column system, large bending moment at the middle of the slab increases exponentially with the increase in span. With the understanding that flexural stresses kills the efficiency of the beam column system,

---

[6] Leonards, L. and Mehmel, A. "The Lecture hall of the Science Faculty of the University of Mainz" Der Baungenieur (1969): p87-92

the objective for the shell design is to form a membrane oriented structure where bending is avoided to the greatest possible extent. This objective could be met by varying the shape of the structure, given a loading condition and boundary conditions. The first part of the paper discusses the behavior of shell structures and characteristics that make edge-free shells possible. With this background understanding, the paper proceeds to illustrate experimental and computer form-finding methods. Different optimal shapes will be obtained for different loading and boundary conditions. Since design is not always driven by structural efficiency, but a combination of factors such as cost, aesthetic and manufacturing requirements, these optimization results should always be treated as just an aid to design, and not a deterministic factor.

References

1. Billington, David P. Thin Shell Concrete Structures (New York: Mc Graw-Hill Book Company, 1965)
2. Ekkehard Ramm. "On Shape Finding Methods and Ultimate Load Analyses of Reinforced Concrete Shells" Engineering Structure Vol. 13, April (1991): p178-197
3. Faber, Colin. Candela/The Shell Builder (London: The Architectural Press, 1963)
4. Flugge, W. Stresses in Shells (Berlin: Springer-Verlag OHG, 1960)
5. Joedicke, Jurgen. Shell Architecture (London: Alec Tiranti Ltd, 1963)
6. John L. Meek and Xiaoyan Xia. " Computer Shape Finding of Form Structures," International Journal of Space Structures Vol. 14 No.1 (1999): p35-54
7. Kollegger, J. and Mehlhornm G. 'Traglastversuch an einer freigeformten Stahlbetonschale'. Res Rep. no. 7, Div. of Reinforced Concrete, University of Kassel, 1989.
8. Leonards, L. and Mehmel, A. "The Lecture hall of the Science Faculty of the University of Mainz" Der Baungenieur (1969): p87-92
9. http://www.mathematik-21.de/statussem/poster/camprubi.pdf

Note: This version of the paper does not include the figures.