**5–6.** Refer to Problem 5–3. Determine $\mathscr{U}_Q^q$ corresponding to $\mathscr{U}_P^p = \{1/2, -1/4, 1/3, -1/10, 1/10, 0\}$. Verify that

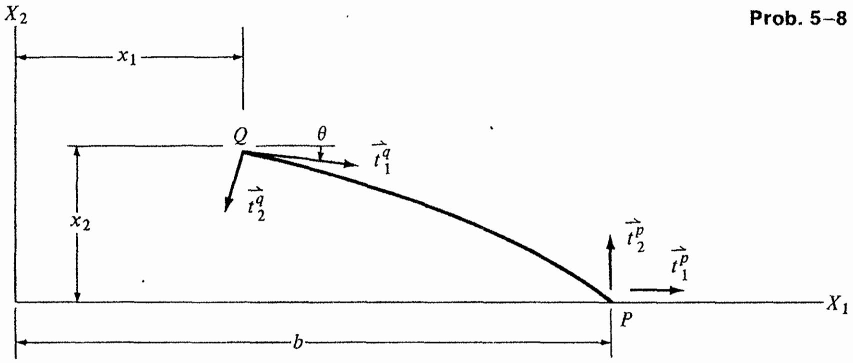
$$\mathscr{F}_Q^{q,T}\,\mathscr{U}_Q^q = \mathscr{F}_P^{p,T}\,\mathscr{U}_P^p$$

**5–7.** Verify that (5–27) and (5–28) are equivalent forms. Note that

$$\cdots\left[\begin{array}{c|c} \mathbf{I}_3 & \mathbf{X}_{PQ}^1 \\ \hline \mathbf{0} & \mathbf{I}_3 \end{array}\right] = \left[\begin{array}{c|c} \mathbf{I}_3 & \mathbf{0} \\ \hline \mathbf{X}_{QP} & \mathbf{I}_3 \end{array}\right]^T = (\mathscr{X}_{QP}^1)^T$$

**5–8.** Consider the plane member shown. The reference axis is defined by $x_2 = f(x_1)$.

**Prob. 5–8**



(a) Determine $\mathscr{T}_{PQ}^{pq}$. Note that the local frame at $P$ coincides with the basic frame whereas the local frame at $Q$ coincides with the natural frame at $Q$.

(b) Specialize part (a) for the case where

$$x_2 = \frac{4a}{b^2}(x_1 b - x_1^2)$$

and the $x_1$ coordinate of point $Q$ is equal to $b/4$. Use the results of Prob. 4–2.

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# Part II

# ANALYSIS OF AN IDEAL TRUSS

# 6

# Governing Equations for an Ideal Truss

## 6–1. GENERAL

A system of bars* connected at their ends by frictionless hinges to joints and subjected only to forces applied at the joint centers is called an *ideal* truss.† The bars are assumed to be weightless and so assembled that the line connecting the joint centers at the ends of each bar coincides with the centroidal axis. Since the bars are weightless and the hinges are frictionless, it follows that each bar is in a state of direct stress. There is only one force unknown associated with each bar, namely, the magnitude of the axial force; the direction of the force coincides with the line connecting the joint centers. If the bars lie in one plane, the system is called a *plane* or *two-dimensional* truss. There are two displacement components associated with each joint of a plane truss. Similarly, a general system is called a *space* or *three-dimensional* truss, and there are three displacement components associated with each joint.

We suppose there are $m$ bars (members) and $j$ joints. We define $i$ as

$$\begin{aligned} i &= 2 \qquad \text{for a plane truss} \\ i &= 3 \qquad \text{for a space truss} \end{aligned} \tag{6–1}$$

Using this notation, there are $ij$ displacement quantities associated with the $j$ joints. In general, some of the joint-displacement components are prescribed. Let $r$ be the number of prescribed displacement components (displacement restraints) and $n_d$ the total number of unknown joint displacements. It follows that

$$n_d = ij - r \tag{6–2}$$

Corresponding to each joint displacement restraint is an unknown joint force

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* A prismatic member is conventionally referred to as a bar in truss analysis.

† See Ref. 1.

(reaction). We let $n_f$ be the total number of force unknowns. Then,

$$n_f = m + r \tag{6-3}$$

Finally, the total number of unknowns, $n$, for an ideal truss is

$$n = n_f + n_d = ij + m \tag{6-4}$$

The equilibrium equations for the bars have been used to establish the fact that the force in each bar has the direction of the line connecting the joint centers at the ends of the bar. There remains the equilibrium equations for the joints. Since each joint is subjected to a concurrent force system, there are $ij$ scalar force-equilibrium equations relating the bar forces, external joint loads, and direction cosines for the lines connecting the joint centers in the deformed state. In order to solve the problem, that is, to determine the bar forces, reactions, and joint displacements, $m$ additional independent equations are required. These additional equations are referred to as the bar force–joint displacement relations and are obtained by combining the bar force–bar elongation relation and bar elongation–joint displacement relation for each of the $m$ bars.

In this chapter, we first derive the elongation–joint displacement relation for a single bar and then express the complete set of $m$ relations as a single matrix equation. This procedure is repeated for the bar force-elongation relations and the joint force-equilibrium equations. We then describe a procedure for introducing the joint-displacement restraints and summarize the governing equations. Finally, we briefly discuss the solvability of the governing equations for the linear case. In this case, the question of initial instability is directly related to the solvability.

In Chapter 7, we develop variational principles for an ideal truss. The two general procedures for solving the governing equations are described in Chapters 8 and 9. We refer to these procedures as the *displacement* and *force* methods. They are also called the *stiffness* and *flexibility* methods in some texts.

The basic concepts employed in formulating and solving the governing equations for an ideal truss are applicable, with slight extension, to a member system having moment resisting connections. Some authors start with the general system and then specialize the equations for the case of an ideal truss. We prefer to proceed from the truss to the general system since the basic formulation techniques for the ideal truss can be more readily described. To adequately describe the formulation for a general system requires introducing a considerable amount of notation which tends to overpower the reader.

## 6–2. ELONGATION–JOINT DISPLACEMENT RELATION FOR A BAR

We number the joints consecutively from 1 through $j$. It is convenient to refer the coordinates of a joint, the joint-displacement components, and the external joint load components to a common right-handed cartesian reference frame. Let $X_j$, $\vec{\imath}_j$ ($j = 1, 2, 3$) be the axes and corresponding orthogonal unit vectors for the basic frame. The initial coordinates, displacement components, and components of the resultant external force for joint $k$ are denoted by $x_{kj}$, $u_{kj}$, $p_{kj}$ ($j = 1, 2, 3$) and the corresponding vectors are written as

$$\begin{aligned} \vec{r}_k &= \sum_{j=1}^{3} x_{kj}\vec{\imath}_j = \mathbf{x}_k^T\mathbf{i} \\ \vec{u}_k &= \mathbf{u}_k^T\mathbf{i} \\ \vec{p}_k &= \mathbf{p}_k^T\mathbf{i} \end{aligned} \tag{6-5}$$

The coordinates and position vector for joint $k$ in the deformed state are $\eta_{kj}$, $\vec{\rho}_k$.

$$\begin{aligned} \vec{\rho}_k &= \boldsymbol{\eta}_k^T\mathbf{i} = \vec{r}_k + \vec{u}_k \\ \boldsymbol{\eta}_k &= \mathbf{x}_k + \mathbf{u}_k \end{aligned} \tag{6-6}$$

Figure 6–1 illustrates the notation associated with the joints.

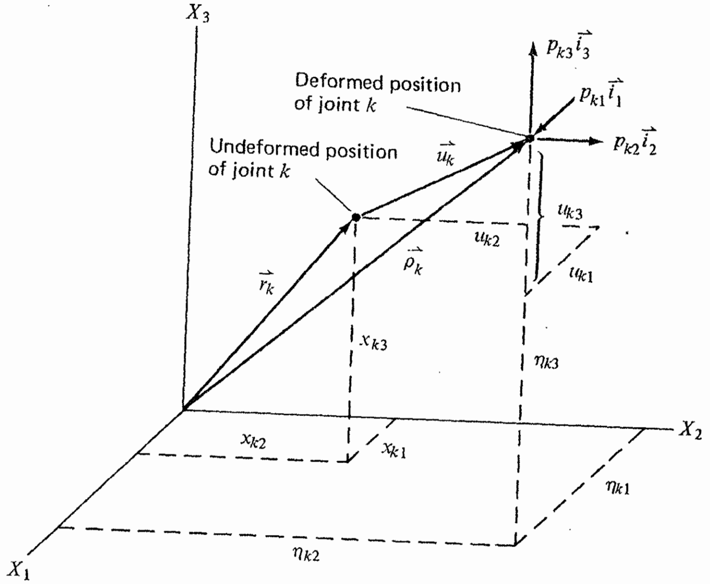

**Fig. 6–1.** Notation for joints.

We number the bars from 1 through $m$ and consider bar $n$ to be connected to joints $k$ and $s$. The centroidal axis of bar $n$ coincides with the line connecting joints $k$ and $s$. From Fig. 6–2 the initial length of bar $n$, denoted by $L_n$, is equal to the magnitude of the vector $\Delta\vec{r} = \vec{r}_s - \vec{r}_k$:

$$L_n^2 = \Delta\vec{r} \cdot \Delta\vec{r} \tag{6-7}$$

Since the basic frame is orthogonal, (6–7) reduces to

$$L_n^2 = (\mathbf{x}_s - \mathbf{x}_k)^T(\mathbf{x}_s - \mathbf{x}_k) = \sum_{j=1}^{3} (x_{sj} - x_{kj})^2 \tag{6–8}$$

Before the orientation of the bar can be specified, a positive sense or direction must be selected. We take the positive sense for bar $n$ to be *from* joint $k$ *to* joint $s$ and define $\alpha_{nj}$ as the direction cosine for the positive sense of bar $n$ in the *undeformed* state with respect to the $X_j$ direction:

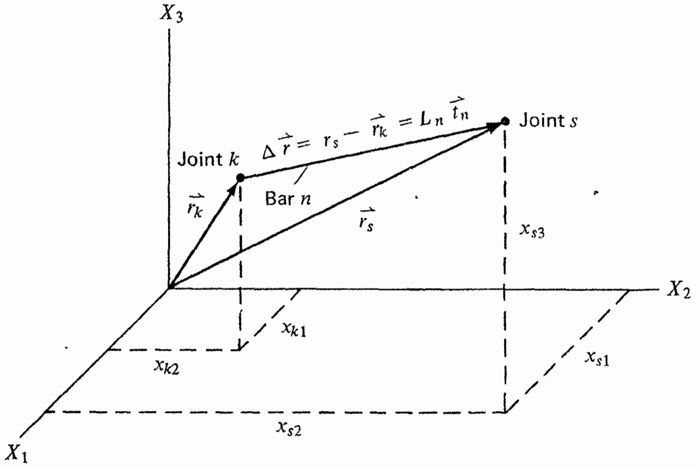

**Fig. 6–2.** Undeformed position of Bar $n$.

$$\alpha_{nj} = \frac{1}{L_n}(\Delta\bar{r} \cdot \bar{i}_j) = \frac{1}{L_n}(x_{sj} - x_{kj}) \tag{6–9}$$

It is convenient to list the direction cosines in a row matrix, $\boldsymbol{\alpha}$.

$$\boldsymbol{\alpha}_n = [\alpha_{n1}\alpha_{n2}\alpha_{n3}] = \frac{1}{L_n}(\mathbf{x}_s - \mathbf{x}_k)^T \tag{6–10}$$

Note that $\boldsymbol{\alpha}_n\boldsymbol{\alpha}_n^T = 1$, due to the orthogonality of the reference frame. Finally, we let $\bar{t}_n$ be the unit vector associated with the positive direction of bar $n$ in the *undeformed* state. By definition,

$$\bar{t}_n = \frac{1}{L_n}\Delta\bar{r} = \boldsymbol{\alpha}_n\bar{\mathbf{i}} \tag{6–11}$$

The deformed position of bar $n$ is shown in Fig. 6–3. The length and direction cosines for bar $n$ are equal to the magnitude and direction cosines for the vector, $\Delta\bar{\rho} = \bar{\rho}_s - \bar{\rho}_k$. Let $L_n + e_n$ be the deformed length, $\bar{v}_n$ the unit vector associated with the positive direction in the deformed state, and $\boldsymbol{\beta}_n$ the corresponding direction cosine matrix. These quantities are defined by

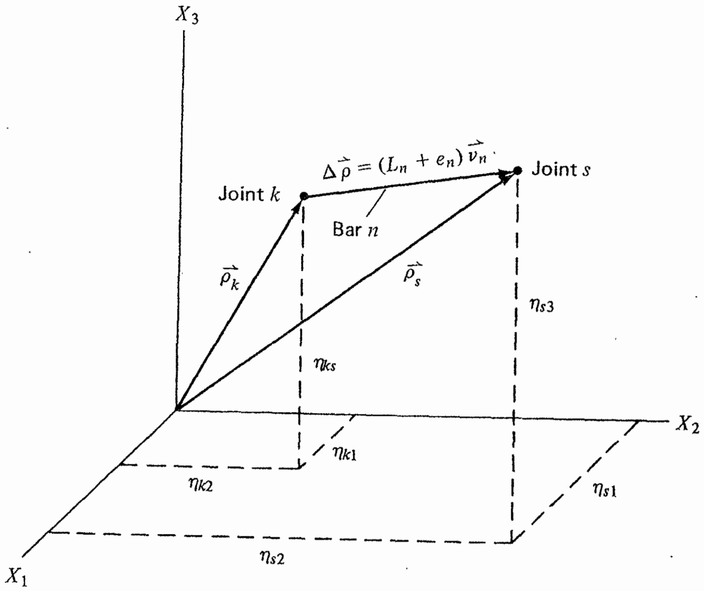

**Fig. 6–3.** Deformed position of Bar $n$.

$$(L_n + e_n)^2 = \Delta\bar{\rho} \cdot \Delta\bar{\rho} \tag{6–12}$$

$$\bar{v}_n = \frac{1}{L_n + e_n}\Delta\bar{\rho} = \boldsymbol{\beta}_n\bar{\mathbf{i}} \tag{6–13}$$

$$\beta_{nj} = \frac{1}{L_n + e_n}(\Delta\bar{\rho} \cdot \bar{i}_j) \tag{6–14}$$

We consider first (6–12). Substituting for $\Delta\bar{\rho}$,

$$\Delta\bar{\rho} = \Delta\bar{r} + (\bar{u}_s - \bar{u}_k) \tag{a}$$

and noting (6–7), (6–11), we obtain, after dividing both sides by $L_n^2$,

$$\left(1 + \frac{e_n}{L_n}\right)^2 = 1 + \frac{2}{L_n}\boldsymbol{\alpha}_n(\mathbf{u}_s - \mathbf{u}_k) + \frac{1}{L_n^2}(\mathbf{u}_s - \mathbf{u}_k)^T(\mathbf{u}_s - \mathbf{u}_k) \tag{6–15}$$

The expression for the direction cosine, $\beta_{nj}$, expands to

$$\beta_{nj} = \frac{1}{1 + \dfrac{e_n}{L_n}}\left[\alpha_{nj} + \frac{1}{L_n}(u_{sj} - u_{kj})\right] \tag{6–16}$$

We list the $\beta$'s in a row matrix, $\boldsymbol{\beta}$.

$$\boldsymbol{\beta}_n = \frac{1}{1 + \dfrac{e_n}{L_n}}\left[\boldsymbol{\alpha}_n + \frac{1}{L_n}(\mathbf{u}_s - \mathbf{u}_k)^T\right] \tag{6–17}$$

By definition, $e_n$ is the change in length of bar $n$. Then, $e_n/L_n$ is the extensional strain which is considerably less than unity for most engineering materials. For example, the strain is only $10^{-3}$ for steel at a stress level of $3 \times 10^4$ ksi. The relations simplify if we introduce the assumption of small strain,

$$e_n/L_n \ll 1 \tag{6–18}$$

Expanding the left-hand side of (6–15), and noting (6–18), we obtain

$$e_n \approx \boldsymbol{\alpha}_n(\mathbf{u}_s - \mathbf{u}_k) + \frac{1}{2L_n}(\mathbf{u}_s - \mathbf{u}_k)^T(\mathbf{u}_s - \mathbf{u}_k) \tag{6–19}$$

The direction cosines for the deformed orientation reduce to

$$\boldsymbol{\beta}_n \approx \boldsymbol{\alpha}_n + \frac{1}{L_n}(\mathbf{u}_s - \mathbf{u}_k)^T \tag{6–20}$$

To simplify the expression for $e_n$ further, we need to interpret the quadratic terms. Using (6–20), we can write (6–19) as

$$e_n = \left(\boldsymbol{\alpha}_n + \frac{1}{2}(\boldsymbol{\beta}_n - \boldsymbol{\alpha}_n)\right)(\mathbf{u}_s - \mathbf{u}_k) \tag{a}$$

This form shows that the second-order terms are related to the *change in orientation* of the bar. If the initial geometry is such that the bar cannot experience a *significant* change in orientation, then we can neglect the nonlinear terms. We use the term *linear geometry* for this case. The linearized relations are

$$\begin{aligned} e_n &\approx \boldsymbol{\alpha}_n(\mathbf{u}_s - \mathbf{u}_k) \\ \boldsymbol{\beta}_n &\approx \boldsymbol{\alpha}_n \end{aligned} \tag{6–21}$$

We discuss this reduction in greater detail in Chapter 8. Since we are concerned in this chapter with the formulation of the governing equations, we will retain the nonlinear rotation terms. However, we will assume small strain, i.e., we work with (6–19), (6–20).

## 6–3. GENERAL ELONGATION–JOINT DISPLACEMENT RELATION

We have derived expressions for the direction cosines and elongation of a bar in terms of the initial coordinates and displacement components of the joints at the ends of the bar. By considering the truss as a system or network, the geometric relations for *all* the bars can be expressed as a single matrix equation. The relations for bar $n$, which is connected to joints $s$ and $k$ (positive direction from $k$ to $s$) are summarized below for convenience:

$$\begin{aligned} L_n^2 &= (\mathbf{x}_s - \mathbf{x}_k)^T(\mathbf{x}_s - \mathbf{x}_k) \\ \boldsymbol{\alpha}_n &= \frac{1}{L_n}(\mathbf{x}_s - \mathbf{x}_k)^T \\ e_n &= \boldsymbol{\gamma}_n(\mathbf{u}_s - \mathbf{u}_k) \\ \boldsymbol{\gamma}_n &= \boldsymbol{\alpha}_n + \frac{1}{2L_n}(\mathbf{u}_s - \mathbf{u}_k)^T \\ \boldsymbol{\beta} &= \boldsymbol{\alpha}_n + \frac{1}{L_n}(\mathbf{u}_s - \mathbf{u}_k)^T \end{aligned} \tag{a}$$

Up to this point, we have considered joints $s$ and $k$ as coinciding with the positive and negative ends of member $n$. Now we introduce new notation which is more convenient for generalization of the geometric relations. Let $n_+$, $n_-$ denote the joint numbers for the joints at the *positive* and *negative* ends of member $n$. The geometric relations take the form (we replace $s$ by $n_+$ and $k$ by $n_-$ in (a)):

$$\begin{aligned} L_n^2 &= (\mathbf{x}_{n_+} - \mathbf{x}_{n_-})^T(\mathbf{x}_{n_+} - \mathbf{x}_{n_-}) \\ \boldsymbol{\alpha}_n &= \frac{1}{L_n}(\mathbf{x}_{n_+} - \mathbf{x}_{n_-})^T \\ e_n &= \boldsymbol{\gamma}_n(\mathbf{u}_{n_+} - \mathbf{u}_{n_-}) \\ \boldsymbol{\gamma}_n &= \boldsymbol{\alpha}_n + \frac{1}{2L_n}(\mathbf{u}_{n_+} - \mathbf{u}_{n_-})^T \\ \boldsymbol{\beta}_n &= \boldsymbol{\alpha}_n + \frac{1}{L_n}(\mathbf{u}_{n_+} - \mathbf{u}_{n_-})^T \end{aligned} \tag{6–22}$$

To proceed further, we must relate the bars and joints of the system, that is, we must specify the *connectivity* of the truss. The *connectivity* can be defined by a table having $m$ rows and three columns. In the first column, we list the bar numbers in ascending order, and in the other two columns the corresponding numbers, $n_+$ and $n_-$, of the joints at the positive and negative ends of the members. This table is referred to as the *branch-node incidence table* in network theory.* For structural systems, a branch corresponds to a member and a node to a joint, and we shall refer to this table as the member-joint incidence table or simply as the connectivity table. It should be noted that the connectivity depends *only* on the numbering of the bars and joints, that is, it is independent of the initial geometry and distortion of the system.

**Example 6–1**

As an illustration, consider the two-dimensional truss shown. The positive directions of the bars are indicated by arrowheads and the bar numbers are encircled. The connectivity

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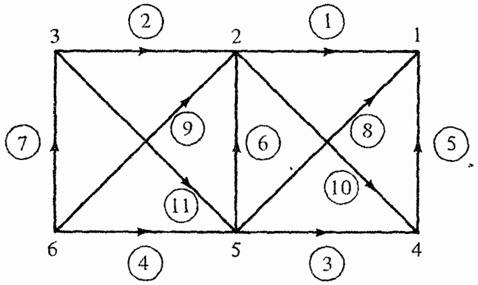
* See Ref. 8.

table (we list it horizontally to save space) for this numbering scheme takes the following form:

| Bar, $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| +Joint ($n_+$) | 1 | 2 | 4 | 5 | 1 | 2 | 3 | 1 | 2 | 4 | 5 |
| −Joint ($n_-$) | 2 | 3 | 5 | 6 | 4 | 5 | 6 | 5 | 6 | 2 | 3 |

**Fig. E6-1**



With the connectivity table, the evaluation of the initial length and direction cosines can be easily automated. The initial data consists of the $j$ coordinate matrices, $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_j$. To compute $L_n$ and $\boldsymbol{\alpha}_n$, we first determine $n_+$ and $n_-$ from the connectivity table and then use the first two equations of (6–22). For example, for bar 8, $8_+ = 1$, $8_- = 5$, and

$$\begin{aligned}\mathbf{x}_{8_+} - \mathbf{x}_{8_-} &= \mathbf{x}_1 - \mathbf{x}_5\\ L_8^2 &= (\mathbf{x}_1 - \mathbf{x}_5)^T(\mathbf{x}_1 - \mathbf{x}_5)\\ \boldsymbol{\alpha}_8 &= \frac{1}{L_8}(\mathbf{x}_1 - \mathbf{x}_5)^T\end{aligned}$$

We define $\mathbf{e}$ and $\mathscr{U}$ as the system elongation and joint-displacement matrices,

$$\begin{aligned}\mathbf{e} &= \{e_1, e_2, \ldots, e_m\}\\ \mathscr{U} &= \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_j\}\end{aligned} \tag{6–23}$$

and express the $m$ elongation-displacement relations as a single matrix equation

$$\overset{e}{\mathbf{u}} = \mathscr{A}\mathscr{U} \tag{6–24}$$

where $\mathscr{A}$ is of order $m \times ij$. The elements in the $n$th row of $\mathscr{A}$ involve only the elements of $\gamma_n$. Then, partitioning $\mathscr{A}$ into submatrices, $\mathscr{A}_{k\ell}$ of order $1 \times i$, where $k = 1, 2, \ldots, m$ and $\ell = 1, 2, \ldots, j$, it follows that the only nonvanishing submatrices for row $n$ are the *two* submatrices whose *column* number corresponds to the *joint* number at the positive or negative end of member $n$, namely, $n_+$ and $n_-$:

$$\begin{aligned}\mathscr{A}_{nn_+} &= +\gamma_n\\ \mathscr{A}_{nn_-} &= -\gamma_n\\ \mathscr{A}_{n\ell} = \mathbf{0} \quad &\text{when} \quad \ell \neq n_+ \text{ or } n_-\end{aligned} \tag{6–25}$$

**Example 6–2**

The $\mathscr{A}$ matrix can be readily established by using the connectivity table. For row $n$, one puts $+\gamma_n$ at column $n_+$, $-\gamma_n$ at column $n_-$, and null matrices at the other locations. The general form of the $\mathscr{A}$ matrix for the truss treated in Example 6–1 is listed below. We have also listed the elongations and joint displacement matrices to emphasize the significance of the rows and partitioned columns of $\mathscr{A}$.

$$\begin{array}{c|c|c|c|c|c|c|} & \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 & \mathbf{u}_4 & \mathbf{u}_5 & \mathbf{u}_6 \\ \hline e_1 & \gamma_1 & -\gamma_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline e_2 & \mathbf{0} & \gamma_2 & -\gamma_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline e_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \gamma_3 & -\gamma_3 & \mathbf{0} \\ \hline e_4 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \gamma_4 & -\gamma_4 \\ \hline e_5 & \gamma_5 & \mathbf{0} & \mathbf{0} & -\gamma_5 & \mathbf{0} & \mathbf{0} \\ \hline e_6 & \mathbf{0} & \gamma_6 & \mathbf{0} & \mathbf{0} & -\gamma_6 & \mathbf{0} \\ \hline e_7 & \mathbf{0} & \mathbf{0} & \gamma_7 & \mathbf{0} & \mathbf{0} & -\gamma_7 \\ \hline e_8 & \gamma_8 & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\gamma_8 & \mathbf{0} \\ \hline e_9 & \mathbf{0} & \gamma_9 & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\gamma_9 \\ \hline e_{10} & \mathbf{0} & -\gamma_{10} & \mathbf{0} & \gamma_{10} & \mathbf{0} & \mathbf{0} \\ \hline e_{11} & \mathbf{0} & \mathbf{0} & -\gamma_{11} & \mathbf{0} & \gamma_{11} & \mathbf{0} \\ \hline \end{array}$$

The $\mathscr{A}$ matrix depends on both the geometry and the topology. It is of interest to express $\mathscr{A}$ in a form where these two effects are segregated. The form of (6–25) suggests that we list the $\gamma$'s in a quasi-diagonal matrix,

$$\gamma = \begin{bmatrix} \gamma_1 & & & \\ & \gamma_2 & & \\ & & \ddots & \\ & & & \gamma_m \end{bmatrix} \tag{6–26}$$

and define $\mathbf{C}$ as

$$\begin{aligned} \mathbf{C} &= [\mathbf{C}_{k\ell}] \qquad & k &= 1, 2, \dots, m \\ & & \ell &= 1, 2, \dots, j \\ \mathbf{C}_{nn_+} &= +\mathbf{I}_i \qquad & \mathbf{C}_{nn_-} &= -\mathbf{I}_i \\ \mathbf{C}_{n\ell} &= 0 & \ell &\neq n_+ \text{ or } n_- \end{aligned} \tag{6-27}$$

Then,

$$\mathscr{A} = \gamma \mathbf{C} \tag{6-28}$$

The network terminology* for $\mathbf{C}$ is *augmented branch-node incidence matrix.* We shall refer to it simply as the connectivity matrix.

### Example 6–3

The connectivity matrix for Example 6–1 is listed below. The unit matrices are of order 2 since the system is two-dimensional.

| | Joint Numbers | | | | | |
|---|---|---|---|---|---|---|
| Bar Numbers | 1 | 2 | 3 | 4 | 5 | 6 |
| 1 | $+\mathbf{I}_2$ | $-\mathbf{I}_2$ | | | | |
| 2 | | $+\mathbf{I}_2$ | $-\mathbf{I}_2$ | | | |
| 3 | | | | $+\mathbf{I}_2$ | $-\mathbf{I}_2$ | |
| 4 | | | | | $+\mathbf{I}_2$ | $-\mathbf{I}_2$ |
| 5 | $+\mathbf{I}_2$ | | | $-\mathbf{I}_2$ | | |
| 6 | | $+\mathbf{I}_2$ | | | $-\mathbf{I}_2$ | |
| 7 | | | $+\mathbf{I}_2$ | | | $-\mathbf{I}_2$ |
| 8 | $+\mathbf{I}_2$ | | | | $-\mathbf{I}_2$ | |
| 9 | | $+\mathbf{I}_2$ | | | | $-\mathbf{I}_2$ |
| 10 | | $-\mathbf{I}_2$ | | $+\mathbf{I}_2$ | | |
| 11 | | | $-\mathbf{I}_2$ | | $+\mathbf{I}_2$ | |

One can consider row $n$ of $\mathbf{C}$ to define the two joints associated with bar $n$. It follows that *column* $k$ of $\mathbf{C}$ defines the *bars* associated with *joint* $k$. This association is usually referred to as *incidence.* We say that a joint is positive incident on a bar when it is at the *positive* end of the bar. Similarly, a bar is positive incident on a joint when its positive end is at the joint. For example, we see that joints 1 and 4 are incident on bar 5 and bars 3, 4, 6, 8, and 11 are incident on joint 5. We will use this property of the connectivity matrix later to generalize the joint force-equilibrium equations.

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* See Prob. 6–6. See also Ref. 8.

## 6–4. FORCE-ELONGATION RELATION FOR A BAR

By definition, each bar of an ideal truss is *prismatic* and subjected only to *axial* load applied at the centroid of the *end* cross sections. It follows that the only nonvanishing stress component is the axial stress, $\sigma$, and also, $\sigma$ is constant throughout the bar. We will consider each bar to be homogeneous but we will not require that all the bars be of the same material. The strain, $\varepsilon$, will be constant when the bar is homogeneous and the force-elongation relation will be similar in form to the uniaxial stress-strain curve for the material.

A typical $\sigma$-$\varepsilon$ curve is shown in Fig. 6–4. The initial portion of the curve is essentially straight for engineering materials such as steel and aluminum. A material is said to be elastic when the stress-strain curve is unique, that is, when the curves corresponding to increasing and decreasing $\sigma$ coincide (OAB and BAO in Fig. 6–4). If the behavior for decreasing $\sigma$ is different, the material is said to be inelastic. For ductile materials, the unloading curve (BC) is essentially parallel to the initial curve.*

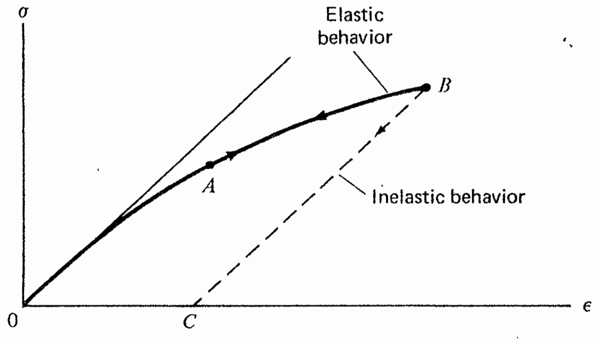

**Fig. 6–4.** Stress-strain curves for elastic and inelastic behavior.

We introduce the following notation:

$A$ = cross sectional area
$F$ = axial force, positive when tension
$e_0$ = initial elongation, i.e., elongation not associated with stress

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* A detailed discussion of the behavior of engineering materials is given in Chap. 5 of Ref. 2.

Since the stress and strain are constant throughout the bar,

$$\begin{aligned} F &= \sigma A \\ e &= L\varepsilon \\ e_0 &= L\varepsilon_0 \end{aligned} \tag{6–29}$$

We convert the $\sigma$-$\varepsilon$ relation for the material to the force-elongation relation for the bar by applying (6–29).

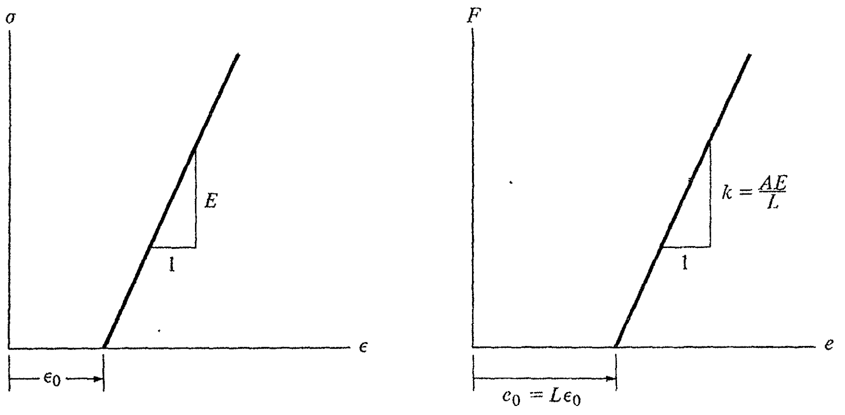


**Fig. 6–5.** Linear elastic behavior.

We consider first the case where the stress-strain relation is linear, as shown in Fig. 6–5. A material having this property is called *Hookean.* The initial and transformed relations are

$$\begin{aligned} \sigma &= E(\varepsilon - \varepsilon_0) \\ &\Downarrow \\ F &= \frac{AE}{L}(e - e_0) = k(e - e_0) \\ e &= \frac{L}{AE}F + e_0 = fF + e_0 \end{aligned} \tag{6–30}$$

We call $k, f$ the stiffness and flexibility factors for the bar. Physically, $k$ is the force required per unit elongation and $f$, which is the inverse of $k$, is the elongation due to a unit force.

We consider next the case where the stress-strain relation is approximated by a series of straight line segments. The material is said to be *piecewise linear.* Figure 6–6 shows this idealization for two segments. A superscript $(j)$ is used to identify the modulus and limiting stress for segment $j$. The force-elongation relation will still be linear, but now we have to determine what segment the deformation corresponds to and also whether the strain is increasing (loading) or decreasing (unloading). For unloading, the curve is assumed to be parallel to the initial segment.* The relations for the various possibilities are listed below.

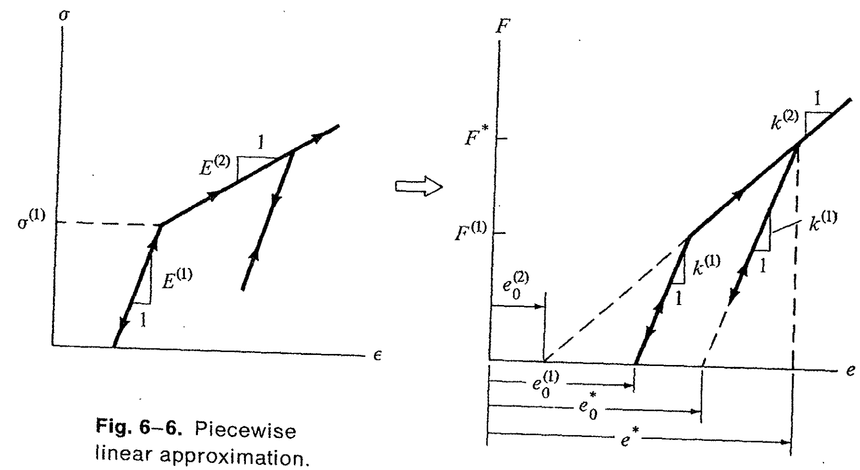


**Fig. 6–6.** Piecewise linear approximation.

### *1. Loading or Unloading—Initial Segment*

$$\begin{aligned} F &\leqslant A\sigma^{(1)} = F^{(1)} \\ F &= k^{(1)}(e - e_0^{(1)}) \end{aligned} \tag{6–31}$$

### *2. Loading—Second Segment*

$$\begin{aligned} F^{(1)} < F &\leqslant A\sigma^{(2)} = F^{(2)} \\ F &= k^{(2)}(e - e_0^{(2)}) \\ e_0^{(2)} &= e_0^{(1)} + (f^{(1)} - f^{(2)})F^{(1)} \end{aligned} \tag{6–32}$$

### *3. Unloading—Second Segment*

$$\begin{aligned} F^{(1)} < F^* &\leqslant F^{(2)} \\ F &= k^{(1)}(e - e_0^*) \\ e_0^* &= e^* - f^{(1)}F^* \end{aligned} \tag{6–33}$$

One can readily generalize these relations for the $n$th segment.†

---

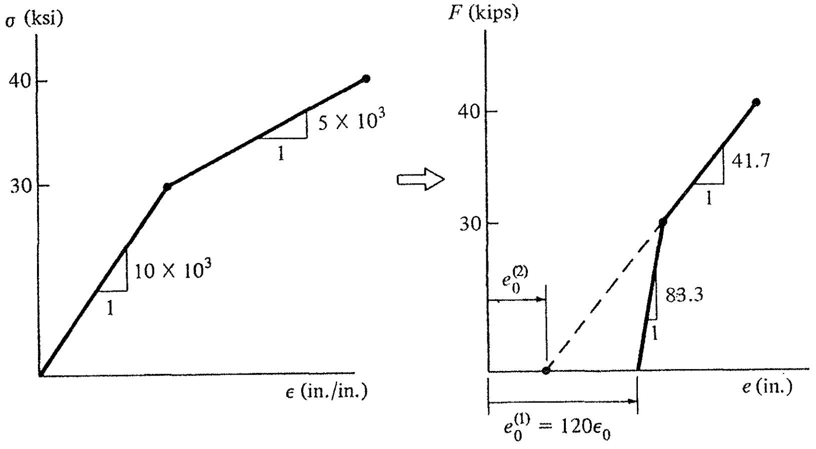
* We are neglecting the Bauschinger effect. See Ref. 2, Sec. 5.9, or Ref. 3, Art. 74.

† See Prob. 6–8.

**Example 6–4**

We consider a bilinear approximation, shown in Fig. E6–4.

Fig. E6–4



(1)

Taking

$L = 10$ ft $= 120$ in $\qquad A = 1$ in.²

we obtain

$$k^{(1)} = \frac{AE^{(1)}}{L} = 83.3 \text{ kips/in.} \qquad f^{(1)} = 1/k^{(1)} = 12 \times 10^{-3} \text{ in./kip}$$

$$k^{(2)} = \frac{AE^{(2)}}{L} = 41.7 \text{ kips/in.} \qquad f^{(2)} = 24 \times 10^{-3} \text{ in./kip}$$

$$F^{(1)} = A\sigma^{(1)} = 30 \text{ kips}$$

$$e_0^{(2)} = e_0^{(1)} + (f^{(1)} - f^{(2)})F^{(1)} = 120\,\varepsilon_0 - 0.36 \text{ in.}$$

Segment 1 $\qquad F = (83.3)(e - 120\,\varepsilon_0)$

Segment 2 $\qquad F = (41.7)(e - e_0^{(2)})$

(2)

Suppose a force of 35 kips is applied and the bar is unloaded. The equivalent initial strain is (see Equation 6–33 and Fig. 6–6):

$$e_0^* = e^* - f^{(1)}F^*$$
$$e_0^* = e_0^{(2)} + (f^{(2)} - f^{(1)})F^* = e_0^{(1)} + 0.06 \text{ in.}$$

The procedure described above utilizes the segment stiffness, which can be interpreted as an average tangent stiffness for the segment. We have to modify the stiffness and equivalent initial elongation only when the limit of the segment is reached. An alternate procedure is based on using the initial linear stiffness for all the segments. In what follows, we outline the *initial stiffness* approach.

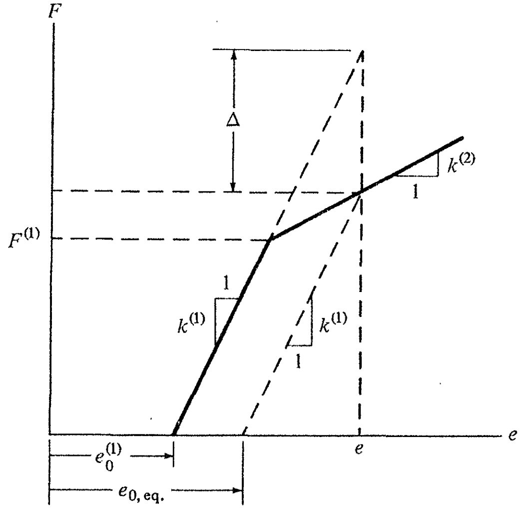


**Fig. 6–7.** Notation for the initial stiffness approach.

Consider Fig. 6–7. We write the force-elongation relation for segment 2 as

$$\begin{aligned} F &= k^{(1)}(e - e_0^{(1)}) - \Delta \\ &= k^{(1)}(e - e_{0,\text{eq}}) \end{aligned} \tag{6–34}$$

where $e_{0,\text{eq}}$ is interpreted as the equivalent linear initial strain and is given by

$$\begin{aligned} e_{0,\text{eq}} &= e_0^{(1)} + f^{(1)}\Delta \\ \Delta &= (k^{(1)} - k^{(2)})(e - e_0^{(1)} - f^{(1)}F^{(1)}) \end{aligned} \tag{6–35}$$

The equivalent initial strain, $e_{0,\text{eq}}$, depends on $e$, the actual strain. Since $e$ in turn depends on $F$, one has to iterate on $e_{0,\text{eq}}$ regardless of whether the segment limit has been exceeded. This disadvantage is offset somewhat by the use of $k^{(1)}$ for all the segments.

The notation introduced for the piecewise linear case is required in order to distinguish between the various segments and the two methods. Rather than continue with this detailed notation, which is too cumbersome, we will drop all the additional superscripts and write the force-deformation relations for bar $n$ in the simple linear form

$$\begin{aligned} F_n &= k_n(e_n - e_{0,n}) \\ e_n &= e_{0,n} + f_nF_n \end{aligned} \tag{6–36}$$

where $k$, $f$, and $e_0$ are defined by (6–31) through (6–35) for the physically nonlinear case.

## 6–5. GENERAL BAR FORCE–JOINT DISPLACEMENT RELATION

The force-deformation and deformation-displacement relations for bar $n$ are given by (6–22) and (6–36). Combining these two relations leads to an expression for the bar force in terms of the displacement matrices for the joints at the ends of the bar. The two forms are:

$$F_n = k_n(e_n - e_{0,n}) = F_{0,n} + k_n\gamma_n\mathbf{u}_{n_+} - k_n\gamma_n\mathbf{u}_{n_-} \qquad (6\text{–}37)$$
$$F_{0,n} = -k_n e_{0,n}$$

and

$$\gamma_n(\mathbf{u}_{n_+} - \mathbf{u}_{n_-}) = e_n = e_{0,n} + f_nF_n \qquad (6\text{–}38)$$

We can express the force-displacement relations for the "$m$" bars as a single matrix equation by defining

$$\mathbf{F} = \{F_1F_2\cdots F_m\} \qquad (6\text{–}39)$$
$$\mathbf{k} = \begin{bmatrix} k_1 & & & \\ & k_2 & & \\ & & \ddots & \\ & & & k_m \end{bmatrix} = \mathbf{f}^{-1}$$

and noting (6–24). The generalized forms of (6–37) and (6–38) are:

$$\mathbf{F} = \mathbf{k}(\mathbf{e} - \mathbf{e}_0) = \mathbf{F}_0 + \mathbf{k}\mathscr{A}\mathscr{U} \qquad (6\text{–}40)$$

and

$$\mathscr{A}\mathscr{U} = \mathbf{e}_0 + \mathbf{fF} \qquad (6\text{–}41)$$

## 6–6. JOINT FORCE-EQUILIBRIUM EQUATIONS

Let $\vec{F}_n$ be the axial force vector for bar $n$ (see Fig. 6–8). The force vector has the direction of the unit vector, $\vec{v}_n$, which defines the orientation of the bar in the deformed state. Now, $\vec{v}_n = \boldsymbol{\beta}_n\mathbf{i}$. Then,

$$\vec{F}_n = F_n\vec{v}_n = F_n\boldsymbol{\beta}_n\mathbf{i} \qquad (6\text{–}42)$$

When $F_n$ is positive, the sense of $\vec{F}_n$ is the same as the positive sense for the bar. Continuing, we define $\vec{F}_{nn_+}$ and $\vec{F}_{nn_-}$ as the forces exerted *by* bar $n$ *on* the joints at the positive and negative ends of the bar. From Fig. 6–8,

$$\begin{aligned}\vec{F}_{nn_+} &= -\vec{F}_n = -F_n\boldsymbol{\beta}_n\mathbf{i}\\ \vec{F}_{nn_-} &= +\vec{F}_n = +F_n\boldsymbol{\beta}_n\mathbf{i}\end{aligned} \qquad (6\text{–}43)$$

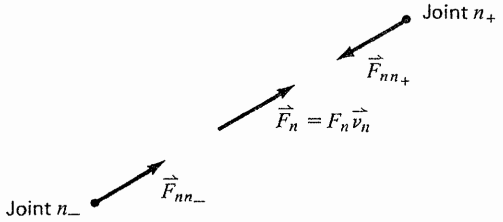


**Fig. 6–8.** Notation for barforce.

We consider next joint $k$. The external joint load vector is $\vec{p}_k$, where $\vec{p}_k = \mathbf{p}_k^T\mathbf{i}$. For equilibrium, the resultant force vector must equal zero. Then,

$$\vec{p}_k = -\sum_{j_+=k}\vec{F}_{jj_+} - \sum_{\ell_-=k}\vec{F}_{\ell\ell_-} \qquad (a)$$

The first summation involves the bars which are positive incident on joint $k$ (positive end at joint $k$) and the second the bars which are negative incident. Using (6–43), the matrix equilibrium equation for joint $k$ takes the form:

$$\mathbf{p}_k = \sum_{j_+=k}\vec{F}_j(\boldsymbol{\beta}_j^T) - \sum_{\ell_-=k}\vec{F}_\ell(\boldsymbol{\beta}_\ell^T) \qquad (6\text{–}44)$$

Let $\mathscr{P}$ be the general external joint load matrix:

$$\mathscr{P} = \{\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_j\}\ (ij \times 1) \qquad (6\text{–}45)$$

We write the complete set of joint force-equilibrium equations as:

$$\mathscr{P} = \mathscr{B}\mathbf{F} \qquad (6\text{–}46)$$

Note that the rows of $\mathscr{B}$ pertain to the *joints* and the columns to the *bars*. We partition $\mathscr{B}$ into submatrices of order $i \times 1$.

$$\begin{gathered}\mathscr{B} = [\mathscr{B}_{\ell k}] \qquad (ij \times m)\\ \ell = 1, 2, \ldots, j \qquad \text{and} \qquad k = 1, 2, \ldots, m\end{gathered} \qquad (6\text{–}47)$$

Since a bar is incident only on *two* joints, there will be only *two* elements in any column of $\mathscr{B}$. From (6–44), we see that, for column $n$,

$$\begin{aligned}\mathscr{B}_{n_+n} &= +\boldsymbol{\beta}_n^T\\ \mathscr{B}_{n_-n} &= -\boldsymbol{\beta}_n^T\\ \mathscr{B}_{\ell n} = \mathbf{0} &\qquad \text{when} \qquad \ell \neq n_+ \text{ or } n_-\end{aligned} \qquad (6\text{–}48)$$

The $\mathscr{B}$ matrix can be readily developed using the connectivity table. It will have the same form as $\mathscr{A}^T$ with $\gamma_n$ replaced by $\boldsymbol{\beta}_n$. When the geometry is linear, $\boldsymbol{\beta}_n = \boldsymbol{\gamma}_n = \boldsymbol{\alpha}_n$, and $\mathscr{B} = \mathscr{A}^T$.

**Example 6–5**

The $\mathscr{B}$ matrix for the truss of Example 6–1 has the following general form:

Bar Numbers

| Joint Numbers | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | $+\boldsymbol{\beta}_1^T$ | | | | $+\boldsymbol{\beta}_5^T$ | | | $+\boldsymbol{\beta}_8^T$ | | | |
| 2 | $-\boldsymbol{\beta}_1^T$ | $+\boldsymbol{\beta}_2^T$ | | | | $+\boldsymbol{\beta}_6^T$ | | | $+\boldsymbol{\beta}_9^T$ | $-\boldsymbol{\beta}_{10}^T$ | |
| 3 | | $-\boldsymbol{\beta}_2^T$ | | | | | $+\boldsymbol{\beta}_7^T$ | | | | $-\boldsymbol{\beta}_{11}^T$ |
| 4 | | | $+\boldsymbol{\beta}_3^T$ | | $-\boldsymbol{\beta}_5^T$ | | | | | $+\boldsymbol{\beta}_{10}^T$ | |
| 5 | | | $-\boldsymbol{\beta}_3^T$ | $+\boldsymbol{\beta}_4^T$ | | $-\boldsymbol{\beta}_6^T$ | | $-\boldsymbol{\beta}_8^T$ | | | $+\boldsymbol{\beta}_{11}^T$ |
| 6 | | | | $-\boldsymbol{\beta}_4^T$ | | | $-\boldsymbol{\beta}_7^T$ | | $-\boldsymbol{\beta}_9^T$ | | |

---

We could have also utilized the connectivity* matrix $\mathbf{C}$ to develop $\mathscr{B}$. It was pointed out in Example 6–3 that the elements of the $k$th column of $\mathbf{C}$ define the incidence of the bars on joint $k$. Using this property, we can write the generalized form of (6–44) as

$$\mathscr{P} = \mathbf{C}^T\boldsymbol{\beta}^T\mathbf{F} \tag{a}$$

where

$$\boldsymbol{\beta} = \begin{bmatrix} \boldsymbol{\beta}_1 & 0 & \cdots & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\beta}_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \boldsymbol{\beta}_m \end{bmatrix} \quad (m \times im) \tag{6–49}$$

Finally, we have

$$\mathscr{B} = \mathbf{C}^T\boldsymbol{\beta}^T = (\boldsymbol{\beta}\mathbf{C})^T \tag{6–50}$$

## 6–7. INTRODUCTION OF DISPLACEMENT RESTRAINTS; GOVERNING EQUATIONS

We have developed the following equations relating $\mathbf{F}$, $\mathbf{e}$, $\mathscr{P}$, and $\mathscr{U}$,

$$\begin{aligned} \mathbf{e} &= \mathscr{A}\mathscr{U} = \mathbf{e}_0 + \mathbf{fF} \\ \mathscr{P} &= \mathscr{B}\mathbf{F} \end{aligned} \tag{a}$$

where the elements of $\mathscr{U}$ and $\mathscr{P}$ are the external joint-displacement and external joint-load matrices arranged in ascending order. Also, in our derivation, we have considered the components to be referred to a *basic* reference frame. Now, when joint displacement restraints are imposed, there will be a reduction in the number of joint displacement unknowns and a corresponding increase in the number of force unknowns. This will require a rearrangement of $\mathscr{U}$, $\mathscr{A}$, $\mathscr{P}$ and $\mathscr{B}$.

Let $r$ be the number of displacement restraints and $n_d$ the number of displacement unknowns. There will be $n_d$ *prescribed* joint loads and $r$ *unknown* joint loads (usually called reactions) corresponding to the $n_d$ *unknown* joint displacements and the $r$ *known* joint displacements. We let $\mathbf{U}_1$, $\bar{\mathbf{U}}_2$ be the column matrices of *unknown* and *prescribed* joint displacement components and $\bar{\mathbf{P}}_1$, $\mathbf{P}_2$ the corresponding *prescribed* and *unknown* joint load matrices. The rearranged system joint displacement and joint load matrices are written as $\mathbf{U}$, $\mathbf{P}$:

$$\begin{aligned} \mathbf{U} &= \begin{Bmatrix} \mathbf{U}_1 \\ \bar{\mathbf{U}}_2 \end{Bmatrix} \begin{matrix} (n_d \times 1) \\ (r \times 1) \end{matrix} \\ \mathbf{P} &= \begin{Bmatrix} \bar{\mathbf{P}}_1 \\ \mathbf{P}_2 \end{Bmatrix} \begin{matrix} (n_d \times 1) \\ (r \times 1) \end{matrix} \\ n_d + r &= ij \end{aligned} \tag{6–51}$$

We point out that the components contained in $\mathbf{U}$ (and $\mathbf{P}$) may be referred to local reference frames at the various joints rather than to the basic frames. This is necessary when the restraint direction at a joint does not coincide with one of the directions of the basic frame. Finally, we let $\mathbf{A}$ and $\mathbf{B}$ be the transformation matrices associated with $\mathbf{U}$ and $\mathbf{P}$. Then, (a) takes the form:

$$\begin{aligned} \mathbf{e} &= \mathscr{A}\mathscr{U} = \mathbf{AU} = \mathbf{e}_0 + \mathbf{fF} \\ \mathbf{P} &= \mathbf{BF} \end{aligned} \tag{b}$$

We partition $\mathbf{A}$, $\mathbf{B}$ consistent with the partitioning of $\mathbf{U}$, $\mathbf{P}$:

$$\begin{aligned} \mathbf{A} &\Rightarrow \Big[ \underset{(m \times n_d)}{\mathbf{A}_1} \;\Big|\; \underset{(m \times r)}{\mathbf{A}_2} \Big] \\ \mathbf{B} &\Rightarrow \left[ \begin{array}{c} \mathbf{B}_1 \\ \hline \mathbf{B}_2 \end{array} \right] \begin{matrix} (n_d \times m) \\ (r \times m) \end{matrix} \end{aligned} \tag{6–52}$$

and write (b) in expanded form:

$$\mathbf{e} = \mathbf{A}_1\mathbf{U}_1 + \mathbf{A}_2\bar{\mathbf{U}}_2 = \mathbf{e}_0 + \mathbf{fF} \tag{6–53}$$

$$\bar{\mathbf{P}}_1 = \mathbf{B}_1\mathbf{F} \tag{6–54}$$

$$\mathbf{P}_2 = \mathbf{B}_2\mathbf{F} \tag{6–55}$$

Equation (6–53) represents $m$ equations relating the $m$ unknown bar forces, the $n_d$ unknown displacements, and the $r$ prescribed displacements. Equation (6–54) represents $n_d$ equations involving the $m$ unknown bar forces and the $n_d$ prescribed joint loads. Lastly, Equation (6–55) represents $r$ equations for the $r$ reactions in terms of the $m$ bar forces. When the geometry is nonlinear, $\mathbf{A}$ and $\mathbf{B}$ involve the joint displacements. If the geometry is linear, $\mathbf{A} = \mathbf{B}^T$, and

$$\mathbf{B}_j = \mathbf{A}_j^T \qquad j = 1, 2 \tag{6–56}$$

---

* See Sec. 6–3, Eq. 6–27.

We have introduced the displacement restraints into the formulation by replacing $\mathscr{A}$, $\mathscr{B}$ with **A**, **B**. It remains to discuss how one determines **A**, **B** from $\mathscr{A}$, $\mathscr{B}$. In the following section, we treat the case of an arbitrary restraint direction. We also describe how one can represent the introduction of displacement restraints as a matrix transformation.

## 6–8. ARBITRARY RESTRAINT DIRECTION

When all the restraint directions are parallel to the direction of the global reference frame, we obtain **U** from $\mathscr{U}$ by simply rearranging the *rows* of $\mathscr{U}$ such that the elements in the first $n_d$ rows are the unknown displacements and the last $r$ rows contain the prescribed displacements. To obtain **A**, we perform the same operations on the *columns* of $\mathscr{A}$. Finally, since **P** corresponds to **U**, we obtain **B** by operating on the *rows* of $\mathscr{B}$ or alternately, by operating on the *columns* of $\mathscr{B}^T$ and then transposing the resulting matrix.

When the restraint at a joint does not coincide with one of the directions of the basic frame, it is necessary first to transform the joint displacement and external load components from the *basic* frame to a *local* frame associated with the restraint at the joint. Suppose there is a displacement restraint at joint $k$. Let $Y_j^k$ ($j = 1, 2, 3$) be the orthogonal directions for the local reference frame associated with the displacement restraint at joint $k$. Also, let $u_{kj}^k$ and $p_{kj}^k$ be the corresponding displacement and external joint load components. Finally, let $\mathbf{R}^{ok}$ be the rotation transformation matrix for the *local* frame at joint $k$ with respect to the *basic* frame (frame $o$). The components are related by:

$$\begin{aligned}\mathbf{u}_k^k &= \mathbf{R}^{ok}\mathbf{u}_k \\ \mathbf{p}_k^k &= \mathbf{R}^{ok}\mathbf{p}_k\end{aligned} \tag{6–57}$$

where

$$\mathbf{R}^{ok} = [\cos{(Y_i^k, X_j)}] \tag{6–58}$$

We have omitted the frame superscript ($o$) for quantities referred to the basic frame ($u_{kj}^o$, $p_{kj}^o$) to simplify the notation.

We define $\mathscr{U}^J$, $\mathscr{P}^J$ as the system joint-displacement and -force matrices referred to the local joint reference frames,

$$\begin{aligned}\mathscr{U}^J &= \{u_1^1, u_2^2, \ldots, u_j^j\} \\ \mathscr{P}^J &= \{p_1^1, p_2^2, \ldots, p_j^j\}\end{aligned} \tag{6–59}$$

and $\mathscr{R}^{oJ}$ as the system joint-rotation matrix,

$$\mathscr{R}^{oJ} = \begin{bmatrix} \mathbf{R}^{o1} & & & \\ & \mathbf{R}^{o2} & & \\ & & \ddots & \\ & & & \mathbf{R}^{oj} \end{bmatrix} \tag{6–60}$$

Then,

$$\begin{aligned}\mathscr{U} &= (\mathscr{R}^{oJ})^T\mathscr{U}^J \\ \mathscr{P}^J &= \mathscr{R}^{oJ}\mathscr{P}\end{aligned} \tag{a}$$

Operating on the initial equations with (a),

$$\begin{aligned}(\mathbf{e} = \mathscr{A}\mathscr{U}) &\Rightarrow (\mathbf{e} = \mathscr{A}^J\mathscr{U}^J) \\ (\mathscr{P} = \mathscr{B}\mathbf{F}) &\Rightarrow (\mathscr{P}^J = \mathscr{B}^J\mathbf{F})\end{aligned} \tag{b}$$

leads to

$$\mathscr{B}^J = \mathscr{R}^{oJ}\mathscr{B} \qquad \mathscr{A}^J = \mathscr{A}(\mathscr{R}^{oJ})^T \tag{6–61}$$

The transformation of $\mathscr{B}^J$ to $\mathscr{B}$ is the same as for the case where the restraint directions are parallel to the directions of the basic frame, that is, it will involve only a rearrangement of the rows of $\mathscr{B}^J$. Similarly, we obtain **A** by rearranging the columns of $\mathscr{A}^J$. The steps are

$$\begin{aligned}&\mathscr{A} \to \mathscr{A}^J \to \mathbf{A} \to [\mathbf{A}_1 \mid \mathbf{A}_2] \\ &\mathscr{B} \to \mathscr{B}^J \to \mathbf{B} \to \begin{bmatrix}\mathbf{B}_1 \\ \hline \mathbf{B}_2\end{bmatrix}\end{aligned}$$

**Example 6–6**

To obtain the submatrices in column $k$ of $\mathscr{A}^J$ we postmultiply the submatrices in column $k$ of $\mathscr{A}$ by $\mathbf{R}^{ok,T}$. We can perform the same operation on $\mathscr{B}^T$ and then transpose the resulting matrix or, alternately, we can premultiply the submatrices in row $k$ of $\mathscr{B}$ by $\mathbf{R}^{ok}$. As an illustration, see the $\mathscr{B}^J$ matrix for Example 6–5 on page 136. The $\mathscr{A}^J$ matrix can be determined by transposing $\mathscr{B}^J$ and replacing $\boldsymbol{\beta}_n$ by $\gamma_n$.

One can visualize the introduction of displacement restraints as a matrix transformation. We represent the operations

$$\mathscr{U} \to \mathbf{U} \qquad \text{and} \qquad \mathscr{P} \to \mathbf{P} \tag{6–62}$$

as

$$\mathbf{U} = \mathbf{D}\mathscr{U} \qquad \mathbf{P} = \mathbf{D}\mathscr{P}$$

and call **D** the displacement-restraint transformation matrix.

When the restraint directions are parallel to the directions of the basic frame, **D** is a permutation matrix which rearranges the rows of $\mathscr{U}$. We obtain **D** by applying the *same* row rearrangement to a unit matrix of order $ij$. Postmultiplication by $\mathbf{D}^T$ effects the same rearrangements on the columns. Also,* $\mathbf{D}^T = \mathbf{D}^{-1}$.

For the general case of arbitrary restraint directions, we first determine $\mathscr{U}^J$ and then **U**. Now,

$$\mathscr{U}^J = \mathscr{R}^{oJ}\mathscr{U} \tag{a}$$

The step, $\mathscr{U}^J \to \mathbf{U}$, involves only a permutation of the rows of $\mathbf{U}^J$ and can be represented as

$$\mathbf{U} = \boldsymbol{\Pi}\mathscr{U}^J \tag{6–63}$$

where $\boldsymbol{\Pi}$ is the permutation matrix corresponding to the displacement restraints.

---

* See Prob. 1–36 for a discussion of permutation matrices.

$\mathscr{B}^J$ **Matrix for Example 6–6**

| $\mathscr{P}^J$ | $F_1$ | $F_2$ | $F_3$ | $F_4$ | $F_5$ | $F_6$ | $F_7$ | $F_8$ | $F_9$ | $F_{10}$ | $F_{11}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $p_1^1$ | $\mathbf{R}^{o1}\boldsymbol{\beta}_1^T$ | | | | $\mathbf{R}^{o1}\boldsymbol{\beta}_5^T$ | | | $\mathbf{R}^{o1}\boldsymbol{\beta}_8^T$ | | | |
| $p_2^2$ | $-\mathbf{R}^{o2}\boldsymbol{\beta}_1^T$ | $\mathbf{R}^{o2}\boldsymbol{\beta}_2^T$ | | | | $\mathbf{R}^{o2}\boldsymbol{\beta}_6^T$ | | | $\mathbf{R}^{o2}\boldsymbol{\beta}_9^T$ | $-\mathbf{R}^{o2}\boldsymbol{\beta}_{10}^T$ | |
| $p_3^3$ | | $-\mathbf{R}^{o3}\boldsymbol{\beta}_2^T$ | | | | | $\mathbf{R}^{o3}\boldsymbol{\beta}_7^T$ | | | | $-\mathbf{R}^{o3}\boldsymbol{\beta}_{11}^T$ |
| $p_4^4$ | | | $\mathbf{R}^{o4}\boldsymbol{\beta}_3^T$ | | $-\mathbf{R}^{o4}\boldsymbol{\beta}_5^T$ | | | | | $\mathbf{R}^{o4}\boldsymbol{\beta}_{10}^T$ | |
| $p_5^5$ | | | $-\mathbf{R}^{o5}\boldsymbol{\beta}_3^T$ | $\mathbf{R}^{o5}\boldsymbol{\beta}_4^T$ | | $-\mathbf{R}^{o5}\boldsymbol{\beta}_6^T$ | | $-\mathbf{R}^{o5}\boldsymbol{\beta}_8^T$ | | | $\mathbf{R}^{o5}\boldsymbol{\beta}_{11}^T$ |
| $p_6^6$ | | | | $-\mathbf{R}^{o6}\boldsymbol{\beta}_4^T$ | | | $-\mathbf{R}^{o6}\boldsymbol{\beta}_7^T$ | | $-\mathbf{R}^{o6}\boldsymbol{\beta}_9^T$ | | |

Combining (a) and (6–63), we have

$$\mathbf{U} = \boldsymbol{\Pi}\mathscr{R}^{oJ}\mathscr{U} \tag{b}$$

and it follows that

$$\mathbf{D} = \boldsymbol{\Pi}\mathscr{R}^{oJ} \tag{6–64}$$

Since both $\boldsymbol{\Pi}$ and $\mathscr{R}^{oJ}$ are orthogonal matrices, $\mathbf{D}$ is also an orthogonal matrix. Using (6–62),

$$\begin{aligned} \mathbf{B} &= \mathbf{D}\mathscr{B} \\ \mathbf{A} &= \mathscr{A}\mathbf{D}^T \end{aligned} \tag{c}$$

and then substituting for $\mathscr{A}$, $\mathscr{B}$, and $\mathbf{D}$ in terms of the geometrical, connectivity, local rotation matrices lead to

$$\begin{aligned} \mathbf{B} &= \boldsymbol{\Pi}\mathscr{R}^{oJ}(\boldsymbol{\beta}\mathbf{C})^T \\ \mathbf{A} &= \boldsymbol{\gamma}\mathbf{C}(\boldsymbol{\Pi}\mathscr{R}^{oJ})^T \end{aligned} \tag{6–65}$$

Equation (6–65) is of interest since the various terms are isolated. However, one would not generate $\mathbf{A}$, $\mathbf{B}$ with it.

## 6–9. INITIAL INSTABILITY

The force equilibrium equations relating the prescribed external joint forces and the (internal) bar forces has been expressed as (see Equation 6–54):

$$\bar{\mathbf{P}}_1 = \mathbf{B}_1\mathbf{F} \tag{a}$$

where $\bar{\mathbf{P}}_1$ is $(n_d \times 1)$ and $\mathbf{F}$ is $(m \times 1)$. When the geometry is nonlinear, $\mathbf{B}_1$ depends on the joint displacements as well as on the initial geometry and restraint directions. In this section, we are concerned with the behavior under an infinitesimal loading. Since the nonlinear terms depend on the load intensity, they will be negligible in comparison to the linear terms for this case, i.e., we take $\mathbf{B}_1$ as constant. Then, (a) represents $n_d$ linear equations in $m$ unknowns. If these equations are inconsistent for an arbitrary infinitesimal loading, we say the system is initially unstable.

When the geometry is linear, $\mathbf{B}_1$ is independent of the loading and the initial stability criterion is also applicable for a finite loading. This is not true for a nonlinear system. We treat stability under a finite loading in Chapter 7.

Consider a set of $j$ linear algebraic equations in $k$ unknowns.

$$\mathbf{ax} = \mathbf{c} \tag{b}$$

In general, (b) can be solved only if $\mathbf{a}$ and $[\mathbf{a} \mid \mathbf{c}]$ have the same rank.* It follows that the equations are consistent for an arbitrary right-hand side only when the rank of $\mathbf{a}$ is equal to $j$, the total number of equations. Applying this condition

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* See Sec. 1–13; see also Prob. 1–45.

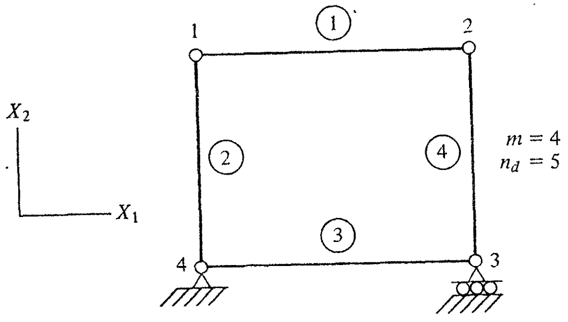
to (a), we see that the truss is initially unstable when the rank of $\mathbf{B}_1$ is less than $n_d$.

For the truss to be initially stable under an arbitrary loading, $\mathbf{B}_1$ must be of rank $n_d$. This requires $m \geqslant n_d$. That is, the number of bars must be at least equal to the number of unknown displacement components. Since the rank may still be less than $n_d$, this condition is necessary but not sufficient for initial stability. In order to determine whether a truss is initially stable, one must actually find the rank of $\mathbf{B}_1$. The following examples illustrate various cases of initial instability.

### Example 6–7

The force-equilibrium equations for the accompanying sketch are:

Fig. E6–7



**F**

| $\mathbf{B}_1$ | $F_1$ | $F_2$ | $F_3$ | $F_4$ |
|---|---|---|---|---|
| $p_{11}$ | $-1$ | | | |
| $p_{12}$ | | $+1$ | | |
| $\bar{\mathbf{P}}_1$ $p_{21}$ | $+1$ | | | |
| $p_{22}$ | | | | $+1$ |
| $p_{31}$ | | | $+1$ | |

Row 3 is $(-1)$ times row 1. The equations are consistent only if $p_{21} = -p_{11}$. Since $m < n_d$, we know the system is unstable for an arbitrary loading without actually finding $r(\mathbf{B}_1)$.

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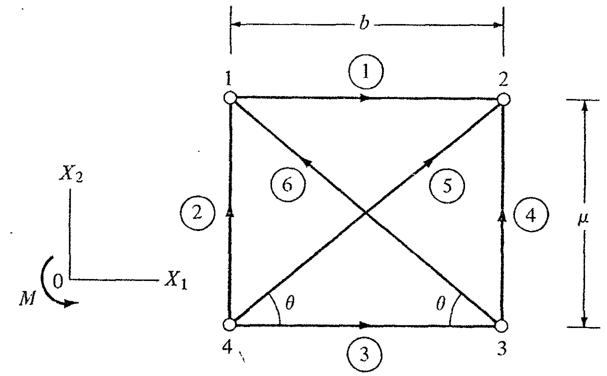
### Example 6–8

We first develop the $\mathscr{B}$ matrix for the truss shown in Fig. E6–8A and then specialize it for various restraint conditions.

Fig. E6–8A



**F**

| $\mathscr{B}$ | | $F_1$ | $F_2$ | $F_3$ | $F_4$ | $F_5$ | $F_6$ |
|---|---|---|---|---|---|---|---|
| ① | $p_{11}$ | $-1$ | | | | | $-\cos\theta$ |
| ② | $p_{12}$ | | $+1$ | | | | $\sin\theta$ |
| ③ | $p_{21}$ | $+1$ | | | | $\cos\theta$ | |
| ④ | $p_{22}$ | | | | $+1$ | $\sin\theta$ | |
| $\mathscr{P}$ ⑤ | $p_{31}$ | | | $+1$ | | | $\cos\theta$ |
| ⑥ | $p_{32}$ | | | | $-1$ | | $-\sin\theta$ |
| ⑦ | $p_{41}$ | | | $-1$ | | $-\cos\theta$ | |
| ⑧ | $p_{42}$ | | $-1$ | | | $-\sin\theta$ | |

There are three relations between the rows of $\mathscr{B}$:

(1) row ① + row ③ + row ⑤ = −row ⑦
(2) row ② + row ④ + row ⑥ = −row ⑧
(3) $(\sin\theta)$(row ① + row ③) − $\cos\theta$ (row ④) = $\cos\theta$ (row ⑥)

The first two relations correspond to the scalar force equilibrium conditions for the external joint loads:

$$\begin{aligned} \sum_{k=1}^{4} p_{k1} &= p_{11} + p_{21} + p_{31} + p_{41} = 0 \\ \sum_{k=1}^{4} p_{k2} &= p_{12} + p_{22} + p_{32} + p_{42} = 0 \end{aligned} \qquad \text{(a)}$$

The third relation corresponds to the scalar moment equilibrium condition:

$$\sum_{k=1}^{4} M_k = \sum_{k=1}^{4} (-x_{k2}p_{k1} + x_{k1}p_{k2}) = 0 \qquad \text{(b)}$$

where $M_k$ is the moment of the external force vector acting at joint $k$ with respect to point $O$, the origin of the basic frame. We obtain relation (3) by taking $O$ at joint 4. Equation (b) reduces to

$$-d(p_{11} + p_{21}) + b(p_{22} + p_{32}) = 0 \qquad \text{(c)}$$

Using

$$\begin{aligned} d &= L \sin\theta \\ b &= L \cos\theta \end{aligned} \qquad \text{(d)}$$

we can write (c) as

$$\cos\theta\, p_{32} = \sin\theta(p_{11} + p_{21}) - \cos\theta\, p_{22} \qquad \text{(e)}$$

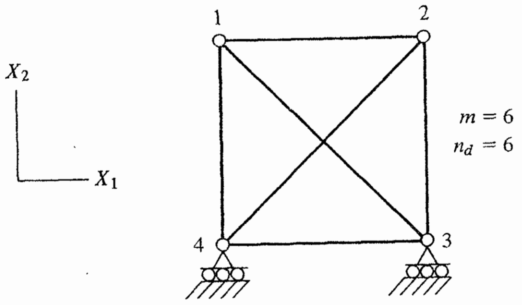
which is relation 3.

We see that rows 2 and 5 are independent. The remaining set (rows 1, 3, 4, 6, 7, 8) contains only three independent rows. Now, we obtain $B_1$ from $\mathcal{B}$ by first taking a linear combination of the rows (when the restraints are not parallel to the basic frame) and then deleting the rows corresponding to the joint forces associated with the prescribed joint displacements. Since $\mathcal{B}$ has *three* linear dependent rows, it follows that we must introduce at least *three* restraints. Initial instability will occur if—

1. An insufficient number of restraints are introduced ($n_d > 5$).
2. A sufficient number of restraints are introduced ($n_d = 5$) but the rows of $\mathbf{B}_1$ are not linearly independent. We say the restraints are *not* independent in this case. These cases are illustrated below.

*Case 1*

**Fig. E6–8B**

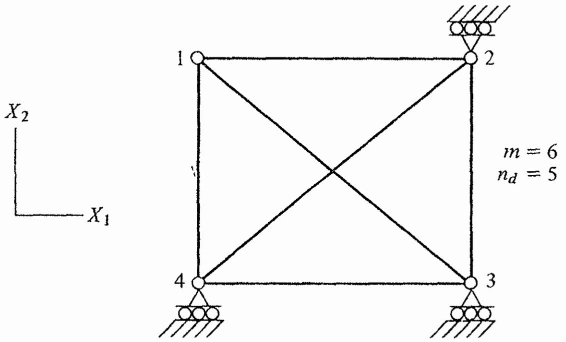


We obtain $\mathbf{B}_1$ by deleting rows 6 and 8 (corresponding to $p_{32}$ and $p_{42}$). The system is stable only when the applied joint loads satisfy the condition

$$p_{11} + p_{21} + p_{31} = -p_{41}$$

*Case 2*

**Fig. E6–8C**



We delete rows 4, 6, and 8. The number of restraints is sufficient ($n_d = 5$) but the restraints are not independent since $r(\mathbf{B}_1) < 5$. Actually, $r(\mathbf{B}_1) = 4$. To make the system stable, at least one horizontal restraint must be introduced.

---

In Example 6–8, we showed that there are three relations between the rows of $\mathcal{B}$ for a two-dimensional truss. These relations correspond to the force- and moment-equilibrium conditions for the complete truss.

To establish the relations for the three-dimensional case, we start with the equilibrium equations,

$$\sum_{\ell=1}^{j} \mathbf{p}_\ell = \overset{(i\times 1)}{\mathbf{0}} \qquad \text{(a)}$$

$$\sum_{\ell=1}^{j} \mathbf{M}_\ell = \overset{(2i-3)\times 1}{\mathbf{0}} \qquad \text{(b)}$$

where $\mathbf{M}_\ell$ is the moment of $\mathbf{p}_\ell$ with respect to an arbitrary moment center, 0. For convenience, we take 0 at the origin of the basic reference frame. Partitioning $\mathcal{B}$,

$$\mathcal{B} \Rightarrow \left\{\begin{array}{c} \mathcal{B}_1 \\ \hdashline \mathcal{B}_2 \\ \hdashline \vdots \\ \hdashline \mathcal{B}_j \end{array}\right\} \qquad (6\text{–}66)$$

where $\mathcal{B}_k$ is of order $(i \times m)$ and using the matrix notation introduced in

Sec. 5–2 for the moment,* the equilibrium equations take the form

$$\sum_{\ell=1}^{j} \mathscr{B}_\ell = \mathbf{0} \tag{6–67}$$

$$\sum_{\ell=1}^{j} \mathbf{X}_{\ell 0}\mathscr{B}_\ell = \mathbf{0} \tag{6–68}$$

Equation (6–67) represents $i$ relations between the rows of $\mathscr{B}$,

$$\text{row } q + \text{row } (q + i) + \cdots + \text{row } [i(j-2) + q] = \text{row } [i(j-1) + q] \qquad q = 1, 2, \ldots i \tag{6–69}$$

and (6–68) corresponds to $(2i - 3)$ relations.

We have shown that there are at least $3(i - 1)$ relations between the rows of $\mathscr{B}$. Now, we obtain $\mathbf{B}$ by combining and rearranging the rows of $\mathscr{B}$. It follows that $\mathbf{B}$ will also have at least $3(i - 1)$ relations between its rows. Finally, we obtain $\mathbf{B}_i$ by deleting the rows corresponding to the restraints. For the system to be initially stable, we must introduce at least $3(i - 1)$ restraints:

$$r = \text{no. of restraints} \geqslant 3(i - 1) \tag{6–70}$$

Note that this requirement is independent of the number of bars. Also, it is a necessary but not sufficient condition for initial stability.

The number of restraints must also satisfy the necessary condition $n_d \leqslant m$. This requires

$$r = (ij - n_d) \geqslant (ij - m) \tag{6–71}$$

Both (6–70) and (6–71) *must be satisfied.* Either condition may control $r$, depending on the arrangement of the bars.

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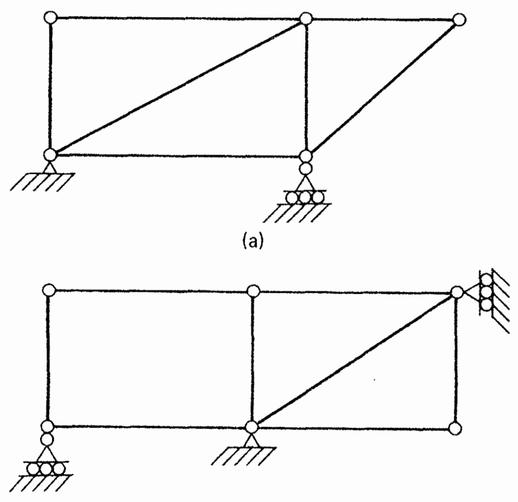
* See Eq. 5–11.

## PROBLEMS

**6–1.** Determine $m, j, r$, and $n_d$ for the following plane trusses:

**Prob. 6–1**



(a)

(b)

**6–2.** Suppose bar $n$ is connected to joints $s$ and $k$ where

$$\mathbf{x}_k = \{1, 1, 0\} \text{ (ft)} \qquad \mathbf{x}_s = \{5, -5, -2\} \text{ (ft)}$$

(a) Take the positive direction of bar $n$ from $k$ to $s$. Determine $L_n$, $\boldsymbol{\alpha}_n$, and $\vec{t}_n$.

(b) Suppose

$$\mathbf{u}_k = \{1/10, 1/20, 1/10\} \qquad \text{(inches)}$$
$$\mathbf{u}_s = \{1/20, -1/10, -1/30\} \qquad \text{(inches)}$$

Find $\boldsymbol{\eta}_k$ and $\bar{\rho}_k$. Note that the units of $\mathbf{x}$ and $\mathbf{u}$ must be consistent. Determine $e_n$ and $\boldsymbol{\beta}_n$, using the exact expressions (Equations 6–15, 6–17), the expressions specialized for the case of small strain (Equations 6–19, and 6–20), and the expressions for the linear geometric case (Equation 6–21). Compare the results for the three cases.

**6–3.** Discuss when the linear geometric relations are valid and develop the appropriate nonlinear elongation-displacement relations for the trusses shown. Assume no support movements.

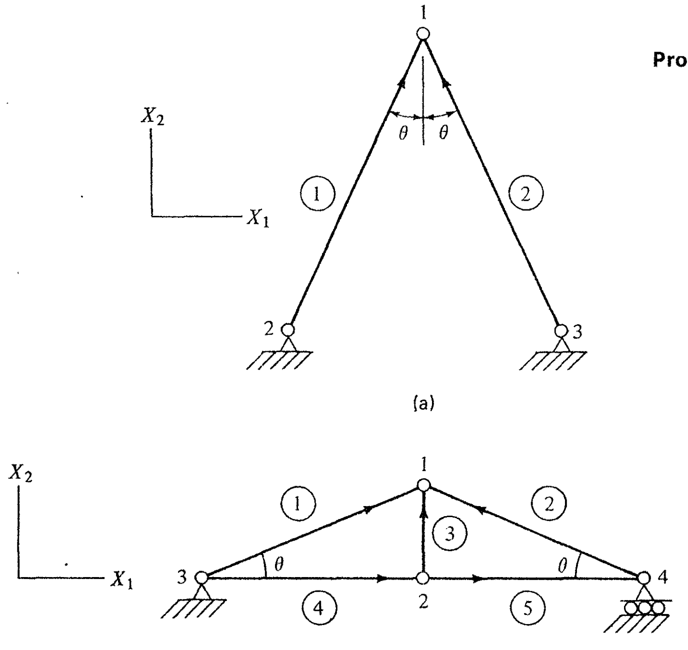
**6–4.** Consider the truss shown:

(a) Establish the connectivity table.

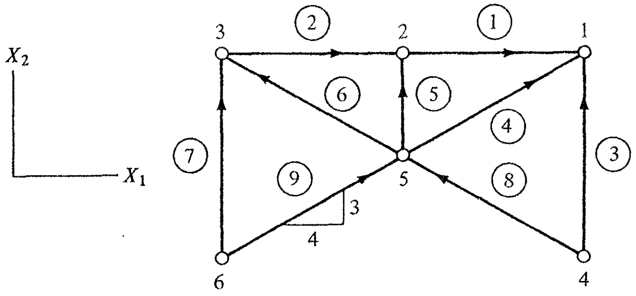
(b) List the initial direction cosines. Do we have to include nonlinear geometric terms for this truss?

(c) Locate the nonzero submatrices in $\mathscr{A}$, using the connectivity table. Determine the complete form of $\mathscr{A}$.

**Prob. 6–3**



**Prob. 6–4**



(d) Determine **C**.
(e) Verify that $\mathscr{A} = \alpha\mathbf{C}$.

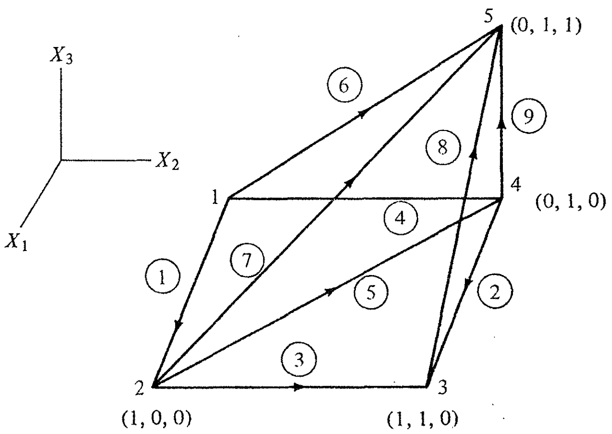
**6–5.** Determine $\mathscr{A}$ for the three-dimensional truss shown.

**6–6.** Consider the d-c network shown. The junctions are generally called nodes, and the line connecting two nodes is called a branch. The encircled numbers refer to the branches and the arrowheads indicate the positive sense (of the current) for each branch.

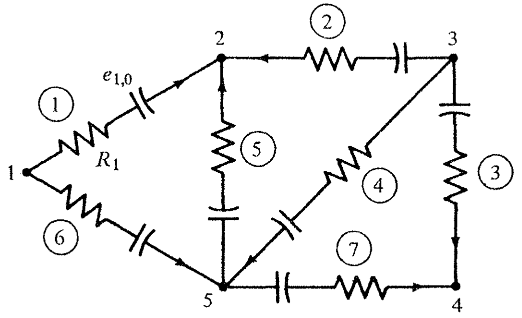
Let $v_j$ ($j = 1, 2, \ldots, 5$) denote the potential at node $j$. Also, let $n_+$ and $n_-$ denote the nodes at the positive and negative ends of branch $n$. The potential *drop* for branch $n$, indicated by $e_n$, is given by

$$e_n = v_{n_-} - v_{n_+}$$

**Prob. 6–5**



**Prob. 6–6**



We define **v** and **e** as

$\mathbf{v} = \{v_1, v_2, \ldots, v_5\}$ = general node potential matrix
$\mathbf{e} = \{e_1, e_2, \ldots, e_7\}$ = general branch potential difference matrix

and write the system of branch potential difference–node potential relations as

$$\mathbf{e} = \mathscr{A}\mathbf{v}$$

Determine $\mathscr{A}$, using the branch-node connectivity table. Discuss how the truss problem differs from the electrical network problem with respect to the form of $\mathscr{A}$. How many independent columns does $\mathscr{A}$ have? In network theory, $\mathscr{A}$ is called the augmented branch node incidence matrix.

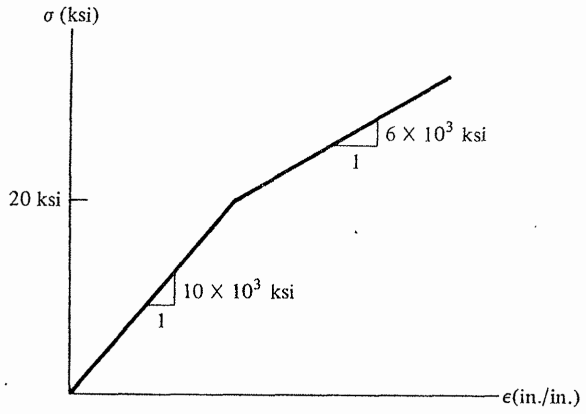
**6–7.** Take $L = 20$ ft, $A = 2$ in$^2$, and the $\sigma$-$\varepsilon$ curve shown.

(a) Develop the piecewise linear force-elongation relations.

(b) Suppose a force of +60 kips is applied and then removed. Determine the force-elongation relation for the inelastic case.

(c) Suppose the bar experiences a temperature increase of 100° F. Determine the initial elongation. Consider the material to be aluminum.

**Prob. 6–7**



**6–8.** Generalize Equation 6–32 for segment $j$. Start with

$$e = e_0^{(j)} + f^{(j)}F$$

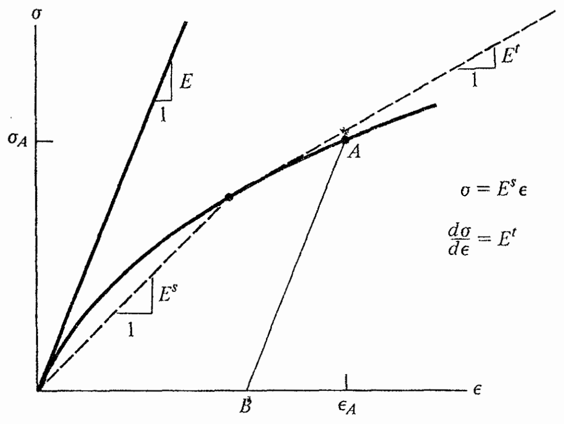
and express $e_0^{(j)}$ in terms of quantities associated with segment $(j - 1)$.

**6–9.** Generalize Equation 6–35 for segment $j$.

**6–10.** Suppose the stress-strain relation for initial loading is approximated, as in the sketch, by

$$\sigma = E(\varepsilon - b\varepsilon^3)$$

**Prob. 6–10**



(a) Determine expressions for $E^s$ and $E^t$, the secant and tangent moduli.

(b) Determine expressions for $k^s$ and $k^t$.

(c) Suppose the material behaves inelastically for decreasing $|\sigma|$. Consider the unloading curve to be parallel to the initial tangent. Determine the force-elongation relation for AB.

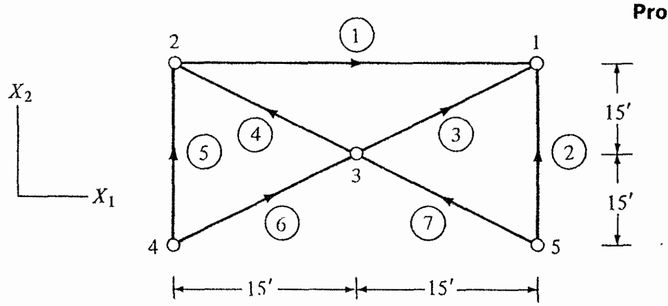
**6–11.** Repeat Prob. 6–10, using the stress-strain relation

$$\varepsilon = \frac{1}{E}(\sigma + c|\sigma|^n)$$

where $E$, $c$, and $n$ are constants.

**6–12.** For the accompanying sketch:

**Prob. 6–12**

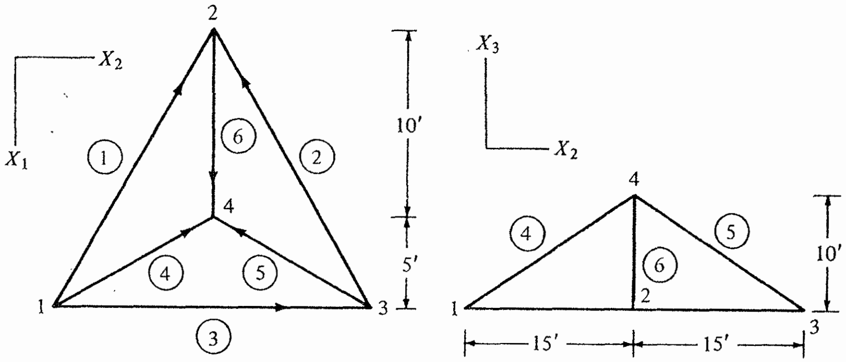


(a) Locate the nonzero submatrices in $\mathscr{B}$.

(b) Assemble $\mathscr{B}$ for the linear geometric case.

**6–13.** Repeat Prob. 6–12 for the three-dimensional truss shown.

**Prob. 6–13**



**6–14.** Consider the electrical network of Prob. 6–6.

(a) Let $i_n$ be the current in branch $n$. The positive sense of $i_n$ is from node $n_-$ to node $n_+$. Now, the total current flowing into a node must equal the total current flowing out of the node. This requirement leads to one equation for each node involving the branch currents incident on the node. Let

$$\mathbf{i} = \{i_1, i_2, \ldots, i_7\} = \text{general branch-current matrix}$$

Show that the complete system of node equations can be written as

$$\mathscr{A}^T\mathbf{i} = \overset{(5\times 1)}{\mathbf{0}} \tag{a}$$

where $\mathscr{A}$ is given in Prob. 6–6.

(b) How many independent equations does (a) represent? (*Hint*: $\mathscr{A}$ has only four independent columns).

(c) When the resistance is linear, the current and potential drop for a branch are related by

$$e_n = e_{0,n} + R_n i_n \tag{b}$$

where $e_{0,n}$ is the branch emf and $R_n$ is the branch resistance. An alternate form is

$$i_n = R_n^{-1}(e_n - e_{0,n})$$

Note the similarity between (b) and the linear elastic member force-elongation relation. Show that the complete system of branch current–node potential relations can be written as

$$\begin{aligned} \mathbf{e} &= \mathscr{A}\mathbf{v} = \mathbf{e}_0 + \mathbf{R}\mathbf{i} \\ \mathbf{i} &= \mathbf{R}^{-1}(\mathbf{e} - \mathbf{e}_0) = \mathbf{R}^{-1}\mathscr{A}\mathbf{v} - \mathbf{R}^{-1}\mathbf{e}_0 \end{aligned} \tag{c}$$

Equations (a) and (c) are the governing unpartitioned equations for a linear-resistance d-c network. The partitioned equations are developed in Prob. 6–23. It should be noted that the network problem is one-dimensional, that is, it does not involve geometry. The $\mathscr{A}$ matrix depends only on the topology (connectivity) of the system. Actually, $\mathscr{A}$ corresponds to the **C** matrix used in Sec. 6–3 with $i = 1$.

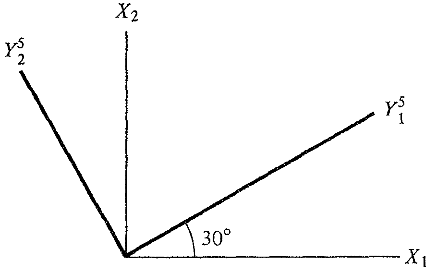
**6–15.** Refer to Prob. 6–12. Suppose $u_{11}$, $u_{42}$, $u_{52}$ are prescribed. Identify $\mathbf{B}_1$ and $\mathbf{B}_2$.

**6–16.** Refer to Prob. 6–12.

(a) Develop the general form of $\mathscr{B}^J$.

(b) Suppose $u_{21}$, $u_{42}$, $u_{52}^5$ are prescribed. The orientation of the local frame at joint 5 is shown in the sketch. Determine $\mathbf{B}_1$ and $\mathbf{B}_2$.

**Prob. 6–16**



**6–17.** Refer to Prob. 6–13

(a) Develop the general form of $\mathbf{B}^J$.

(b) Determine $\mathbf{B}_1$ and $\mathbf{B}_2$ corresponding to the following prescribed displacements:

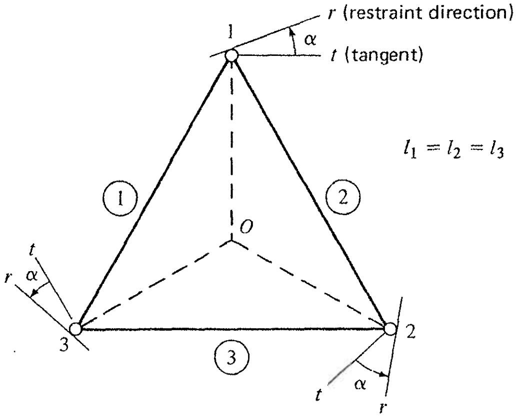
$$u_{11}, \quad u_{12}, \quad u_{31}, \quad u_{33}, \quad u_{23}^2, \quad u_{13}$$

The local frame at joint 2 is defined by the following direction cosine table.

| | $X_1$ | $X_2$ | $X_3$ |
|---|---|---|---|
| $Y_1^2$ | $-1/\sqrt{2}$ | $1/\sqrt{2}$ | 0 |
| $Y_2^2$ | $1/2$ | $1/2$ | $-1/\sqrt{2}$ |
| $Y_3^2$ | $1/2$ | $1/2$ | $1/\sqrt{2}$ |

**6–18.** Consider the two-dimensional truss shown. The bars are of equal length and 0 is the center of the circumscribed circle. The restraint direction is $\alpha$ degrees counterclockwise from the tangent at each joint. Investigate the initial stability of this system. Repeat for the case of four bars.

**Prob. 6–18**



**6–19.** Suppose $n_d = m$. Then, $\mathbf{B}_1$ is of order $m \times m$. The equilibrium equations for $\bar{\mathbf{P}}_1 = \mathbf{0}$ are

$$\overset{(m\times m)}{\mathbf{B}_1}\ \overset{(m\times 1)}{\mathbf{F}} = \overset{m\times 1}{\mathbf{0}} \tag{a}$$

If (a) has a *nontrivial* solution, the rank of $\mathbf{B}_1$ is *less* than $m$ and the system is initially unstable (see Prob. 1–45). Rather than operate on $\mathbf{B}_1$, to determine $r(\mathbf{B}_1)$, we can proceed as follows:

(1) We take the force in some bar, say bar $k$, equal to $C$:

$$F_k = C$$

(2) Using the joint force-equilibrium equations, we express the remaining bar forces in terms of $C$.

(3) The last equilibrium equation leads to an expression for $F_k$ in terms of $C$. If this reduces to an identity, $r(\mathbf{B}_1) < n_d$ since a nontrivial solution for $\mathbf{F}$ exists. This procedure is called the *zero load test*.

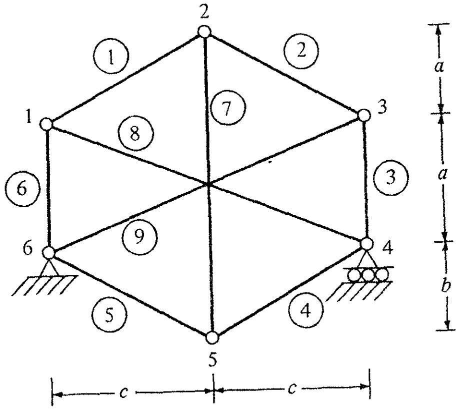
(a) Apply this procedure to Prob. 6–18. Take $F_1 = C$ and determine $F_2$, $F_3$, and then $F_1$ using the equilibrium condition (summation of forces normal to $r$ must equal zero) for joints 1, 2, 3.

(b) When $n_d = m$ and the geometry is linear, the truss is said to be statically determinate. In this case, we can determine $\mathbf{F}$, using only the equations of static equilibrium, since the system, $\bar{\mathbf{P}}_1 = \mathbf{B}_1\mathbf{F}$, is square. Do initial elongations and support settlements introduce forces in the bars of a statically determinate truss?

**6–20.** Modify the zero load test for the case where $n_d < m$. Note that the general solution of $\mathbf{B}_1\mathbf{F} = \mathbf{0}$ involves $m - r(\mathbf{B}_1)$ arbitrary constants.

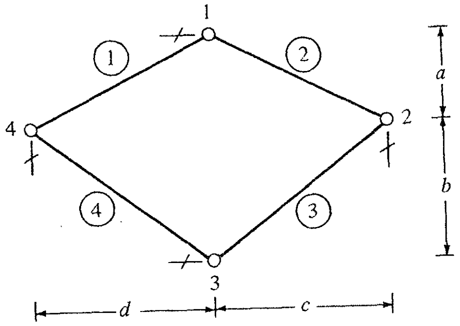
**6–21.** Investigate the initial stability of the two-dimensional truss shown. Use the zero load test.

**Prob. 6–21**



**6–22.** Investigate the initial stability of the system shown. The restraint directions are indicated by the slashed lines.

**Prob. 6–22**



**6–23.** We generalize the results of Probs. 6–6 and 6–14 for a network having $b$ branches and $n$ nodes. Let

$$\mathbf{e} = \text{branch potential diff. matrix} = \{e_1, e_2, \ldots, e_b\}$$
$$\mathbf{i} = \text{branch current matrix} = \{i_1, i_2, \ldots, i_b\}$$
$$\mathbf{v} = \text{node potential matrix} = \{v_1, v_2, \ldots, v_n\}$$

The general relations are (1) node equations ($n$ equations)

$$\underset{}{\overset{(n\times b)}{\mathscr{A}^T}}\ \overset{(b\times 1)}{\mathbf{i}} = \overset{(n\times 1)}{\mathbf{0}} \qquad \text{(a)}$$

and (2) branch equations ($b$ equations)

$$\mathbf{e} = \mathscr{A}v = \mathbf{e}_0 + \mathbf{R}\mathbf{i} \qquad \text{(b)}$$

Now, $\mathscr{A}^T$ has only $n - 1$ independent rows. One can easily show that the rows of $\mathscr{A}^T$ are related by

$$\text{row } n = -\sum_{k=1}^{n-1} \text{row } k \qquad \text{(c)}$$

It follows that (a) represents only $n - 1$ independent equations, and one equation must be disregarded. Suppose we delete the last equation. This corresponds to deleting the last column of $\mathscr{A}$ (last row of $\mathscr{A}^T$). We partition $\mathscr{A}$,

$$\overset{(b\times n)}{\mathscr{A}} = [\ \overset{b\times(n-1)}{\mathscr{A}_1}\ \mid\ \overset{b\times 1}{\mathscr{A}_2}] \qquad \text{(d)}$$

and let $\mathscr{A}_1 = \mathbf{A}$. The reduced system of node equations has the form

$$\mathbf{A}^T\mathbf{i} = \mathbf{0} \qquad \text{(e)}$$

Note that $\mathbf{A}^T$ corresponds to $\mathbf{B}_1$ for the truss problem.

Equation (e) represents $(n - 1)$ equations. Since $\mathbf{v}$ is of order $n$, one of the node potentials must be specified. That is, we can only determine the potential difference for the nodes with respect to an arbitrary node. We have deleted the last column of $\mathscr{A}$ which corresponds to node $n$. Therefore, we take $v_n$ as the reference potential.

(a) Let

$$\underset{(n-1)\times 1}{\mathbf{V}} = \{v_1 - v_n, v_2 - v_n, \ldots, v_{n-1} - v_n\}$$

Show that

$$\mathscr{A}\mathbf{v} = \mathbf{A}\mathbf{V}$$

Summarize the governing equations for the network.

(b) The operation

$$\mathscr{A} \to \mathbf{A}$$
$$\mathbf{v} \to \mathbf{V}$$

corresponds to introducing displacement restraints in the truss problem. Compare the necessary number of restraints required for the network and truss problems.