in (a) and (17–131):

$$\Pi_p = V + (\bar{\mathbf{P}}_{I,1}^T - \bar{\mathbf{P}}_1^T)(\mathbf{BU} + \mathbf{H}_3)$$
$$V = \tfrac{1}{2}(\mathscr{V} - \mathscr{V}_o)^T\mathbf{k}'(\mathscr{V} - \mathscr{V}_o) \qquad (17\text{–}132)$$
$$\mathscr{V} = \mathbf{A}_1\mathbf{BU} + \mathbf{A}_1\mathbf{H}_3 + \mathbf{A}_2\bar{\mathbf{U}}_2$$

The variation of Π_p considering $\mathbf{U}$ as the independent variable is

$$\begin{aligned} d\Pi_p &= \Delta\mathbf{U}^T[\mathbf{B}^T(\bar{\mathbf{P}}_{I,1} - \bar{\mathbf{P}}_1) + (\mathbf{B}^T\mathbf{A}_1^T\mathbf{k}'\mathbf{A}_1\mathbf{B})\mathbf{U} \\ &\quad + \mathbf{B}^T\mathbf{A}_1^T\mathbf{k}'(\mathbf{A}_1\mathbf{H}_3 + \mathbf{A}_2\bar{\mathbf{U}}_2 - \mathscr{V}_o)] \\ &= \Delta\mathbf{U}^T[(\mathbf{B}^T\mathbf{K}_{11}\mathbf{B})\mathbf{U} - \mathbf{B}^T\mathbf{H}_4] \end{aligned} \qquad (g)$$

Requiring Π_p to be stationary for arbitrary $\Delta\mathbf{U}$ results in (17–126). Note that we could have used the reduced form for V, i.e., equation (d). Also, we still have to determine the constraint forces.

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18 Analysis of Geometrically Nonlinear Systems

18–1. INTRODUCTION

In this chapter, we extend the displacement formulation to include geometric nonlinearity. The derivation is restricted to *small rotation*, i.e., where squares of rotations are negligible with respect to unity. We also consider the material to be linearly elastic and the member to be *prismatic*.

The first phase involves developing appropriate member force-displacement relations by integrating the governing equations derived in Sec. 13–9. We treat first planar deformation, since the equations for this case are easily integrated and it reveals the essential *nonlinear* effects. The three-dimensional problem is more formidable and one has to introduce numerous approximations in order to generate an explicit solution. We will briefly sketch out the solution strategy and then present a *linearized* solution applicable for doubly symmetric cross-sections.

The direct stiffness method is employed to assemble the system equations. This phase is essentially the same as for the linear case. However, the governing equations are now nonlinear.

Next, we described two iterative procedures for solving a set of nonlinear algebraic equations, successive substitution and Newton-Raphson iteration. These methods are applied to the system equations and the appropriate recurrence relations are developed. Finally, we utilize the *classical* stability criterion to investigate the stability of an equilibrium position.

18–2. MEMBER EQUATIONS—PLANAR DEFORMATION

Figure 18–1 shows the initial and deformed positions of the member. The centroidal axis initially coincides with the X_1 direction and X_2 is an axis of symmetry for the cross section. We work with displacements (u_1, u_2, ω_3), distributed external force (b_2), and end forces ($\bar{F}_1, \bar{F}_2, \bar{M}_3$) referred to the *initial* (X_1-X_2-X_3) member frame. The rotation of the chord is denoted by ρ_3 and is related to the end displacements by

$$\rho_3 = \frac{u_{B2} - u_{A2}}{L} \tag{18-1}$$

The governing equations follow from (13–88). For convenience, we drop the subscript on x_1, and M_3, ω_3, I_3. Also, we consider $b_1 = m_3 = 0$.

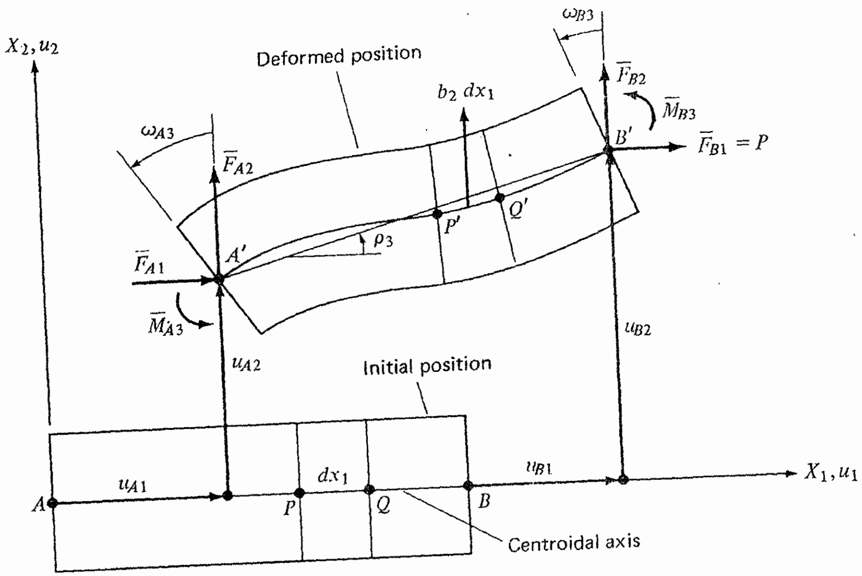


Fig. 18–1. Notation for planar bending.

Equilibrium Equations

$$\begin{aligned} F_{1,x} &= 0 \\ \frac{d}{dx}(F_1 u_{2,x} + F_2) + b_2 &= 0 \\ F_2 &= -M_{,x} \end{aligned} \tag{a}$$

Force-Displacement Relations

$$\begin{aligned} \frac{F_1}{AE} &= u_{1,x} + \tfrac{1}{2}(u_{2,x})^2 \\ \frac{F_2}{GA_2} &= u_{2,x} - \omega \\ \frac{M}{EI} &= \omega_{,x} \end{aligned} \tag{b}$$

Boundary Conditions

For $x = 0$:

$$\begin{aligned} u_1 &= \bar{u}_{A1} \quad \text{or} \quad \left|F_1\right|_0 = -\bar{F}_{A1} \\ u_2 &= \bar{u}_{A2} \quad \text{or} \quad \left|F_2 + F_1 u_{2,x}\right|_0 = -\bar{F}_{A2} \\ \omega &= \bar{\omega}_{A3} \quad \text{or} \quad \left|M\right|_0 = -\bar{M}_{A3} \end{aligned} \tag{c}$$

For $x = L$:

$$\begin{aligned} u_1 &= \bar{u}_{B1} \quad \text{or} \quad \left|F_1\right|_L = +\bar{F}_{B1} \\ u_2 &= \bar{u}_{B2} \quad \text{or} \quad \left|F_2 + F_1 u_{2,x}\right|_L = \bar{F}_{B2} \\ \omega &= \bar{\omega}_{B3} \quad \text{or} \quad \left|M\right|_L = +\bar{M}_{B3} \end{aligned} \tag{d}$$

Integrating (a) leads to

$$\begin{aligned} F_1 &= \bar{F}_{B1} \equiv P \\ F_2 + Pu_{2,x} &= PC_2 - \textstyle\int_x b_2\,dx \\ M_3 - Pu_2 &= -C_3 P - C_2 P x + \textstyle\int_x(\int_x b_2\,dx)dx \end{aligned} \tag{e}$$

where C_2, C_3 are integration constants. We include the factor P so that the dimensions are consistent. The axial displacement u_1 is determined from the first equation in (a),

$$u_{B1} - u_{A1} = \frac{PL}{AE} - \frac{1}{2}\int_0^L (u_{2,x})^2\,dx \tag{18-2}$$

Combining the remaining two equations in (a), we obtain

$$M = EI\left(1 + \frac{P}{GA_2}\right)u_{2,xx} + \frac{EI}{GA_2}b_2 \tag{f}$$

Finally, the governing equation for u_2 follows from the third equation in (e),

$$u_{2,xx} + \mu^2 u_2 = \mu^2(C_2 x + C_3) + \frac{\mu^2}{P}\left\{\frac{EI}{GA_2}b_2 - \int_x\left(\int_x b_2\,dx\right)dx\right\}$$

where

$$\mu^2 = \frac{-P}{EI\left(1 + \dfrac{P}{GA_2}\right)} \tag{18-3}$$

The solutions for u_2 and M are

$$\begin{aligned} u_2 &= C_4 \cos\mu x + C_5 \sin\mu x + C_2 x + C_3 + u_{2b} \\ \omega &= \mu\left(1 + \frac{P}{GA_2}\right)(-C_4 \sin\mu x + C_5\cos\mu x) \\ &\quad + C_2 + \frac{1}{GA_2}\int_x b_2\,dx + \left(1 + \frac{P}{GA_2}\right)u_{2b,x} \end{aligned} \tag{18-4}$$

where u_{2b} denotes the particular solution due to b_2. If b_2 is constant,

$$u_{2b} = \frac{b}{P}\left\{\frac{EI}{GA_2} - \frac{1}{2}\left(x^2 - \frac{2}{\mu^2}\right)\right\} \tag{18-5}$$

Enforcing the boundary conditions on u_2, ω at $x = 0, L$ leads to four linear equations relating $(C_2 \cdots C_5)$. When the coefficient matrix is singular, the member is said to have *buckled*. In what follows, we exclude member buckling. We also neglect transverse shear deformation since its effect is small for a homogeneous cross section.

We consider the case where the end displacements are prescribed. The *net* displacements are

$$\begin{aligned} u_{\text{net}} &= u' = (\bar{u} - u_{2b})_{x=0,L} \\ \omega_{\text{net}} &= \omega' = (\bar{\omega} - u_{2b,x})_{x=0,L} \end{aligned} \tag{18-6}$$

Evaluating (18–4) with $A_2 = \infty$, we obtain

$$\begin{aligned} C_2 &= \omega'_A - \mu C_5 \\ C_3 &= u'_A - C_4 \\ C_4 &= -C_5 \frac{1 - \cos \mu L}{\sin \mu L} - \frac{\omega'_B - \omega'_A}{\mu \sin \mu L} \\ C_5 &= \frac{1}{D}\left\{(u'_B - u'_A - \omega'_A L)\sin \mu L - \frac{1 - \cos \mu L}{\mu}(\omega'_B - \omega'_A)\right\} \\ D &= 2(1 - \cos \mu L) - \mu L \sin \mu L \end{aligned} \tag{18-7}$$

Note that $D \to 0$ as $\mu L \to 2\pi$. This defines the upper limit on P, i.e., the member buckling load:

$$-P\big|_{\max} = \frac{4\pi^2 EI}{L^2} \tag{18-8}$$

The end forces can be obtained with (*c–e*). We omit the algebraic details since they are obvious and list the final form below.

$$\begin{aligned} \bar{M}_{A3} &= M^i_{A3} + \frac{EI_3}{L}\left[\phi_1\omega_{A3} + \phi_2\omega_{B3} - \frac{\phi_3}{L}(u_{B2} - u_{A2})\right] \\ \bar{M}_{B3} &= M^i_{B3} + \frac{EI_3}{L}\left[\phi_2\omega_{A3} + \phi_1\omega_{B3} - \frac{\phi_3}{L}(u_{B2} - u_{A2})\right] \\ \bar{F}_{A2} &= F^i_{A2} + \frac{\phi_3 EI_3}{L^2}\left[\omega_{B3} + \omega_{A3} - \frac{2}{L}(u_{B2} - u_{A2})\right] - \frac{P}{L}(u_{B2} - u_{A2}) \\ \bar{F}_{B2} &= F^i_{B2} - \frac{\phi_3 EI_3}{L^2}\left[\omega_{B3} + \omega_{A3} - \frac{2}{L}(u_{B2} - u_{A2})\right] + \frac{P}{L}(u_{B2} - u_{A2}) \\ P &= \bar{F}_{B1} \qquad \bar{F}_{A1} = -P \\ u_{B1} - u_{A1} &= \frac{PL}{AE} - \int_0^L \frac{1}{2}(u_{2,x})^2\,dx = \frac{PL}{AE} - e_r L \end{aligned} \tag{18-9}$$

where

$$\begin{aligned} D &= 2(1 - \cos \mu L) - \mu L \sin \mu L \\ D\phi_1 &= \mu L(\sin \mu L - \mu L \cos \mu L) \\ D\phi_2 &= \mu L(\mu L - \sin \mu L) \\ D\phi_3 &= D(\phi_1 + \phi_2) = (\mu L)^2(1 - \cos \mu L) \end{aligned}$$

The ϕ_i functions were introduced by Livesley (Ref. 7), and are plotted in Fig. 18–2. They degenerate rapidly as $\mu L \to 2\pi$. The initial end forces depend on the transverse loading, b_2. If b_2 is constant,

$$\begin{aligned} \bar{F}^i_{A2} &= \bar{F}^i_{B2} = -\frac{bL}{2} \\ \bar{M}^i_{B3} &= \frac{bL^2}{(\mu L)^2}\left(1 - \tfrac{1}{2}(\phi_1 - \phi_2)\right) \\ M^i_{A3} &= -\bar{M}^i_{B3} \end{aligned} \tag{18-10}$$

In order to evaluate the stiffness coefficients, P has to be known. If one end, say B, is *unrestrained* with respect to axial displacement, there is no difficulty since $\bar{F}_{B1}$ is now prescribed. The relative displacement is determined from

$$\left.\begin{aligned} P &= \bar{F}_{B1} \\ u_{B1} &= u_{A1} + \frac{PL}{AE} - Le_r \\ e_r &= \frac{1}{2L}\int_0^L (u_{2,x})^2\,dx = e_r(\mu L, u_{A2}, u_{B2}, \omega_A, \omega_B) \\ 2e_r &= \phi_5(C'_5)^2 + \left[4\left(\frac{u_{B2} - u_{A2}}{L}\right) + \phi_6(\omega_{B3} - \omega_{A3})\right]C'_5 \\ &\quad + \phi_7(\omega_{B3} - \omega_{A3})^2 + \left(\frac{u_{B2} - u_{A2}}{L}\right)^2 \\ D\phi_4 &= \mu L \sin \mu L \\ C'_5 &= -\frac{\phi_3}{(\mu L)^2}(\omega_{B3} - \omega_{A3}) + \phi_4\left(\frac{u_{B2} - u_{A2}}{L} - \omega_{A3}\right) \\ \phi_7 &= \frac{1}{(\mu L \sin \mu L)^2}\left\{\mu L(\mu L - \sin \mu L \cos \mu L) + 2(1 - \cos \mu L)^2\right\} \\ \phi_6 &= (1 - \cos \mu L)\left\{\phi_7 - \frac{2}{(\mu L)^2}\right\} \\ \phi_5 &= \left(\frac{1 - \cos \mu L}{2}\right)\left\{\phi_6 + \frac{4}{(\mu L)^2}\right\} + \frac{1}{2}\left\{9 - \frac{\sin \mu L \cos \mu L}{\mu L}\right\} \end{aligned}\right\} \tag{18-11}$$

We call e_r the relative end shortening due to *rotation*. However, when both axial displacements are prescribed, we have to resort to iteration in order to evaluate P since e_r is a nonlinear function of P. The simplest iterative scheme is

$$P^{(i+1)} = \frac{AE}{L}(u_{B1} - u_{A1}) + AEe_r^{(i)} \tag{18-12}$$

and convergence is rapid when μL is not close to 2π.

Expressions for the incremental end forces due to increments in the end displacements are needed in the Newton-Raphson procedure and also for stability analysis. If μL is not close to 2π, we can assume the stability functions

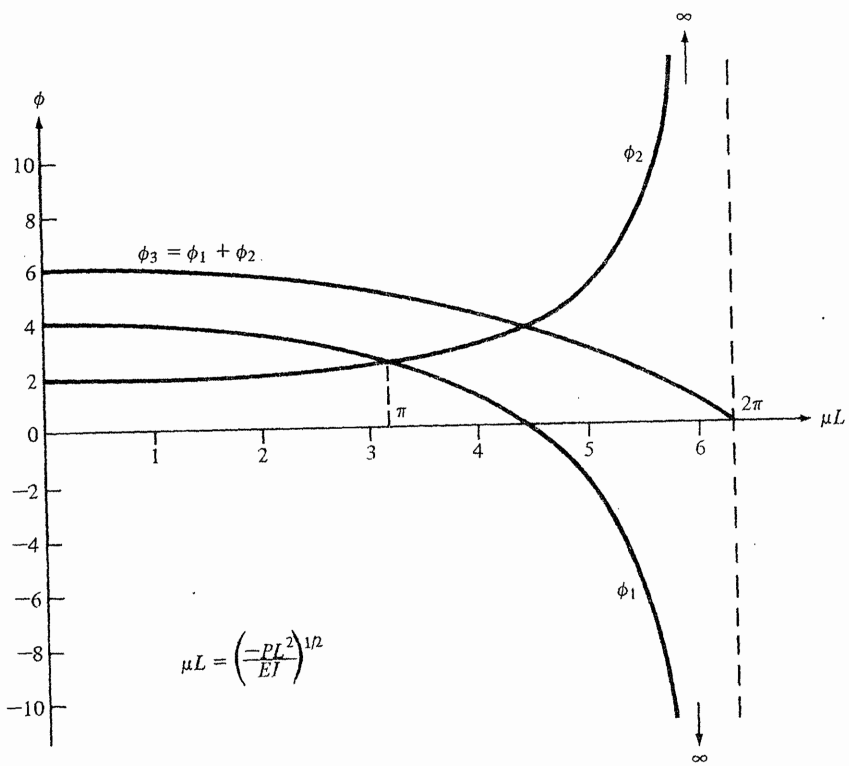


Fig. 18–2. Plot of the ϕ functions.

are constant and equal to their values at the initial position, when operating on (18–9). The resulting expressions are

$$
\begin{aligned}
d\bar{M}_{A3} &= dM^i_{A3} + \frac{EI_3}{L}\left[\phi_1\,\Delta\omega_{A3} + \phi_2\,\Delta\omega_{B3} - \frac{\phi_3}{L}(\Delta u_{B2} - \Delta u_{A2})\right] \\
d\bar{M}_{B3} &= dM^i_{B3} + \frac{EI_3}{L}\left[\phi_2\,\Delta\omega_{A3} + \phi_1\,\Delta\omega_{B3} - \frac{\phi_3}{L}(\Delta u_{B2} - \Delta u_{A2})\right] \\
d\bar{F}_{A2} &= dF^i_{A2} + \frac{\phi_3 EI_3}{L^2}\left[\Delta\omega_{B3} + \Delta\omega_{A3} - \frac{2}{L}(\Delta u_{B2} - \Delta u_{A2})\right] \\
&\quad - \frac{P}{L}(\Delta u_{B2} - \Delta u_{A2}) - \frac{u_{B2} - u_{A2}}{L}\,dP \\
d\bar{F}_{B2} &= dF^i_{B2} - (d\bar{F}_{A2} - dF^i_{A2}) \\
d\bar{F}_{B1} &= dP \qquad d\bar{F}_{A1} = -dP \\
dP &= \frac{AE}{L}(\Delta u_{B1} - \Delta u_{A1}) + AE\,de_r
\end{aligned}
\tag{18–13}
$$

where the incremental initial end forces are due to loading, Δb_2. We can obtain an estimate for de_r by assuming $\Delta u_{2,x}$ is constant.

$$
de_r = \frac{1}{L}\int_0^L u_{2,x}\,\Delta u_{2,x}\,dx \approx \left(\frac{u_{B2} - u_{A2}}{L}\right)\left(\frac{\Delta u_{B2} - \Delta u_{A2}}{L}\right) \tag{18–14}
$$

The coefficients in (18–13) are tangent stiffnesses. They are not exact since we have assumed ϕ_i and $\Delta u_{2,x}$ constant. To obtain the *exact* coefficients, we have to add

$$
\begin{aligned}
&\frac{EI_3}{L}\left[\omega_A\phi_1' + \omega_B\phi_2' - \frac{u_{B2} - u_{A2}}{L}\phi_3'\right]d(\mu L) \\
&\phi_i' = \frac{d}{d(\mu L)}\phi_i \\
&d(\mu L) = -\frac{L}{2EI_3\,\mu}\,dP
\end{aligned}
\tag{18–15}
$$

to $d\bar{M}_A$ and similar terms to $d\bar{M}_B, \ldots, d\bar{F}_B$. The derivatives of the stability functions are listed below for reference:

$$
\begin{aligned}
\phi_3' &= \mu L + \frac{2(\mu L)^2 \sin \mu L}{D} - \frac{\phi_1\phi_3}{\mu L} \\
\phi_3'\big|_{2\pi} &= -2\pi \\
\phi_1' &= \frac{\phi_1}{\mu L}\{1 - \phi_1 + \tfrac{1}{2}\phi_3\} + \tfrac{1}{2}(\phi_3' - \mu L) \\
\phi_2' &= \phi_3' - \phi_1'
\end{aligned}
\tag{18–16}
$$

We also have to use the exact expression for de_r,

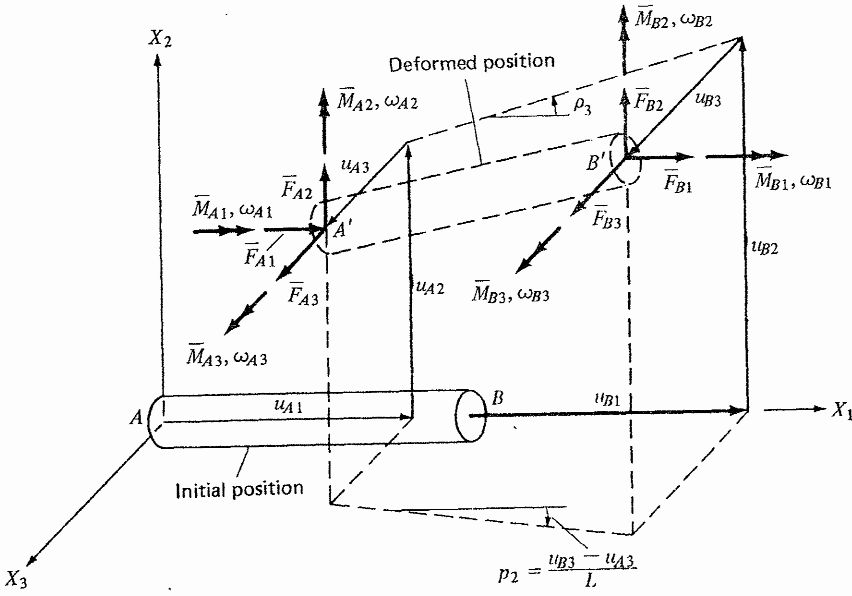
$$
de_r = \frac{\partial e_r}{\partial u_{A2}}\Delta u_{A2} + \frac{\partial e_r}{\partial u_{B2}}\Delta u_{B2} + \frac{\partial e_r}{\partial \omega_A}\Delta\omega_A + \frac{\partial e_r}{\partial \omega_B}\Delta\omega_B + \frac{\partial e_r}{\partial(\mu L)}\Delta(\mu L) \tag{18–17}
$$

in the equation for dP. An improvement on (18–14) is obtained by operating on (18–11), and assuming μL is constant.

18–3. MEMBER EQUATIONS—ARBITRARY DEFORMATION

The positive sense of the end forces for the three-dimensional case is shown in Fig. 18–3. Note that the force and displacement measures are referred to the fixed member frame. The governing equations for *small* rotations were derived in Sec. 13–9. They are nonlinear, and one must resort to an approximate method such as the Galerkin scheme,† in which the displacement measures are expressed in terms of prescribed functions (of x) and parameters. The problem is transformed into a set of nonlinear algebraic equations relating the parameters. Some applications of this technique are presented in Ref. 5.

† This method is outlined in Sec. 10–6.



Note: The centroidal axis coincides with X_1. X_2 and X_3 are principal inertia directions.

Fig. 18–3. Notation for three-dimensional behavior.

If we consider $b_1 = 0$, the axial force F_1 is constant along the member and the nonlinear terms involve ω_1 and coupling terms such as $F_2 u_{s3,1}$; $\omega_1 M_2$; etc. Neglecting these terms results in *linearized* equations, called the *Kappus equations.* Their form is:

Equilibrium Equations

$$
\begin{aligned}
F_1 &= P \\
\frac{d}{dx_1}[P(u_{s2,1} + \bar{x}_3\omega_{1,1}) + F_2] + b_2 &= 0 \\
\frac{d}{dx_1}[P(u_{s3,1} - \bar{x}_2\omega_{1,1}) + F_3] + b_3 &= 0 \\
M^u_{T,1} + M^r_{T,1} + m_T + \frac{d}{dx_1}[P(\bar{x}_3 u_{s2,1} - \bar{x}_2 u_{s3,1} + \bar{\beta}_1\omega_{1,1})] &= 0 \\
M_{2,1} - F_3 + m_2 &= 0 \\
M_{3,1} + F_2 + m_3 &= 0 \qquad (18\text{–}18) \\
M_{\phi,1} - M^r_T + m_\phi &= 0 \\
\bar{\beta}_1 = \frac{I_1}{A} + \bar{x}_2^2 + \bar{x}_3^2
\end{aligned}
$$

Force-Displacement Relations

$$
\begin{aligned}
\frac{P}{AE} &= u_{1,1} + \tfrac{1}{2}(u^2_{s2,1} + u^2_{s3,1} + \bar{\beta}_1\omega^2_{1,1}) + \omega_{1,1}(\bar{x}_3 u_{s2,1} - \bar{x}_2 u_{s3,1}) \\
u_{s2,1} - \omega_3 &= \frac{1}{G}\left(\frac{F_2}{A_2} + \frac{F_3}{A_{23}} + \frac{x_{3r}}{J}M^r_T\right) \\
u_{s3,1} + \omega_2 &= \frac{1}{G}\left(\frac{F_2}{A_{23}} + \frac{F_3}{A_3} + \frac{x_{2r}}{J}M^r_T\right) \\
\omega_{1,1} &= \frac{M^u_T}{GJ} \qquad f_{,1} = \frac{M_\phi}{E_r I_\phi} \\
\omega_{2,1} &= \frac{M_2}{EI_2} \qquad \omega_{3,1} = \frac{M_3}{EI_3} \\
f + \omega_{1,1} &= \frac{1}{GJ}(C_r M^r_T + x_{3r}F_2 + x_{2r}F_3)
\end{aligned}
$$

Boundary Conditions (+ for $x = L$, − for $x = 0$)

$$
\begin{aligned}
P &= \pm\bar{F}_1 \\
P(u_{s2,1} + \bar{x}_3\omega_{1,1}) + F_2 &= \pm\bar{F}_2 \\
P(u_{s3,1} - \bar{x}_2\omega_{1,1}) + F_3 &= \pm\bar{F}_3 \\
M^u_T + M^r_T + P(\bar{x}_3 u_{s2,1} - \bar{x}_2 u_{s3,1} + \bar{\beta}_1\omega_{1,1}) &= \pm\bar{M}_T \\
M_2 = \pm\bar{M}_2 \qquad M_3 = \pm\bar{M}_3 \qquad M_\phi &= \pm\bar{M}_\phi
\end{aligned}
$$

To interpret the *linearization*, we consider (13–81). If one neglects the nonlinear terms in the shearing strains,

$$
\begin{aligned}
\gamma_{12} &\approx \hat{u}_{1,2} + \hat{u}_{2,1} \\
\gamma_{13} &\approx \hat{u}_{1,3} + \hat{u}_{3,1}
\end{aligned} \qquad \text{(a)}
$$

takes the extensional strain as

$$
\varepsilon_1 \approx \hat{u}_{1,1} + \tfrac{1}{2}|\hat{u}^2_{2,1} + \hat{u}^2_{3,1}|_{x_2 = x_3 = 0} + (x_2^2 + x_3^2)\omega^2_{1,1} \qquad \text{(b)}
$$

and assumes

$$
\begin{aligned}
\iint x_2(x_2^2 + x_3^2)dA &= 0 \\
\iint x_3(x_2^2 + x_3^2)dA &= 0 \\
\iint \phi(x_2^2 + x_3^2)dA &= 0
\end{aligned} \qquad \text{(c)}
$$

one obtains (13–81). Equations (c) are exact when the section is doubly symmetric. Assumptions (a) and (b) are reasonable if $\omega_{1,1}$ is small w.r.t. to $u_{2,1}$ and $u_{3,1}$. However, they introduce considerable error when $\omega_{1,1}$ is the dominant term. This has been demonstrated by Black (Ref. 5).

When the cross section is doubly symmetric,

$$
\begin{aligned}
\bar{x}_2 = \bar{x}_3 = x_{2r} = x_{3r} = \frac{1}{A_{23}} &= 0 \\
\bar{\beta}_1 = \frac{I_1}{A} &= r^2
\end{aligned} \qquad (18\text{–}19)
$$

(r is the radius of gyration with respect to the centroid) and the problem uncouples to—

1. Flexure in X_1-X_2 plane
2. Flexure in X_1-X_3 plane
3. Restrained torsion

We have already determined the solution for flexure in the X_1-X_2 plane. If we introduce a subscript for μ and ϕ_j,

$$(\mu_2)^2 = \frac{-P}{EI_3} \qquad (\mu_3)^2 = \frac{-P}{EI_2}$$
$$\phi_{21} = \phi_1(\mu_2 L) \qquad \phi_{31} = \phi_1(\mu_3 L) \tag{18-20}$$

and then replace

$$\begin{aligned} u_2 &\to u_3 & \omega_2 &\to \omega_3 \\ u_3 &\to -u_2 & \omega_3 &\to -\omega_2 \\ F_2 &\to F_3 & M_2 &\to M_3 \\ F_3 &\to -F_2 & M_3 &\to -M_2 \end{aligned} \tag{18-21}$$

in (18–9), (18–13), we obtain the member relations for flexure in the X_1-X_3 frame. For example,

$$\bar{M}_{A3} = M^i_{A3} + \frac{EI_3}{L}\left[\phi_{21}\omega_{A3} + \phi_{22}\omega_{B3} - \frac{\phi_{23}}{L}(u_{B2} - u_{A2})\right]$$
$$\Downarrow \tag{18-22}$$
$$\bar{M}_{A2} = M^i_{A2} + \frac{EI_2}{L}\left[\phi_{31}\omega_{A2} + \phi_{32}\omega_{B2} + \frac{\phi_{33}}{L}(u_{B3} - u_{A3})\right]$$

and

$$\bar{F}_{A2} = \cdots$$
$$\Downarrow$$
$$\bar{F}_{A3} = F^i_{A3} + \frac{\phi_{33}EI_2}{L^2}\left[-\omega_{B2} - \omega_{A2} - \frac{2}{L}(u_{B3} - u_{A3})\right] - \frac{P}{L}(u_{B3} - u_{A3})$$

The expressions for the axial end forces expands to

$$F_{B1} = P \qquad F_{A1} = -P$$
$$P = \frac{AE}{L}(u_{B1} - u_{A1}) + AE(e_{r1} + e_{r2} + e_{r3})$$
$$e_{r1} = \frac{r^2}{2L}\int_0^L \omega_{1,1}^2\,dx_1 \qquad e_{r2} = \frac{1}{2L}\int_0^L (u_{2,x})^2\,dx_1 \tag{18-23}$$
$$e_{r3} = \frac{1}{2L}\int_0^L (u_{3,x})^2\,dx_1$$

where e_{r3} is obtained from e_{r2} by applying (18–21).

We generate the restrained torsion solution following the procedure described in Example 13–7. If the joints are moment resisting (i.e., rigid), it is reasonable to assume no warping, which requires $f = 0$ at $x = 0, L$. The corresponding solution is summarized below:

$$\bar{P} = \frac{r^2P}{GJ} \qquad u^2 = \frac{GJ}{E_r I_\phi}\frac{1+\bar{P}}{1 + C_r(1+\bar{P})}$$
$$\bar{M}_{B1} = \frac{GJ}{L}\phi_t(\omega_{B1} - \omega_{A1})$$
$$\bar{M}_{A1} = -\bar{M}_{B1}$$
$$\phi_t = \frac{1+\bar{P}}{1 + \dfrac{1-\cosh\mu L}{\sinh\mu L}\left[\dfrac{2}{\mu L(1 + C_r(1+\bar{P}))}\right]} \tag{18-24}$$
$$\omega_1 = \omega_{A1} + \frac{\bar{M}_{B1}}{GJ(1+P)}\left\{x + \frac{1}{\mu(1 + C_r(1+\bar{P}))}\left[-\sinh\mu x + \frac{1-\cosh\mu L}{\sinh\mu L}(1 - \cosh\mu x)\right]\right\}$$

We neglect shear deformation due to restrained torsion by setting $C_r = 0$. If warping restraint is neglected,

$$E_r I_\phi = 0 \quad\Rightarrow\quad \mu^2 = \infty$$
$$\phi_t \Rightarrow 1 + \bar{P} \tag{18-25}$$

At this point, we summarize the member force-displacement relations for a doubly symmetric cross section. For convenience, we introduce matrix notation:

$$\mathscr{F}_B = \{\bar{F}_1\bar{F}_2\bar{F}_3\bar{M}_1\bar{M}_2\bar{M}_3\}_B$$
$$\mathscr{U}_B = \{u_1u_2u_3\omega_1\omega_2\omega_3\}_B$$

etc.

$$\mathscr{F}_B = \mathscr{F}^i_B + \mathbf{k}_{BB}\mathscr{U}_B + \mathbf{k}_{BA}\mathscr{U}_A + \mathscr{F}_r$$
$$\mathscr{F}_A = \mathscr{F}^i_A + (\mathbf{k}_{BA})^T\mathscr{U}_B + \mathbf{k}_{AA}\mathscr{U}_A - \mathscr{F}_r \tag{18-26}$$

where $\mathscr{F}_r$ contains nonlinear terms due to chord rotation and end shortening $= \left\{AE(e_{r1} + e_{r2} + e_{r3}); \frac{P}{L}(u_{B2} - u_{A2}); \frac{P}{L}(u_{B3} - u_{A3}); 0; 0; 0\right\}$; $\mathscr{F}^i$ contains the initial end forces due to member loads; and

$$\mathbf{k}_{BB} = \begin{bmatrix} \frac{AE}{L} & & & & & \\ & 2\phi_{23}\frac{EI_3}{L^3} & & & & -\phi_{23}\frac{EI_3}{L^2} \\ & & 2\phi_{33}\frac{EI_2}{L^3} & & \phi_{33}\frac{EI_2}{L^2} & \\ & & & \phi_t\frac{GJ}{L} & & \\ \text{Sym} & & & & \phi_{31}\frac{EI_2}{L} & \\ & & & & & \phi_{21}\frac{EI_3}{L} \end{bmatrix}$$

$$
\mathbf{k}_{BA} = \left[\begin{array}{c|c|c|c|c|c}
-\frac{AE}{L} & & & & & \\ \hline
 & -2\phi_{23}\frac{EI_3}{L^3} & & & & -\phi_{23}\frac{EI_3}{L^2} \\ \hline
 & & -2\phi_{33}\frac{EI_2}{L^3} & & \phi_{33}\frac{EI_2}{L^2} & \\ \hline
 & & & -\phi_t\frac{GJ}{L} & & \\ \hline
 & & -\phi_{33}\frac{EI_2}{L^2} & & \phi_{32}\frac{EI_2}{L} & \\ \hline
 & \phi_{23}\frac{EI_3}{L^2} & & & & \phi_{22}\frac{EI_3}{L}
\end{array}\right]
$$

$$
\mathbf{k}_{AA} = \left[\begin{array}{c|c|c|c|c|c}
\frac{AE}{L} & & & & & \\ \hline
 & 2\phi_{23}\frac{EI_3}{L^3} & & & & \phi_{23}\frac{EI_3}{L^2} \\ \hline
 & & 2\phi_{33}\frac{EI_2}{L^3} & & -\phi_{33}\frac{EI_2}{L^2} & \\ \hline
 & & & \phi_t\frac{GJ}{L} & & \\ \hline
\text{Sym} & & & & \phi_{31}\frac{EI_2}{L} & \\ \hline
 & & & & & \phi_{21}\frac{EI_3}{L}
\end{array}\right]
$$

Operating on $\mathscr{F}_B$, $\mathscr{F}_A$ leads to the incremental equations, i.e., the three-dimensional form of (18–13). Assuming the stability functions are *constant* and taking

$$
\begin{aligned}
d\phi_t &\approx d\bar{P} = \frac{r^2}{GJ}\,dP \\
de_r &\approx \frac{1}{L^2}\Big\{(u_{B2}-u_{A2})(\Delta u_{B2}-\Delta u_{A2}) + (u_{B3}-u_{A3})(\Delta u_{B3}-\Delta u_{A3}) \\
&\qquad + r^2(\omega_{B1}-\omega_{A1})(\Delta\omega_{B1}-\Delta\omega_{A1})\Big\}
\end{aligned}
\tag{18–27}
$$

we obtain

$$
\begin{aligned}
d\bar{\mathscr{F}}_B &= d\mathscr{F}^i_B + (\mathbf{k}_{BB}+\mathbf{k}_r)\Delta\mathscr{U}_B + (\mathbf{k}_{BA}-\mathbf{k}_r)\Delta\mathscr{U}_A \\
d\bar{\mathscr{F}}_A &= d\mathscr{F}^i_A + (\mathbf{k}_{BA}-\mathbf{k}_r)^T\,\Delta\mathscr{U}_B + (\mathbf{k}_{AA}+\mathbf{k}_r)\Delta\mathscr{U}_A
\end{aligned}
\tag{18–28}
$$

where $\mathbf{k}_r$ is the incremental stiffness matrix due to rotation,

$$
\mathbf{k}_r = \frac{AE}{L}\left[\begin{array}{c|c|c|c|c|c}
0 & \rho_3 & -\rho_2 & r^2\rho_1 & 0 & 0 \\ \hline
 & \rho_3^2 + \frac{L\bar{F}_{B1}}{AE} & -\rho_2\rho_3 & r^2\rho_1\rho_3 & 0 & 0 \\ \hline
 & & \rho_2^2 + \frac{L\bar{F}_{B1}}{AE} & -r^2\rho_1\rho_2 & 0 & 0 \\ \hline
 & \text{Symmetrical} & & (r^2\rho_1)^2 & 0 & 0 \\ \hline
 & & & & 0 & 0 \\ \hline
 & & & & & 0
\end{array}\right]
$$

$$
\rho_3 = \frac{1}{L}(u_{B2}-u_{A2}) \qquad \rho_2 = \frac{-1}{L}(u_{B3}-u_{A3}) \qquad \rho_1 = \frac{1}{L}(\omega_{B1}-\omega_{A1})
$$

If P is close to the *member* buckling load, one must include additional terms due to the variation in the stability functions and use the exact expression for de_r.

Kappas's equations have also been solved explicitly for a monosymmetric section with warping and shear deformation neglected. Since the equations are linear, one can write down the general solution for an arbitrary cross section. It will involve twelve integration constants which are evaluated by enforcing the displacement boundary conditions. The algebra is untractable unless one introduces symmetry restrictions.

18–4. SOLUTION TECHNIQUES; STABILITY ANALYSIS

In this section, we present the mathematical background for two solution techniques, successive substitution and Newton-Raphson iteration, and then apply them to the governing equations for a nonlinear member system.

Consider the problem of solving the nonlinear equation

$$\psi(x) = 0 \tag{18–29}$$

Let $\bar{x}$ represent one of the roots. By definition,

$$\psi(\bar{x}) \equiv \bar{\psi} = 0 \tag{18–30}$$

In the method of successive substitution,† (18–29) is rewritten in an equivalent form,

$$x = g(x) \tag{18–31}$$

and successive estimates of the solution are determined, using

$$x^{(k+1)} = g(x^{(k)}) \equiv g^{(k)} \tag{18–32}$$

where $x^{(k)}$ represents the kth estimate.

The exact solution satisfies

$$\bar{x} = \bar{g} \tag{a}$$

† See Ref. 9.

Then,

$$\bar{x} - x^{(k+1)} = \bar{g} - g^{(k)} \tag{b}$$

Expanding $\bar{g}$ in a Taylor series about $x^{(k)}$,

$$\bar{g} = g^{(k)} + g^{(k)}_{,x}(\bar{x} - x^{(k)}) + \tfrac{1}{2}g^{(k)}_{,xx}(\bar{x} - x^{(k)})^2 + \cdots \tag{c}$$

and retaining only the first two terms lead to the convergence measure

$$\bar{x} - x^{(k+1)} = (\bar{x} - x^{(k)})|g_{,x}|_{x=\xi_k} \tag{18–33}$$

where ξ_k is between $x^{(k)}$ and $\bar{x}$.

In the Newton-Raphson method,† $\psi(\bar{x})$ is expanded in a Taylor series about $x^{(k)}$,

$$\psi(\bar{x}) = \psi^{(k)} + \psi^{(k)}_{,x}\,\Delta x + \tfrac{1}{2}\psi^{(k)}_{,xx}(\Delta x)^2 + \cdots = 0 \tag{a}$$

where Δx is the exact correction to $x^{(k)}$,

$$\Delta x = \bar{x} - x^{(k)} \tag{b}$$

An *estimate* for Δx is obtained by neglecting second- and higher-order terms:

$$\begin{aligned} \Delta x^{(k)} &= -\frac{\psi^{(k)}}{\psi^{(k)}_{,x}} \\ x^{(k+1)} &= x^{(k)} + \Delta x^{(k)} \end{aligned} \tag{18–34}$$

The convergence measure for this method can be obtained by combining (a) and (18–34), and has the form

$$(\bar{x} - x^{(k+1)})\psi^{(k)}_{,x} = -\tfrac{1}{2}(\bar{x} - x^{(k)})^2|\psi_{,xx}|_{x=\xi_k} \tag{18–35}$$

Note that the Newton-Raphson method has *second*-order convergence whereas successive substitution has only *first*-order convergence.

We consider next a set of n nonlinear equations:

$$\begin{aligned} \boldsymbol{\psi} &= \{\psi_1\,\psi_2\cdots\psi_n\} = 0 \\ \psi_i &= \psi(x_1, x_2, \ldots, x_n) \end{aligned} \tag{18–36}$$

An exact solution is denoted by $\bar{\mathbf{x}}$. Also, $\boldsymbol{\psi}(\bar{\mathbf{x}}) = \bar{\boldsymbol{\psi}}$.

In successive substitution, (18–36) is rearranged to

$$\mathbf{ax} = \mathbf{c} - \mathbf{g} \tag{18–37}$$

where $\mathbf{a}$, $\mathbf{c}$ are constant, $\mathbf{g} = \mathbf{g}(\mathbf{x})$, and the recurrence relation is taken as

$$\mathbf{ax}^{(k+1)} = \mathbf{c} - \mathbf{g}^{(k)} \tag{18–38}$$

The exact solution satisfies

$$\mathbf{a}\bar{\mathbf{x}} = \mathbf{c} - \bar{\mathbf{g}} \tag{a}$$

Then,

$$\mathbf{a}(\bar{\mathbf{x}} - \mathbf{x}^{(k+1)}) = -(\bar{\mathbf{g}} - \mathbf{g}^{(k)}) \tag{b}$$

† See Ref. 9.

Expanding $\bar{\mathbf{g}}$ in a Taylor series about $\mathbf{x}^{(k)}$,

$$\bar{\mathbf{g}} = \mathbf{g}^{(k)} + \mathbf{g}^{(k)}_{,x}(\bar{\mathbf{x}} - \mathbf{x}^{(k)}) + \cdots \tag{c}$$

$$\mathbf{g}_{,x} = \left[\frac{\partial g_j}{\partial x_r}\right] = \begin{bmatrix} g_{1,1} & g_{1,2} & \cdots & g_{1,n} \\ g_{2,1} & g_{2,2} & \cdots & g_{2,n} \\ \vdots & \vdots & & \vdots \\ g_{n,1} & g_{n,2} & \cdots & g_{n,n} \end{bmatrix} \tag{d}$$

and retaining only the first two terms results in the convergence measure

$$(\mathbf{x} - \mathbf{x}^{(k+1)}) = \mathbf{a}^{-1}|\mathbf{g}_{,x}|_{\mathbf{x}=\xi_k}(\mathbf{x} - \mathbf{x}^{(k)}) \tag{18–39}$$

where ξ_k lies between $\mathbf{x}^k$ and $\bar{\mathbf{x}}$. For convergence, the norm of $\mathbf{a}^{-1}\mathbf{g}_{,x}$ must be less than unity.

The generalized Newton-Raphson method consists in first expanding $\boldsymbol{\psi}(\bar{\mathbf{x}})$ about $\mathbf{x}^{(k)}$,

$$\boldsymbol{\psi}(\bar{\mathbf{x}}) = \boldsymbol{\psi}^{(k)} + d\boldsymbol{\psi}^{(k)} + \tfrac{1}{2}|d^2\boldsymbol{\psi}|_{\xi_k} = \mathbf{0}$$

where

$$\begin{aligned} d\boldsymbol{\psi}^{(k)} &= \boldsymbol{\psi}^{(k)}_{,x}\,\Delta\mathbf{x} = \left[\frac{\partial \psi_j^{(k)}}{\partial x_r}\right]\{\bar{x}_r - x_r^{(k)}\} \\ d^2\boldsymbol{\psi} &= \{d^2\psi_j\} \\ d^2\psi_j &= (\Delta\mathbf{x})^T\left[\frac{\partial^2\psi_j}{\partial x_r\,\partial x_s}\right](\Delta\mathbf{x}) \end{aligned} \tag{18–40}$$

Neglecting the second differential leads to the recurrence relation

$$\begin{aligned} d\boldsymbol{\psi}^{(k)} &= \boldsymbol{\psi}^{(k)}_{,x}\,\Delta\mathbf{x}^{(k)} = -\boldsymbol{\psi}^{(k)} \\ \mathbf{x}^{(k+1)} &= \mathbf{x}^{(k)} + \Delta\mathbf{x}^{(k)} \end{aligned} \tag{18–41}$$

The corresponding convergence measure is

$$\boldsymbol{\psi}^{(k)}_{,x}(\bar{\mathbf{x}} - \mathbf{x}^{(k+1)}) = -\tfrac{1}{2}\,d^2\boldsymbol{\psi}|_{\xi_k} \tag{18–42}$$

Let us now apply these solution techniques to the structural problem. The governing equations are the nodal force-equilibrium equations referred to the *global* system frame,

$$\boldsymbol{\psi} = \mathscr{P}_e - \mathscr{P}_m = \mathbf{0} \tag{18–43}$$

where $\mathscr{P}_e$ contains the external nodal forces and $-P_{m,i}$ is the summation of the member end forces incident on node i. One first has to rotate the member end forces, (18–26), from the member frame to the global frame using

$$\begin{aligned} \mathscr{F}^o &= (\mathscr{R}^{on})^T\mathscr{F}^n \\ \mathbf{k}^o &= (\mathscr{R}^{on})^T\mathbf{k}^n\mathscr{R}^{on} \end{aligned} \tag{a}$$

In our formulation, the member frame is *fixed*, i.e. $\mathscr{R}^{on}$ is constant. We introduce the displacement restraints and write the final equations as

$$\begin{aligned} \psi &= \mathbf{P}_e - \mathbf{P}_m = \mathbf{0} \\ \mathbf{P}_m &= \mathbf{P}_i + \mathbf{P}_r + \mathbf{KU} \end{aligned} \tag{18–44}$$

Note that $\mathbf{K}$ and $\mathbf{P}_i$ depend on the axial forces while $\mathbf{P}_r$ depends on both the axial force and the member rigid body chord rotation. If the axial forces are small in comparison to the member buckling loads, we can replace $\mathbf{K}$ with $\mathbf{K}_l$, the *linear* stiffness matrix.

Applying successive substitution, we write

$$\mathbf{K}\mathbf{U} = \mathbf{P}_e - \mathbf{P}_i - \mathbf{P}_r \qquad \text{(a)}$$

and iterate on $\mathbf{U}$, holding $\mathbf{K}$ constant during the iteration:

$$\mathbf{K}\mathbf{U}^{(n)} = \mathbf{P}_e - \mathbf{P}_i - \mathbf{P}_r^{(n-1)} \qquad (18\text{–}45)$$

We employ (18–45) together with an incremental loading scheme since $\mathbf{K}$ is actually a variable. The steps are outlined here:

1. Apply the first load increment, $\mathbf{P}_{e(1)}$, and solve for $\mathbf{U}_{(1)}$, using $\mathbf{K} = \mathbf{K}_l$.
2. Update $\mathbf{K}$ using the axial forces corresponding to $\mathbf{P}_{e(1)}$. Then apply $\mathbf{P}_{e(2)}$ and iterate on

$$\mathbf{K}_{(1)}\mathbf{U}^{(n)} = \mathbf{P}_{e(1)} + \mathbf{P}_{e(2)} - \mathbf{P}_r^{(n-1)}$$

3. Continue for successive load increments.

A convenient convergence criterion is the relative change in the Euclidean norm, N, of the nodal displacements.

$$N = (\mathbf{U}^T\mathbf{U})^{1/2}$$
$$\left(\frac{N^{(n+1)}}{N^{(n)}} - 1\right)_{\text{abs. value}} \leqslant \varepsilon \quad \text{(a specified value)} \qquad (18\text{–}46)$$

This scheme is particularly efficient when the member axial forces are small with respect to the Euler loads since, in this case, we can take $\mathbf{K} = \mathbf{K}_l$ during the entire solution phase.

In the Newton-Raphson procedure, we operate on ψ according to (18–41):

$$d\psi^{(n)} = -\psi^{(n)} \qquad \text{(a)}$$

Now, $\mathbf{P}_e$ is prescribed so that

$$\begin{aligned} d\psi^{(n)} &= -d\mathbf{P}_m^{(n)} \qquad \text{due to } \Delta\mathbf{U} \\ &= -|d\mathbf{P}_i + d\mathbf{P}_r + \mathbf{K}\,\Delta\mathbf{U} + (d\mathbf{K})\mathbf{U}|_{\mathbf{U}^{(n)}} \\ &= -|\mathbf{K}_t\,\Delta\mathbf{U}|_{\mathbf{U}^{(n)}} \end{aligned} \qquad (18\text{–}47)$$

where $\mathbf{K}_t$ denotes the *tangent* stiffness matrix. The iteration cycle is

$$\begin{aligned} \mathbf{K}_t^{(n)}\,\Delta\mathbf{U}^{(n)} &= \mathbf{P}_e - \mathbf{P}_m^{(n)} \\ \mathbf{U}^{(n+1)} &= \mathbf{U}^{(n)} + \Delta\mathbf{U}^{(n)} \end{aligned} \qquad (18\text{–}48)$$

We iterate on (18–48) for successive load increments. This scheme is more expensive since $\mathbf{K}_t$ has to be updated for each cycle. However, its convergence rate is more rapid than direct substitution. If we assume the stability functions

are constant in forming $d\mathbf{P}_m$ due to $\Delta\mathbf{U}$, the tangent stiffness matrix reduces to

$$\begin{aligned} d\mathbf{K} \approx \mathbf{0} \qquad d\mathbf{P}_i \approx \mathbf{0} \\ \mathbf{K}_t \approx \mathbf{K} + \mathbf{K}_r \end{aligned} \qquad (18\text{–}49)$$

where $\mathbf{K}_r$ is generated with (18–28). We include the incremental member loads in $\mathbf{P}_e$ at the start of the iteration cycle. Rather than update $\mathbf{K}_t$ at each cycle, one can hold $\mathbf{K}_t$ fixed for a limited number of cycles. This is called *modified* Newton-Raphson. The convergence rate is lower than for regular Newton-Raphson but higher than successive substitution.

We consider next the question of stability. According to the classical stability criterion,† an equilibrium position is classified as:

$$\begin{aligned} &\text{stable} && d^2W_m - d^2W_e > 0 \\ &\text{neutral} && d^2W_m - d^2W_e = 0 \\ &\text{unstable} && d^2W_m - d^2W_e < 0 \end{aligned} \qquad (18\text{–}50)$$

where d^2W_e is the second-order work done by the external forces during a displacement increment $\Delta\mathbf{U}$, and d^2W_m is the second-order work done by the member end forces acting *on the members.* With our notation,

$$\begin{aligned} d^2W_e &= \left(\frac{d}{d\mathbf{U}}\mathbf{P}_e\right)^T \Delta\mathbf{U} \\ d^2W_m &= \left(\frac{d}{d\mathbf{U}}\mathbf{P}_m\right)^T \Delta\mathbf{U} \\ &= \Delta\mathbf{U}^T\mathbf{K}_t\,\Delta\mathbf{U} \end{aligned} \qquad (18\text{–}51)$$

and the criteria transform to

$$(\Delta\mathbf{U})^T\mathbf{K}_t\,\Delta\mathbf{U} - \left(\frac{d}{d\mathbf{U}}\mathbf{P}_e\right)^T \Delta\mathbf{U} \quad \begin{matrix} < 0 & \text{stable} \\ = 0 & \text{neutral} \\ > 0 & \text{unstable} \end{matrix} \qquad (18\text{–}52)$$

The most frequent case is $\mathbf{P}_e$ prescribed, and for a constant loading, the tangent stiffness matrix must be *positive definite.*

To detect instability, we keep track of the sign of the determinant of the tangent stiffness matrix during the iteration. The sign is obtained at *no* cost (i.e., no additional computation) if Gauss elimination or the factor method are used to solve the *correction* equation, (18–48). When the determinant changes sign, we have passed through a stability transition. Another indication of the existence of a bifurcation point ($\mathbf{K}_t$ singular) is the degeneration of the convergence rate for Newton-Raphson. The correction tends to diverge and oscillate in sign and one has to employ a higher iterative scheme.

Finally, we consider the special case where the loading does not produce significant chord rotation. A typical example is shown in Fig. 18–4. Both the

† See Secs. 7–6 and 10–6.

frame and loading are symmetrical and the displacement is due only to shortening of the columns. To investigate the stability of this structure, we delete† the rotation terms in $\mathbf{K}_r$ and write

$$\mathbf{K}_t = \mathbf{K} + \lambda\mathbf{K}_r' = \mathbf{K}_t(\lambda) \qquad (18-53)$$

where $\mathbf{K}_r'$ is due to a unit value of the load parameter λ. The member axial forces are determined from a linear analysis. Then, the bifurcation problem reduces to determining the value of λ for which a nontrivial solution of

$$(\mathbf{K} + \lambda\mathbf{K}_r')\Delta\mathbf{U} = \mathbf{0} \qquad (18-54)$$

exists. This is a nonlinear eigenvalue problem, since $K = K(\lambda)$.

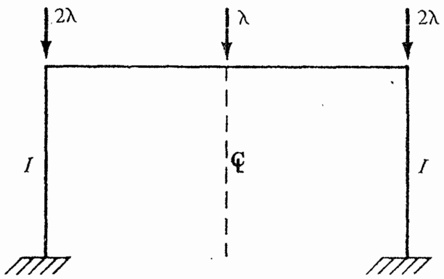


Fig. 18–4. Example of structure and loading for which linearized stability analysis is applicable.

In *linearized* stability analysis, $\mathbf{K}$ is assumed to be $\mathbf{K}_l$ and one solves

$$\mathbf{K}_l\,\Delta\mathbf{U} = -\lambda\mathbf{K}_r'\,\Delta\mathbf{U} \qquad (18-55)$$

Both $\mathbf{K}_l$ and $\mathbf{K}_r'$ are symmetrical. Also, $\mathbf{K}_l$ is positive definite. Usually, only the lowest critical load is of interest, and this can be obtained by applying inverse iteration‡ to

$$(-\mathbf{K}_r')\Delta\mathbf{U} = \bar{\lambda}\mathbf{K}_l\Delta\mathbf{U}$$
$$\bar{\lambda} = \frac{1}{\lambda} \qquad (18-56)$$

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† Set $\rho_1 = \rho_2 = \rho_3 = 0$ in (18–28).
‡ See Refs. 11 and 12 of Chapter 2.

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Index