# 1.224 Case Study2

# Transit Crew Scheduling

## 1. Introduction

The crew scheduling problem is the final step in developing a transit operation plan. The full problem includes five steps: network design, service frequency setting, timetable development, vehicle scheduling, and crew scheduling. In network design, demand and feasible operating links for the network are among the inputs. Outputs are a set of routes to be operated. In the second step, inputs are available vehicles and budget, demand data and service policies and the outputs are service frequencies by time of day for each route. The objective is to provide transit service that meets or exceeds service policies, given the available budget and resources. After setting frequencies, the next step is timetable development. With running time information, departure and arrival times are determined for each vehicle trip. Then, given a timetable, the vehicle scheduling step connects successive trips for each vehicle according to deadhead times, recovery times and other schedule constraints. The objective is to minimize the number of vehicles used while meeting the timetable requirements. Outputs from vehicle scheduling are revenue and deadhead movements for each vehicle starting with the *pull-out* time and ending with the *lay-up* time. The *pull-out* time is when a vehicle leaves the depot and is put into service. A *lay-up* (or pull-in) time is when a vehicle is removed from service and returns to the depot. A pull-out and a corresponding lay-up define the start and end points of a *vehicle block.*

Given the set of vehicle blocks, crews must be assigned to each vehicle so that each vehicle has a crew throughout its block. Since crew cost is a major part of the transit operation costs, obtaining a crew schedule that covers the vehicle block requirement at minimum cost is the ultimate objective.

<u>Terminology</u>:

A *route* is a transit service line consisting of intermediate stops (stations) between two termini. A *trip* is a traversal of a route that is defined by both starting and ending times and locations. It is the basic unit of service in the sense that each trip must be operated by a single vehicle. A *vehicle block* is a sequence of trips starting and ending at a depot. The starting time and location of any trip on a vehicle block must be feasible with respect to the ending time and location of the previous trip. A vehicle block is assigned to a single vehicle.

A vehicle must be operated by a crew throughout the block duration. However, a vehicle block can be too long to be handled by a single crew. Therefore, different crews have to be assigned to operate the vehicle at different times. At some point, a crew is released and another crew takes over the vehicle. A *relief point* within a block provides a time and place for a possible change of crews. Thus, a vehicle block is cut into *pieces of work*. A *piece of work* is defined as a continuous crew work period on a single vehicle between two relief points on a block. The relief points selected form a *partition* of the vehicle block. Since there are usually a large set of possible relief points that may be selected to cut a vehicle block, there are many possible partitions for a vehicle block.

A crew duty, often called a *run,* is a feasible sequence of pieces of work for a single crew to operate in a day. A feasible run must satisfy the work rules, which are part of the labor agreement. Work rules usually include constraints on *work time*, *spread time*, *unpaid break*, and define pay provisions. *Work time* is the total time a crew spends on a vehicle. *Spread time* is time from the start of the first piece of work to the end of the last piece of work on a run. *Unpaid break* is the period that is not paid between two pieces of work. Various cost penalties (or pay premiums) affect the cost of any duty, reflecting spread time, work time etc. Often work rules may restrict the number of pieces of work in a duty to be no more than two.

## 2. Problem Description

Since heaviest transit demand generally occurs for commuting trips, the supply of transit service is at the highest level immediately before and after normal working hours and varies greatly by time of day. Correspondingly, there is often a large variation in vehicle requirements between the peak and off-peak periods. (The ratio of peak to off-peak requirements is often two to one or greater.) Figure 1 illustrates the typical pattern of temporal variation and a few examples of vehicle blocks.

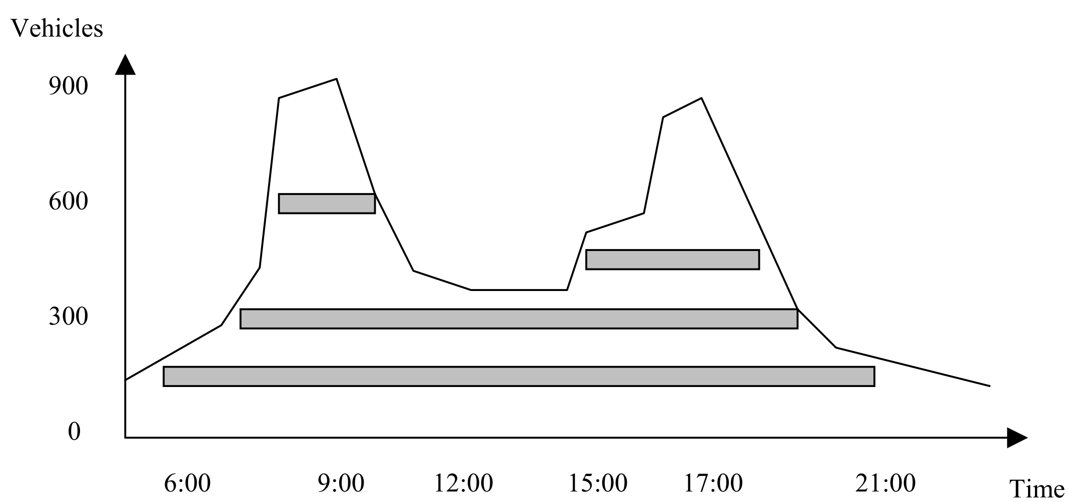


**Figure 1 Temporal Variation in Vehicle Requirements and Vehicle Blocks**

Due to this peaking characteristic, the length of vehicle blocks varies greatly. For example, the longest block in Figure 1 starts at 6:00 and ends at 21:00, while the smallest block only lasts for a few hours in the morning peak period. This large variation is one of the most important characteristics of transit scheduling and a major complicating factor in crew scheduling. Since some blocks are too long to be covered by a single crew, they must be cut at relief points to form feasible pieces of work. Then, different pieces of work are combined to form legal runs for crews.

Given the vehicle schedule embodied in a set of vehicle blocks and work rules, the objective of crew scheduling is to generate the least cost crew schedule.

## 3. Crew Scheduling Procedure

The crew scheduling process consists of two tasks. First, the long vehicle blocks must be cut into pieces of work. Second, the pieces of work must be combined to form legal runs. As a result, the overall process is often called run-cutting and scheduling. Since work rules and policies affect both these two steps, we first introduce typical work rules and policies, then describe each of these steps and formulate the problem mathematically.

### 3.1 Work Rules

Work rules are often of the following types:

(1) **Maximum number of pieces of work assigned to a single crew for a workday**. Often, work rules restrict a crew duty to no more than two pieces of work.

(2) **Maximum work hours and overtime premium**. Maximum work time is usually greater than 8 hours. A full-time operator is usually guaranteed 8-hours pay even if the work time is less than 8 hours. A 50% premium is usually paid for any work time over 8 hours.

(3) **Maximum spread and spread penalties**. If a driver's run requires clocking off at the end of the day more than a specific number of hours after the start of work, a bonus known as a spread penalty is paid. Typically, there is an absolute maximum spread.

(4) **Minimum unpaid break**. If the break between two pieces of work is smaller than some amount, it must be paid time.

(5) **"Swing Bonus"**. A bonus is paid for any driver who does not start and end each piece of work at the same location.

Work rules of the type shown above are hard constraints and must be complied with fully. In addition to these hard constraints, transit agencies may have other policies.

There is often policy that defines the minimum or maximum length for a single piece of work. The two pieces of work should not be too unbalanced: for example, if the first piece has 7 hours work time while the second piece has only one hour, the crew may not show up for the second piece. There may also be a policy that no reliefs are scheduled during the peak period, because if the crew assigned to take over the vehicle does not show up on time, the service impact could be severe.

### 3.2 Cutting Vehicle Blocks

Figure 2 shows an example of a part of a vehicle block. It is a vehicle block for a vehicle operating on a single route between terminals A and B. Let $a_A(a_B)$ denote the arrival time at terminal A (B) and $d_A(d_B)$ denote the departure time at terminal A (B). The vehicle block starts at the depot with the vehicle deadheading to terminal B and entering into service. We will assume a relief can occur only when the vehicle arrives at a terminal.

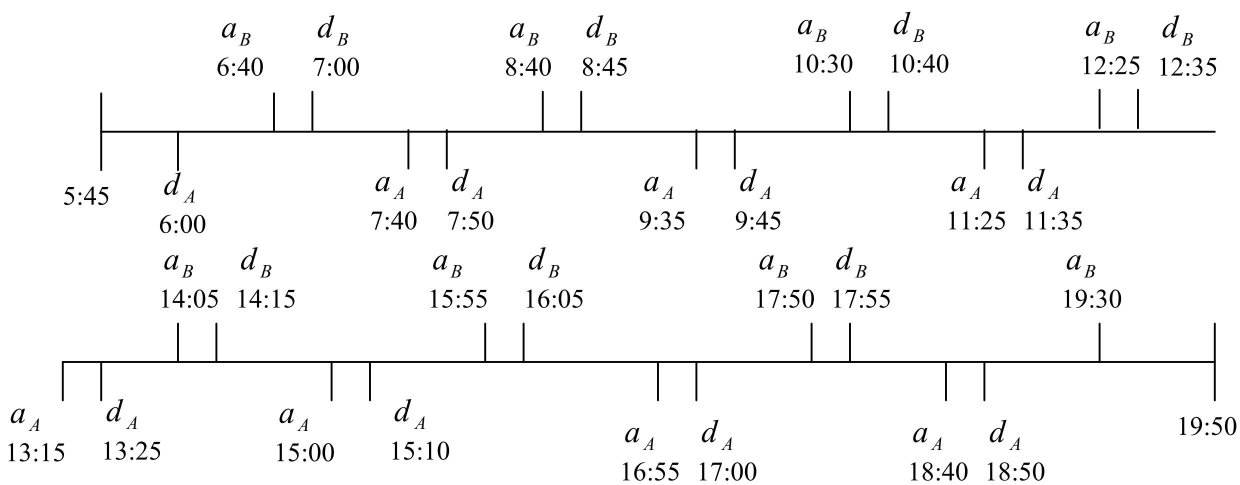


**Figure 2 A Vehicle Block**

If no constraints are placed on the cuts we can make, and each of the 15 arrivals at terminals could be a possible relief point, the number of possible partitions of this single vehicle block could be huge. However, as discussed previously, there is usually a policy defining the minimum length of a piece of work for practical reasons. Suppose the minimum piece length is 2.5 hours, the first possible cut would be at 8:40 when the vehicle arrives at B. In addition, relief at both termini may not be equally attractive. For example, there may not be convenient transportation and other service at one of the termini. Therefore, we may restrict reliefs to one terminus, which further reduces the number of possible partitions of the block.

Even with these policies, a single block may still have 10 to 30 different possible pieces of work. A transit agency with several hundred vehicles operating from a single depot may have thousands of possible pieces of work, and the number of possible runs resulting from combining different pieces may be millions.

### 3.3 Generating Legal Runs

A crew duty, or a run, can usually have up to two pieces of work, depending on the work rules and policies. Obviously, the starting time of the second piece of work can be no earlier than the ending time of the first piece of work. In addition, the maximum work time and maximum spread also constrain the possible combination of pieces.

Figure 3 shows two vehicle blocks B1 and B2 with possible relief points indicated by times. Pieces P1, P2 and P3 result from one possible partition of block B1. Pieces P4, P5 and P6 are from one partition of block B2.

P1 can be combined with P2 to form a *straight run* with 7.5 hours work time. A straight has no break between two pieces of work. Alternatively, P1 could be combined with P5 to form a *split run* with 7.5 hours work time and 9 hours spread time. A split run has a break between two pieces of work. Depending on the maximum allowable work time and spread time, P1 could potentially be combined with P3 to form a split run with 8’40” and 12’20” spread time, or with P6 to form a split run with 7’15”work time and 12’45” spread time. Even if legal, such runs are usually more expensive. Obviously, P1 cannot be combined with P4 because they overlap.

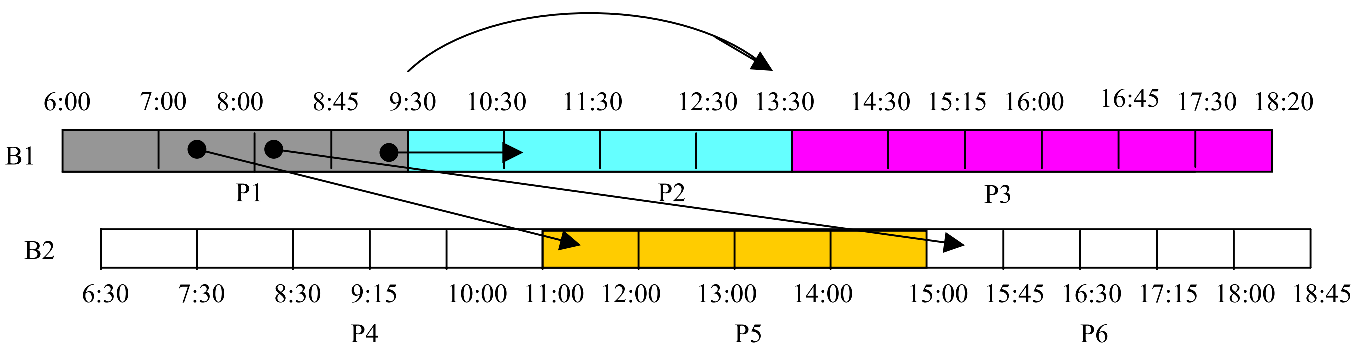


**Figure 3 Combining Pieces of Work to Form Runs**

The set of legal runs is huge even with a small number of vehicle blocks because of the large number of possible pieces and combinations of pieces. For example, several hundred vehicle blocks could result in millions of possible runs.

If it is reasonable to generate all possible runs, the following optimization model, known as set partition model, could be used to select the runs to meet the vehicle schedule requirement with minimum cost.

### 3.3 Mathematical Model

Notation:

| | | |
|---|---|---|
| $P$ | = | The set of trips to be covered |
| $R$ | = | The set of all feasible runs |
| $c_j$ | = | The cost of run $j \in R$ |
| $x_j$ | = | 1 if run $j \in R$ is selected, and 0 otherwise |
| $\delta_i^j$ | = | 1 if trip $i$ is included in run $j \in R$, and 0 otherwise |

$$\text{Min} \quad \sum_{j \in R} c_j x_j \tag{1}$$

Subject to:

$$\sum_{j \in R} x_j \delta_i^j = 1, \ \forall \ i \in P. \tag{2}$$

$$x_j \in \{0, 1\}, \ \forall \ j \in R.$$

Constraint (2) ensures that each trip must be covered by exactly one run that contains the trip. Constraint (2) can also be expressed in matrix form.

$$\boldsymbol{Ax = 1.} \tag{3}$$

Each row (or constraint) of $A$ corresponds to a trip. An element $a_{ij}$ of the matrix $A,$ like $\delta_i^j$, is 1 if trip $i \in P$ is contained in run $j \in R$, 0 otherwise.

Obviously in this problem formulation there is one constraint for each trip which may result in a large number of constraints. For example, for a single route with 6-minute mean headways over a 20-hour service day there would be 400 one-way trips. Even if we restrict relief to one terminal, there would be still 200 round-trips. For a depot serving 50 such routes, there would be about 10,000 such round-trip constraints. In addition, there will be millions of possible runs in this example.

One strategy to reduce the problem size is to form units of multiple trips on a vehicle block that would not be split apart by any partition. For example, a small vehicle block that cannot be partitioned according to the policies must always be covered as a unit, and therefore, we can replace the set of single (round)-trip constraints with a single constraint at the block level. For large blocks that must be split, this technique is more involved. For example, Figure 4 shows a single vehicle block with each small rectangular representing a single trip. Suppose the block has only three possible partitions (as shown) $PP_1$, $PP_2$ and $PP_3$. The resulting unique possible pieces are P1 to P7. Obviously, P1 and P4 both cover the first five trips albeit in two different pieces. To reduce the problem size, we may treat the first five trips as a "compound trip" T1 that must always be covered by a single run. There are no reliefs on T1 within any possible partition. Therefore, all trips included in T1 will be covered as a whole by a piece of work, and hence a run. Similarly, we can define compound trips T2 through T5. Instead of 18 individual trips to be covered, we now have only 5 compound trips.

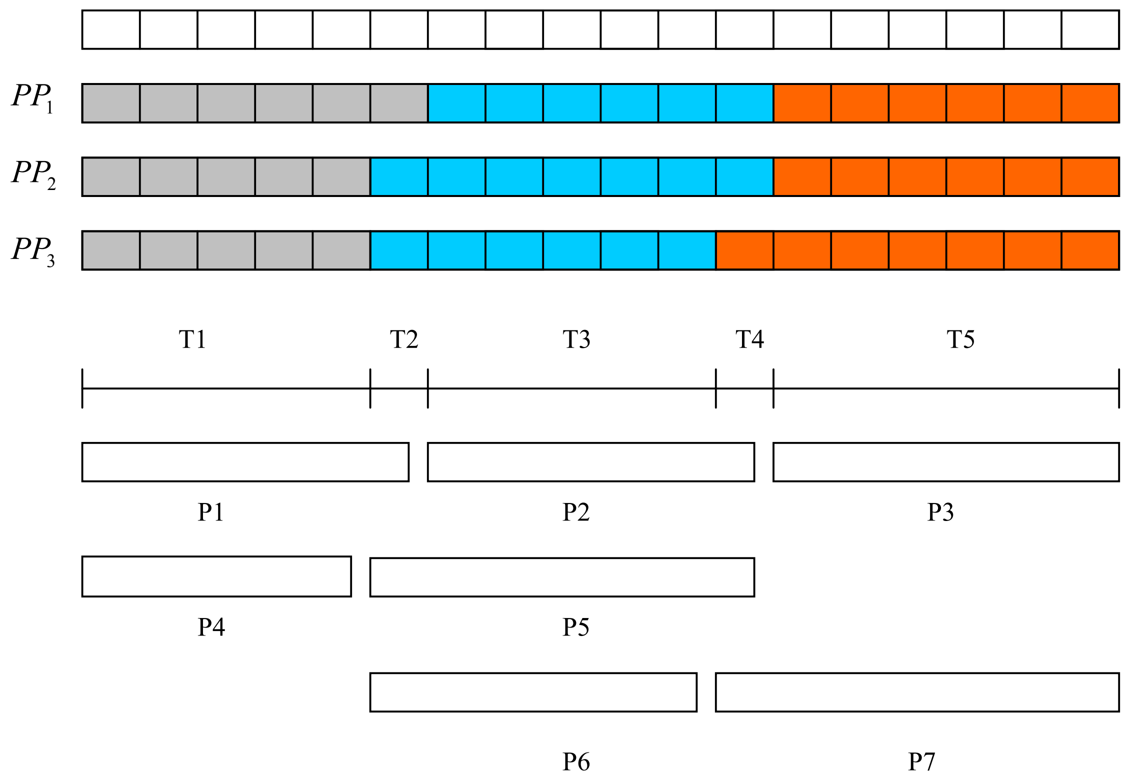


**Figure 4 Partitions of Vehicle Block, Pieces of Work and Compound Trip**

It should be noted that there may be no feasible solution if there are tight constraints on the formulation. For example, if each run must have exactly two pieces of work, as a trivial example, there is no feasible crew solution to the vehicle block in Figure 4.

There are several ways of dealing with these difficulties. First, runs could be generated consisting of a single piece of work combined with a "cover" piece in which the crew is paid to be available, but without any designated assignment. This crew could be used to provide extra service or fill in for absent operators. In practice this would mean that each piece might have to be costed as the only piece of work in a run with crew pay based on the guaranteed day (usually 8 hours).

Alternatively some transit agencies may be allowed to use *trippers* which are simply single pieces of work not built into full duties. Trippers are typically either worked by regular crews on over time, or cover operators, or by part-time operators. There is usually an upper limit on the hours a tripper may work and a limit on the total number of trippers or tripper hours a transit agency may use.

Thus, there are other constraints in addition to the basic constraints in the above formulation. For example, suppose there is a policy or requirement that the total tripper work time should be no more than 25% of the total *timetable time*. *Timetable time* is the sum of all *vehicle block time. Vehicle block time* is the time span from the start of the block to the end of the block. Let $WT$ denote the total timetable work time, $R^T$ denote the set of tripper runs, $t_j$ denote the work time for tripper run $j \in R^T$, The additional constraint is

$$\sum_{j \in R^T} t_j x_j \leq 0.25\ WT. \qquad (5)$$

If the number of possible runs is very large, it will not be practical to solve the model directly. It could be quite complex even to generate all feasible runs. Under these circumstances, there are usually three possible approaches. The first two approaches include heuristic methods, which simplify the problem but are unlikely to provide a global optimal solution. The last approach is a more advanced approach called column generation.

The first approach has two stages. In the first stage, vehicle blocks are cut following rules of thumb and one partition is formed for each block. The resulting pieces are only a small set of all possible pieces, which exactly cover all the trips. For example, in Figure 3, P1, P2 and P3 could be the partition for B1, and P4, P5 and P6 the partition for B2. They exactly cover B1 and B2. Therefore, if each piece of work is covered by a crew, each trip in the two blocks is covered. Thus, in the second stage, pieces of work are combined to form runs. If the number of pieces in a run is restrict to two, a matching model can be used to obtain the optimal solution for the second stage. Since only one partition for each block is considered, the solution is very unlikely to be a global optimal. Furthermore, there might not even be a feasible crew schedule from the selected pieces, depending on the work rules.

Suppose tripper runs are not allowed and that a legal run must have two pieces of work, a matching model can be applied as described below. Let a node represent a piece of work. If two pieces of work can be combined to form a legal run, an arc is generated between the corresponding two nodes. The cost associated with the arc is simply the cost of the run. For example, suppose the maximum work time and spread are 8.5 and 13 hours, respectively. The resulting network for partitions {P1, P2, P3} and {P4, P5, P6} in Figure 3 are shown in Figure 5.

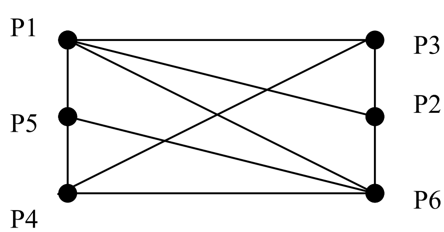


**Figure 5 Network Representation of the Matching Problem**

In Figure 5, each run must have two pieces of work. If trippers are allowed, which means a run can have only one piece of work, network transformation is required to apply the matching model. Such an example is given in Figure 6. Suppose P1, P2, P3 and P4 in Figure 6 (a) can either be matched or left as tripper runs. Correspondingly, the optimal solution may choose to match these nodes or leave them unmatched. In (b), we create a mirror node for each of the nodes that can be left as tripper runs. The cost of the arc connecting the mirror node and the original node is the cost to leave the original node unmatched. The arcs connecting the mirror nodes replicate the original arcs but with 0 cost. The optimal matching for (b) corresponds to an optimal solution to (a). (The proof is included in Appendix 2.)

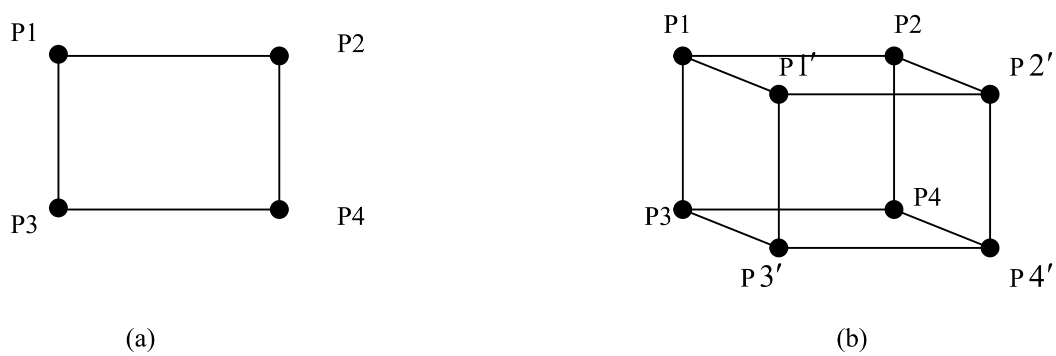


**Figure 6   Network Representation when Tripper Runs are Allowed**

Notation:

| | | |
|---|---|---|
| $A$ | = | The set of arcs in the network |
| $N$ | = | The set of nodes in the network |
| $(i, j)$ | = | An arc between node $i$ and node $j$ |
| $A(i)$ | = | The set of arcs incident at node $i$ |
| $c_{ij}$ | = | The cost of arc $ij$ |
| $x_{ij}$ | = | 1 if arc $ij$ is selected in the matching, and 0 otherwise |

$$\text{Min} \quad \sum_{(i,j)\in A} c_{ij} x_{ij} \tag{6}$$

Subject to:

$$\sum_{(i,j)\in A(i)} x_{ij} = 1, \ \forall \ i \in N. \tag{7}$$

$$x_{ij} \in \{0, 1\}, \ \forall \ ij \in A.$$

The number of pieces resulting from the first stage is relatively small compared with the set of all possible pieces. In addition, the matching problem is one of the integer problems for which efficient solution algorithms exist. Results from the second stage can be fed back to the first stage to change the partition heuristically, and the matching can potentially be improved. Since the simplified problem can be solved in reasonable time, the process can be repeated many times to improve the solution. However, the process may not lead to drastic changes of the partition. A significantly different partition that could potentially improve the solution might not be considered.

The second approach is also a heuristic. It selects a subset of partitions following rules of thumb. Then the resulting pieces of work are also combined to form runs, which represent a subset of all possible runs. Then, these runs are entered into the set partition model introduced earlier to select runs to be assigned to crews.

The third and more advanced approach is column generation. It maintains a small set of possible runs and looks for the optimal crew assignment solution over this set (often called solving a master problem). With the information obtained from solving the master problem, it solves a sub-problem and finds the possible run(s) that could further reduce total costs. It then adds the(se) run(s) to the initial set and re-solves the master problem. The master problem and sub-problem are solved iteratively until no possible run can be identified in the sub-problem to further reduce the cost. At that point, a global optimal solution is obtained.

## 4. Tasks

In this case study, you are provided with the vehicle blocks for the Mattapan High Speed Line of the MBTA (Massachusetts Bay Transportation Authority) along with possible relief points in an Excel file. The following is an example of a vehicle block.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| B3 | 6:40 | 6:45 | 7:15 | 7:45 | 8:15 | 8:45 | 9:13 |
| | | 6:57 | 7:27 | 7:57 | 8:27 | 8:57 | |

The first time shown for a vehicle block is the pull-out time from the depot, and the last time is the lay-up time at the depot. Reliefs are only allowed at one terminus. Therefore, a round trip is the smallest scheduling unit. The upper time shown in each column is the departure time at the relief terminal. You may assume that any cut occurs 5 minutes before the corresponding terminal departure time. For example, if a cut is made before the trip departing at 8:15, the duty time of the crew being relieved ends at 8:10, and the work time of the relieving crew begins at 8:10 as well. The work rules are attached in Appendix 1. The wage rate is $20 for both full-time operators and trippers. Work rules must be complied with fully.

In addition to the given work rules, there are the following policies.

P1. Reliefs should not occur between 7-9 AM and 4-6PM.
P2. No piece of work can be smaller than 3 hours (unless it is associated with a short block smaller than 3 hours) or larger than 5 hours.

A. For this part assume that no trippers runs are allowed.

(1) In the Excel file provided, you have been given all possible partitions complying with the work rules and policies for all vehicle blocks. All legal runs are provided on sheet "Form 8". Calculate the work time, spread time, and cost for each run in Excel.

(2) The trip-run matrix is also provided in sheet "Form 8". Implement the set partition model in XPRESS-MP with the provided runs, trips and trip-run matrix.

(3) Solve the problem as an IP with only these full-timer runs.

B. We now permit some tripper runs.

(4) The unique pieces from the block cuts are provided in sheet "Pieces". Generate all tripper runs – a tripper run is simply a single piece of work. Calculate the cost for each tripper run and augment the trip-run matrix to include tripper runs.

(5) Suppose the transit agency now is allowed to use trippers, but the number of tripper runs cannot exceed 50% of the number of full-timer runs. Write the corresponding constraint. Implement the model with tripper runs under this constraint. Solve the IP and comment on the difference between this solution and the one to (3).

(6) Consider a different form of tripper constraint which specifies that total tripper work time to be no more than 25% of the total timetable time. Write this constraint and solve the corresponding IP. Is it a better choice from the agency's perspective than the constraint given in (5) above?

(7) Can the transit agency improve the solution if it is allowed to use more tripper work time? Provide some quantitative evidence without resolving the problem.

(8) Suppose the transit agency now is able to use tripper work time up to 40% of the total timetable time. With the results obtained solving the LP in (6), can you estimate the change of the optimal cost with this new tripper constraint without re-solving the problem? Solve the IP with the constraint that tripper work time can be up to 40% of the total timetable time. Comment on the difference between the optimal objective function value and your prior estimate.

C. Alternative work rules:

(9) Suppose the hourly wage rate for tripper work is raised to $30, but there is no limit on total tripper work time. At the same time, the maximum allowable work time for full-time operator is relaxed to 8.5 hours. The legal runs (both full-time runs and tripper runs) under the 8.5-hour maximum work time and the trip-run matrix are provided on sheet "Form 8.5". Cost the runs, implement the model and solve the IP problem. Comment on the impact of this change.

(10) Suppose instead of using trippers, the transit agency has another option to use operators working up to 10 hours a day but just 4 days per week. The work rules for 10-hour operators are also included in Appendix 1. In addition, policy P2 for 10-hour runs is that a piece of work should be no smaller than 4 hours and no greater than 6 hours. Remember: NO trippers are allowed.

(i) On Sheet "10-Hour", you are provided with all possible additional partitions for all vehicle blocks except block B5, for which some additional partitions may exist. Possible runs for 8-hour and 10-hour operators associated with the pieces from the provided partitions are on Sheet "Form 10". Enumerate any possible additional partitions for block B5 and generate all possible runs associated with the pieces from the additional partitions for both 8-hour and 10-hour operators. Please mark your partition(s) and runs generated.

(ii) Generate the trip-run matrix, and implement the set partition model and solve the IP to get the crew schedule for a single day. Is any 10-hour run used?

(iii) Evaluate this option. Are there any additional difficulties in going to 10-hour day 4-day week runs?

## 5. Glossary

*Route:* A sequence of stops served by a single vehicle.

*Trip:* A vehicle traversal of a route that is defined by both starting and ending times and locations. A trip is the basic unit of service in the sense that a trip must be operated by a single vehicle.

*Vehicle block:* A feasible sequence of trips starting and ending at any depot.

*Relief point:* A time and place along a vehicle block where a crew can leave or take over the vehicle.

*A Piece of work:* A continuous crew working period on a single vehicle between two relief points on a block.

*Partition:* The set of pieces of work formed by making cuts at a subset of relief points on the vehicle block.

*Compound Trip*: A set of consecutive trips on a vehicle block with no possible cut between these trips for all possible partitions of the vehicle block.

*Run:* A feasible combination of pieces of work for a crew.

*Work time:* The total service time a crew spends on a vehicle.

*Vehicle block time*: The time span from the start of the block to the end of the block.

*Timetable work time*: The sum of all vehicle block time.

*Spread time:* The time from the start of the first piece of work to the end of the last piece of work.

*Straight Run:* A run with no break between two pieces of work.

*Split Run*: A run consisting of two pieces of work separated by an unpaid break.

## Reference


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## Appendix 1 Work Rules

A. For an 8-hour 5-day operator

(1) All full-time drivers are guaranteed 8 hours pay.

(2) Work hours can be no more than 8 hours 15 minutes and any time over 8 hours is paid at an overtime premium of 50% of the wage rate.

(3) Spread penalties: If a driver's run requires clocking off at the end of the day more than a specific number of hours after clocking on at the start, a bonus known as a spread penalty is paid. This results in the driver being paid 1.5 times the basic wage rate for time worked in the $11^{th}$ hour after the run begins and 2 times the basic wage rate for time worked in the $12^{th}$ and $13^{th}$ hours. No run can have spread time more than 13 hours.

(4) No run may have more than two pieces of work.

B. For a 10-hour 4-day operator:

(1) Operators are guaranteed 10-hour pay for 4 days.

(2) Work hours can be no more than 10 hours 15 minutes and any time over 10 hours is paid at an overtime premium of 50% of the wage rate.

(3) Spread penalties: the operator is paid 1.5 times the basic wage rate for time worked in the $13^{th}$ hour after the run begins and 2 times the basic wage rate for time worked in the $14^{th}$ and $15^{th}$ hours. No run can have spread time more than 15 hours.

(4) No run may have more than two pieces of work.

C. For trippers:

(1) Trippers can work up to 5 hours per day in a single piece of work.

## Appendix 2

**Network Transformation for Matching Problem that Allows Nodes to be Unmatched**

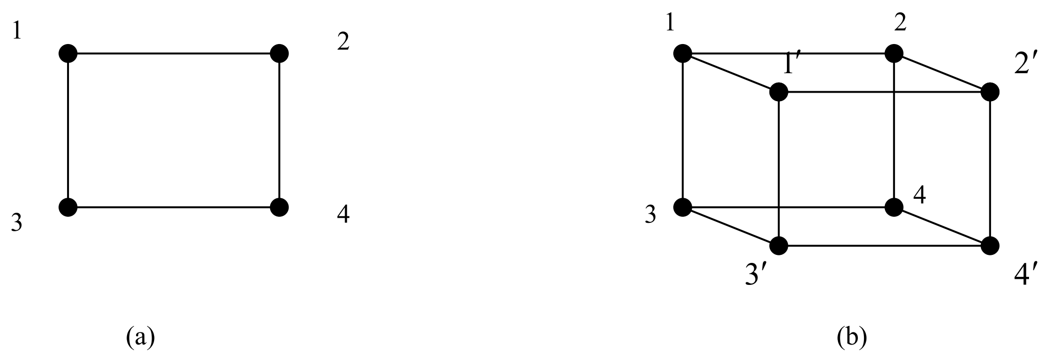


**Figure 7 Network Transformation if Nodes Can be Left Unmatched**

Given a graph $G = (N, A)$, if node $i$ can be matched with node $j$, an arc $(i, j)$ exists and has a matching cost $c_{ij}$. Suppose among the n nodes, a set of nodes $N_1 \subseteq N$ can be left unmatched with associated cost. The original problem $P$ is to find a set of arcs that match (some of) the nodes and leaves a sub-set of $N_1$(the sub-set could be empty) unmatched with the minimum cost. The cost function is the total matching cost plus the total unmatching cost for nodes that are not matched.

Network Transformation:

For each node $i \in N$, create a "mirror" node $i'$. Define this set of new nodes as $N'$. If $i \in N_1$, create an arc $(i, i')$. The cost of arc $(i, i')$ is the cost to leave node $i$ unmatched, and the set of such new arcs as $A''$. For each arc $(i, j) \in A$, create an arc $(i', j')$ with zero cost. Define this set of arcs as $A'$.

The minimum perfect matching problem is to find a set of arcs on a network that match all nodes with minimum cost. Define problem $P'$ as the minimum perfect matching problem on graph $G' = (N \cup N', A \cup A' \cup A'')$. We would like to establish the equivalency of $P$ and $P'$.

Proof:

We would like to show that for any feasible solution to $P$ with a cost, there is a perfect matching on $G'$ with the same cost, and vice versa. Therefore, an solution with minimum cost to $P$ corresponds to a minimum perfect matching to $P'$, and vice versa.

Assume the unmatching cost for each node $i \in N_1$ is $C_i$. Suppose $S$ is the feasible solution to problem $P$. Suppose a subset of nodes $N_2 \subseteq N_1$ are unmatched, the set nodes $N \setminus N_2$ are matched, and the set of arcs $\overline{A} \subseteq A$ are used in the matching. The cost of $S$ the matching cost for nodes in the set $N \setminus N_2$ plus for the unmatching cost for nodes in $N_2$, which is

$$\sum_{(i,j)\in \overline{A}} c_{ij} + \sum_{i \in N_2} C_i$$

We now compose a perfect matching in $G'$ according to $S$. For nodes in the set $N \setminus N_2$, we can use the same set of arcs $\overline{A}$ to match them. Define the set of arcs $\overline{A}' = \{(i', j'):(i, j) \in \overline{A}\}$. Obviously, arcs in $\overline{A}'$ provide a perfect matching for nodes in the set $\{i' : i \in N \setminus N_2\}$. For nodes in the set $\{i : i \in N_2\} \cup \{i' : i' \in N_2'\}$ we can easily match them with arc $(i, i')$. Therefore, we've established a perfect matching on $G'$ according to $S$. The cost is

$$\sum_{(i,j)\in \overline{A}} c_{ij} + \sum_{(i',j')\in \overline{A}} c_{i'j'} + \sum_{i \in N_2} c_{ii'} = \sum_{(i,j)\in \overline{A}} c_{ij} + \sum_{i \in N_2} C_i ,$$

since $c_{i'j'} = 0$ for $(i', j') \in A'$, and $c_{ii'} = C_i$.

Conversely, suppose $S'$ is a perfect matching on $G'$, we can show that we can find a feasible solution to $P$ with the same cost correspondingly. Obviously, on graph $G'$, arc $(i, j) \in A$ can only match nodes $i \in N$, arc $(i', j') \in A'$ can only match nodes $i' \in N'$, arc $(i, i') \in A''$ can only match nodes $i \in N_1$ with nodes $i' \in N_1'$. Suppose $\overline{A}' \subseteq A''$ is the set of arcs used in the perfect matching, we can leave the corresponding nodes $i \in N_1$ that are matched by arcs in $\overline{A}'$ unmatched in $G$. For the rest of the nodes in $N$, we can match them as the perfect matching on $G'$ with arcs in $A$. Thus, we've established a feasible solution to $P$. Obviously, the cost of the feasible solution is the same as the cost of the perfect matching.

Therefore, the equivalency of the two problems has been established.

## Solutions

(3) The LP obj. value is \$1534.77. The IP obj. value is \$1556.00. 9 operators are required.

(6) The LP obj value is \$1433.35. The IP obj. value is \$1452.83. 8 full-timers and 3 trippers are required.

(7) The LP obj. value is \$1420.00. The IP obj. value is \$1426.50. 7 full-timers and 5 trippers are required

(8) The shadow price for the tripper constraint in (7) is -\$1.76. If tripper time is allowed up to 40% of the 70.42 hours timetable time, the estimate of the new IP obj value is 1407.91. The true IP obj. value is \$1416.83.

(9) The LP obj. value is \$1512.88. The IP obj. value is \$1533.50. There are 8 full-timers and 3 trippers.

(10)
(ii) The LP obj. value is \$1496.5. The IP obj. value is \$1496.5. 8 operators are required. There are four 10-hour operators.