2

Characteristic-Value Problems and Quadratic Forms

2–1. INTRODUCTION

Consider the second-order homogeneous system,

$$\begin{aligned}(a_{11} - \lambda)x_1 + a_{12}x_2 &= 0\\ a_{21}x_1 + (a_{22} - \lambda)x_2 &= 0\end{aligned} \tag{2–1}$$

where λ is a scalar. Using matrix notation, we can write (2–1) as

$$\mathbf{ax} = \lambda\mathbf{x} \tag{2–2}$$

or

$$(\mathbf{a} - \lambda\mathbf{I}_2)\mathbf{x} = \mathbf{0} \tag{2–3}$$

The values of λ for which nontrivial solutions of (2–1) exist are called the *characteristic values of* **a**. Also, the problem of finding the characteristic values and corresponding nontrivial solutions of (2–1) is referred to as a second-order characteristic-value problem.*

The characteristic-value problem occurs naturally in the free-vibration analysis of a linear system. We illustrate for the system shown in Fig. 2–1. The equations of motion for the case of no applied forces (the free-vibration case) are

$$\begin{aligned}m_2\frac{d^2y_2}{dt^2} + k_2(y_2 - y_1) &= 0\\ m_1\frac{d^2y_1}{dt^2} + k_1y_1 - k_2(y_2 - y_1) &= 0\end{aligned} \tag{a}$$

* Also called "eigenvalue" problem in some texts. The term "eigenvalue" is a hybrid of the German term *Eigenwerte* and English "value."

Assuming a solution of the form

$$y_1 = A_1e^{i\omega t} \qquad y_2 = A_2e^{i\omega t} \tag{b}$$

and substituting in (a) lead to the following set of algebraic equations relating the frequency, ω, and the amplitudes, A_1, A_2:

$$\begin{aligned}(k_1 + k_2)A_1 - k_2A_2 &= m_1\omega^2A_1\\ -k_2A_1 + k_2A_2 &= m_2\omega^2A_2\end{aligned} \tag{c}$$

We can transform (c) to a form similar to that of (2–1) by defining new amplitude measures,*

$$\begin{aligned}\lambda &= \omega^2\\ \bar{A}_1 &= A_1\sqrt{m_1}\\ \bar{A}_2 &= A_2\sqrt{m_2}\end{aligned} \tag{d}$$

and the final equations are

$$\begin{aligned}\frac{k_1 + k_2}{m_1}\bar{A}_1 - \frac{k_2}{\sqrt{m_1m_2}}\bar{A}_2 &= \lambda\bar{A}_1\\ -\frac{k_2}{\sqrt{m_1m_2}}\bar{A}_1 + \frac{k_2}{m_2}\bar{A}_2 &= \lambda\bar{A}_2\end{aligned} \tag{e}$$

The characteristic values and corresponding nontrivial solutions of (e) are related to the natural frequencies and normal mode amplitudes by (d). Note that the coefficient matrix in (e) is symmetrical. This fact is quite significant, as we shall see in the following sections.

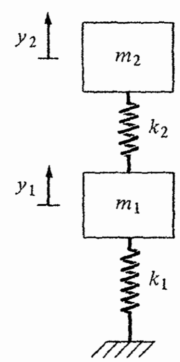

Fig. 2–1. A system with two degrees of freedom.

Although the application to dynamics is quite important, our primary reason for considering the characteristic-value problem is that results obtained for the characteristic value problem provide the basis for the treatment of quadratic

* See Prob. 2–1.

forms which are encountered in the determination of the relative extrema of a function (Chapter 3), the construction of variational principles (Chapter 7), and stability criteria (Chapters 7, 18). This discussion is restricted to the case where **a** is real. Reference 9 contains a definitive treatment of the underlying theory and computational procedures.

2-2. SECOND-ORDER CHARACTERISTIC-VALUE PROBLEM

We know from Cramer's rule that nontrivial solutions of

$$\begin{aligned}(a_{11} - \lambda)x_1 + a_{12}x_2 &= 0 \\ a_{21}x_1 + (a_{22} - \lambda)x_2 &= 0\end{aligned} \tag{2-4}$$

are possible only if the determinant of the coefficient matrix vanishes, that is, when

$$\begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = 0 \tag{2-5}$$

Expanding (2-5) results in the following equation (usually called the characteristic equation) for λ:

$$\lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{21}a_{12}) = 0 \tag{2-6}$$

We let

$$\begin{aligned}\beta_1 &= a_{11} + a_{22} \\ \beta_2 &= a_{11}a_{22} - a_{12}a_{21} = |\mathbf{a}|\end{aligned} \tag{2-7}$$

and the characteristic equation reduces to

$$\lambda^2 - \beta_1\lambda + \beta_2 = 0 \tag{2-8}$$

The roots of (2-8) are the characteristic values of **a**. Denoting the roots by λ_1, λ_2, the solution is

$$\lambda_{1,2} = (\beta_1 \pm \sqrt{\beta_1^2 - 4\beta_2})/2 \tag{2-9}$$

When **a** is symmetrical, $a_{12} = a_{21}$, and

$$\beta_1^2 - 4\beta_2 = (a_{11} - a_{22})^2 + 4(a_{12})^2$$

Since this quantity is never negative, it follows that the characteristic values for a *symmetrical* second-order matrix are always *real*.

Example 2-1

(1)

$$\mathbf{a} = \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}$$

$$\beta_1 = 2 + 5 = 7$$

$$\beta_2 = (2)(5) - (2)(2) = 6$$

The characteristic equation for this matrix is

$$\lambda^2 - 7\lambda + 6 = 0 \tag{a}$$

Solving (a),

$$\lambda_1 = +6 \qquad \lambda_2 = +1$$

(2)

$$\mathbf{a} = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}$$

$$\beta_1 = 0 \qquad \beta_2 = +1$$

$$\lambda_{1,2} = \pm i \qquad \text{where } i = \sqrt{-1}$$

By definition, nontrivial solutions of (2-4) exist only when $\lambda = \lambda_1$ or λ_2. In what follows, we suppose the characteristic values are real. We consider first the case where $\lambda = \lambda_1$. Equation (2-4) becomes

$$\begin{aligned}(a_{11} - \lambda_1)x_1 + a_{12}x_2 &= 0 \\ a_{21}x_1 + (a_{22} - \lambda_1)x_2 &= 0\end{aligned} \tag{a}$$

The second equation is related to the first by

$$\text{second eq.} = \left(\frac{a_{21}}{a_{11} - \lambda_1}\right) \text{times the first eq.} \tag{b}$$

This follows from the fact that the coefficient matrix is singular.

$$(a_{11} - \lambda_1)(a_{22} - \lambda_1) - a_{12}a_{21} = 0 \tag{c}$$

Since only *one* equation is independent and there are *two* unknowns, the solution is *not* unique. We define $x_1^{(1)}, x_2^{(1)}$ as the solution for $\lambda = \lambda_1$. Assuming* that $a_{12} \neq 0$, the solution of the first equation is

$$\begin{aligned}x_1^{(1)} &= c_1 \\ x_2^{(1)} &= -\frac{a_{11} - \lambda_1}{a_{12}} c_1\end{aligned} \tag{d}$$

where c_1 is an arbitrary constant. Continuing, we let

$$\mathbf{x}^{(1)} = \{x_1^{(1)}, x_2^{(1)}\} \tag{e}$$

and take c_1 such that $(\mathbf{x}^{(1)})^T\mathbf{x}^{(1)} = 1$. This operation is called normalization, and the resulting column matrix, denoted by $\mathbf{Q}_1$, is referred to as the characteristic vector for λ_1.

$$\mathbf{Q}_1 = c_1 \left\{+1, -\frac{a_{11} - \lambda_1}{a_{12}}\right\} \tag{2-10}$$

$$\left(\frac{1}{c_1}\right)^2 = 1 + \left[\frac{a_{11} - \lambda_1}{a_{12}}\right]^2$$

By definition,

$$\mathbf{Q}_1^T\mathbf{Q}_1 = 1 \tag{2-11}$$

* If $a_{12} = 0$, we work with the second equation.

Since $\mathbf{Q}_1$ is a solution of (2–4) for $\lambda = \lambda_1$, we see that

$$\mathbf{aQ}_1 = \lambda_1 \mathbf{Q}_1 \tag{2-12}$$

Following the same procedure for $\lambda = \lambda_2$, we obtain

$$\mathbf{Q}_2 = c_2 \left\{ +1, -\frac{a_{11} - \lambda_2}{a_{12}} \right\} \tag{2-13}$$

where

$$\left(\frac{1}{c_2}\right)^2 = 1 + \left[\frac{a_{11} - \lambda_2}{a_{12}}\right]^2$$

Also,

$$\begin{aligned} \mathbf{Q}_2^T\mathbf{Q}_2 &= 1 \\ \mathbf{aQ}_2 &= \lambda_2 \mathbf{Q}_2 \end{aligned} \tag{2-14}$$

It remains to discuss the case where $\lambda_1 = \lambda_2$. If $\mathbf{a}$ is symmetrical, the characteristic values will be equal only when $a_{11} = a_{22}$ and $a_{12} = a_{21} = 0$. Equation (2–4) takes the form

$$\begin{aligned} (a_{11} - \lambda)x_1 + (0)x_2 &= 0 \\ (0)x_1 + (a_{11} - \lambda)x_2 &= 0 \end{aligned} \tag{a}$$

These equations are linearly independent, and the two independent solutions are

$$\begin{aligned} \mathbf{x}^{(1)} &= \{c_1, 0\} \\ \mathbf{x}^{(2)} &= \{0, c_2\} \end{aligned} \tag{b}$$

The corresponding characteristic vectors are

$$\begin{aligned} \mathbf{Q}_1 &= \{+1, 0\} \\ \mathbf{Q}_2 &= \{0, +1\} \end{aligned} \tag{2-15}$$

If $\mathbf{a}$ is *not* symmetrical, there is only *one* independent nontrivial solution when the characteristic values are equal.

It is of interest to examine the product, $\mathbf{Q}_1^T\mathbf{Q}_2$. From (2–10) and (2–13), we have

$$\mathbf{Q}_1^T\mathbf{Q}_2 = -c_1c_2\left(1 + \frac{(a_{11} - \lambda_1)(a_{11} - \lambda_2)}{a_{12}^2}\right) \tag{a}$$

Now, when $\mathbf{a}$ is *symmetrical*, the right-hand term vanishes since

$$a_{11} - \lambda_2 = -(a_{22} - \lambda_1) = -\frac{a_{12}^2}{a_{11} - \lambda_1} \tag{b}$$

and we see that $\mathbf{Q}_1^T\mathbf{Q}_2 = 0$. This result is also valid when the roots are equal. In general, $\mathbf{Q}_1^T\mathbf{Q}_2 \neq 0$ when $\mathbf{a}$ is *unsymmetrical*. Two nth order column vectors $\mathbf{U}, \mathbf{V}$ having the property that

$$\mathbf{U}^T\mathbf{V} = \mathbf{V}^T\mathbf{U} = 0 \tag{2-16}$$

are said to be orthogonal. Using this terminology, $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are orthogonal for the symmetrical case.

Example 2–2

(1)

$$\mathbf{a} = \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}$$

$$\lambda_1 = +6 \qquad \lambda_2 = +1$$

The equations for $\lambda = \lambda_1 = +6$ are

$$\begin{aligned} -4x_1 + 2x_2 &= 0 \\ 2x_1 - x_2 &= 0 \end{aligned}$$

We see that the second equation is $-\frac{1}{2}$ times the first equation. Solving the first equation, we obtain

$$x_1^{(1)} = c_1 \qquad x_2^{(1)} = 2x_1^{(1)} = 2c_1$$

Then,

$$\mathbf{x}^{(1)} = c_1\{1, 2\}$$

and the normalized solution is

$$\mathbf{Q}_1 = \frac{1}{\sqrt{5}}\{1, 2\}$$

Repeating for $\lambda = \lambda_2 = +1$, we find

$$\mathbf{x}^{(2)} = c_2\{1, -\tfrac{1}{2}\}$$

and

$$\mathbf{Q}_2 = \frac{2}{\sqrt{5}}\{1, -\tfrac{1}{2}\} = \frac{1}{\sqrt{5}}\{2, -1\}$$

One can easily verify that

$$\mathbf{aQ}_j = \lambda_j\mathbf{Q}_j \qquad j = 1, 2$$

and

$$\mathbf{Q}_1^T\mathbf{Q}_2 = \mathbf{Q}_2^T\mathbf{Q}_1 = 0$$

(2)

$$\mathbf{a} = \begin{bmatrix} 1 & 8 \\ 1 & 3 \end{bmatrix}$$

The characteristic values and corresponding normalized solutions for this matrix are

$$\lambda_1 = +5 \qquad \lambda_2 = -1$$

$$\mathbf{Q}_1 = \frac{1}{\sqrt{5}}\{2, +1\}$$

$$\mathbf{Q}_2 = \frac{1}{\sqrt{17}}\{4, -1\}$$

We see that $\mathbf{Q}_1^T\mathbf{Q}_2 \neq 0$. Actually,

$$\mathbf{Q}_1^T\mathbf{Q}_2 = \frac{7}{\sqrt{85}}$$

(3)

$$\mathbf{a} = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}$$

$$\lambda_1 = +i \qquad \lambda_2 = -i$$

We have included this example to illustrate the case where the characteristic values are complex. The equations corresponding to $\lambda = \lambda_1$ are

$$(1 - i)x_1 - 2x_2 = 0$$
$$x_1 - (1 + i)x_2 = 0$$

Note that the second equation is $(1 - i)$ times the first equation. The general solution is

$$x^{(1)} = c_1\left\{1, \frac{1 - i}{2}\right\}$$

Repeating for $\lambda = \lambda_2$, we find

$$x^{(2)} = c_2\left\{1, \frac{1 + i}{2}\right\}$$

When the roots are complex, λ_2 is the complex conjugate of λ_1. Now, we take $c_2 = c_1$. Then, $\mathbf{x}^{(2)}$ is the complex conjugate of $\mathbf{x}^{(1)}$. We determine c_1 such that

$$(\mathbf{x}^{(1)})^T\mathbf{x}^{(2)} = 1$$

Finally, the characteristic values and characteristic vectors are

$$\lambda_{1,2} = \pm i$$
$$\mathbf{Q}_{1,2} = \sqrt{2/3}\left\{1, \frac{1 \mp i}{2}\right\}$$

In general, the characteristic values are complex conjugate quantities when the elements of $\mathbf{a}$ are real. Also, the corresponding characteristic vectors are complex conjugates.

2–3. SIMILARITY AND ORTHOGONAL TRANSFORMATIONS

The characteristic vectors for the second-order system satisfy the following relations:

$$\mathbf{a}\mathbf{Q}_1 = \lambda_1\mathbf{Q}_1 \qquad \text{(a)}$$
$$\mathbf{a}\mathbf{Q}_2 = \lambda_2\mathbf{Q}_2$$

We can write (a) as

$$\mathbf{a}[\mathbf{Q}_1 \quad \mathbf{Q}_2] = [\mathbf{Q}_1 \quad \mathbf{Q}_2]\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix} \qquad \text{(b)}$$

Now, we let

$$\mathbf{q} = [\mathbf{Q}_1 \quad \mathbf{Q}_2]$$
$$\boldsymbol{\lambda} = \begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix} \qquad (2\text{–}17)$$

Column j of $\mathbf{q}$ contains the normalized solution for λ_j. We call $\mathbf{q}$ the *normalized modal matrix** for $\mathbf{a}$. With this notation, (b) takes the form

$$\mathbf{a}\mathbf{q} = \mathbf{q}\boldsymbol{\lambda} \qquad (2\text{–}18)$$

* This terminology has developed from dynamics, where the characteristic vectors define the normal modes of vibration for a discrete system.

We have shown that the characteristic vectors are always linearly independent when $\mathbf{a}$ is symmetrical. They are also independent when $\mathbf{a}$ is unsymmetrical, provided that $\lambda_1 \neq \lambda_2$. Then, $|\mathbf{q}| \neq 0$ except for the case where $\mathbf{a}$ is unsymmetrical and the characteristic values are equal. If $|\mathbf{q}| \neq 0$, $\mathbf{q}^{-1}$ exists and we can express (2–18) as

$$\mathbf{q}^{-1}\mathbf{a}\mathbf{q} = \boldsymbol{\lambda} \qquad (2\text{–}19)$$

The matrix operation, $\mathbf{p}^{-1}(\ )\mathbf{p}$ where $\mathbf{p}$ is arbitrary, is called a *similarity transformation*. Equation (2–19) states that the similarity transformation, $\mathbf{q}^{-1}(\ )\mathbf{q}$, reduces $\mathbf{a}$ to a diagonal matrix whose elements are the characteristic values of $\mathbf{a}$.

If $\mathbf{a}$ is symmetrical, the normalized characteristic vectors are orthogonal, that is,

$$\mathbf{Q}_1^T\mathbf{Q}_2 = \mathbf{Q}_2^T\mathbf{Q}_1 = 0$$

Also, by definition,

$$\mathbf{Q}_1^T\mathbf{Q}_1 = \mathbf{Q}_2^T\mathbf{Q}_2 = 1$$

Using these properties, we see that

$$\mathbf{q}^T\mathbf{q} = \begin{bmatrix}\mathbf{Q}_1^T\\ \mathbf{Q}_2^T\end{bmatrix}[\mathbf{Q}_1 \quad \mathbf{Q}_2] = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

and it follows that

$$\mathbf{q}^{-1} = \mathbf{q}^T \qquad (2\text{–}20)$$

A square matrix, say $\mathbf{p}$, having the property that $\mathbf{p}^T = \mathbf{p}^{-1}$ is called an *orthogonal* matrix and the transformation, $\mathbf{p}^T(\ )\mathbf{p}$, is called an orthogonal transformation. Note that an orthogonal transformation is also a similarity transformation. Then, the modal matrix for a symmetrical matrix is orthogonal and we can write

$$\mathbf{q}^T\mathbf{a}\mathbf{q} = \boldsymbol{\lambda} \qquad (2\text{–}21)$$

Example 2–3

(1)

$$\mathbf{a} = \begin{bmatrix}2 & 2\\ 2 & 5\end{bmatrix}$$

$$\lambda_1 = +6 \qquad \mathbf{Q}_1 = \frac{1}{\sqrt{5}}\{1, 2\}$$

$$\lambda_2 = +1 \qquad \mathbf{Q}_2 = \frac{1}{\sqrt{5}}\{2, -1\}$$

$$\boldsymbol{\lambda} = \begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix} = \begin{bmatrix}+6 & 0\\ 0 & +1\end{bmatrix}$$

$$\mathbf{q} = [\mathbf{Q}_1 \quad \mathbf{Q}_2] = \frac{1}{\sqrt{5}}\begin{bmatrix}1 & 2\\ 2 & -1\end{bmatrix}$$

We verify that $\mathbf{q}^T = \mathbf{q}^{-1}$ and $\mathbf{q}^T\mathbf{a}\mathbf{q} = \boldsymbol{\lambda}$:

$$\mathbf{q}^T\mathbf{q} = \frac{1}{5}\begin{bmatrix}1 & 2\\2 & -1\end{bmatrix}\begin{bmatrix}1 & 2\\2 & -1\end{bmatrix} = \frac{1}{5}\begin{bmatrix}5 & 0\\0 & 5\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

$$\mathbf{a}\mathbf{q} = \frac{1}{\sqrt{5}}\begin{bmatrix}2 & 2\\2 & 5\end{bmatrix}\begin{bmatrix}1 & 2\\2 & -1\end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix}6 & 2\\12 & -1\end{bmatrix}$$

$$= \frac{1}{\sqrt{5}}\begin{bmatrix}1 & 2\\2 & -1\end{bmatrix}\begin{bmatrix}6 & 0\\0 & 1\end{bmatrix} = \mathbf{q}\boldsymbol{\lambda}$$

$$\mathbf{q}^T\mathbf{a}\mathbf{q} = \frac{1}{5}\begin{bmatrix}1 & 2\\2 & -1\end{bmatrix}\begin{bmatrix}6 & 2\\12 & -1\end{bmatrix} = \begin{bmatrix}6 & 0\\0 & 1\end{bmatrix} = \boldsymbol{\lambda}$$

(2)

$$\mathbf{a} = \begin{bmatrix}1 & 8\\1 & 3\end{bmatrix}$$

$$\lambda_1 = +5 \qquad \mathbf{Q}_1 = \frac{1}{\sqrt{5}}\{2, +1\}$$

$$\lambda_2 = -1 \qquad \mathbf{Q}_2 = \frac{1}{\sqrt{17}}\{4, -1\}$$

$$\boldsymbol{\lambda} = \begin{bmatrix}5 & 0\\0 & -1\end{bmatrix} \qquad \mathbf{q} = \begin{bmatrix}2/\sqrt{5} & 4/\sqrt{17}\\1/\sqrt{5} & -1/\sqrt{17}\end{bmatrix}$$

Since $\mathbf{a}$ is not symmetrical, $\mathbf{q}^T \neq \mathbf{q}^{-1}$. Actually,

$$\mathbf{q}^{-1} = \left(-\frac{\sqrt{85}}{6}\right)\begin{bmatrix}-1/\sqrt{17} & -4/\sqrt{17}\\-1/\sqrt{5} & 2/\sqrt{5}\end{bmatrix} = \begin{bmatrix}\sqrt{5}/6 & 2\sqrt{5}/3\\\sqrt{17}/6 & -\sqrt{17}/3\end{bmatrix}$$

One can easily verify that

$$\mathbf{q}^{-1}\mathbf{a}\mathbf{q} = \begin{bmatrix}5 & 0\\0 & -1\end{bmatrix} = \boldsymbol{\lambda}$$

(3)

$$\mathbf{a} = \begin{bmatrix}1 & -2\\1 & -1\end{bmatrix}$$

$$\boldsymbol{\lambda} = \begin{bmatrix}+i & 0\\0 & -i\end{bmatrix}$$

$$\mathbf{q} = \sqrt{2/3}\begin{bmatrix}1 & 1\\\dfrac{1-i}{2} & \dfrac{1+i}{2}\end{bmatrix}$$

In this case, $\mathbf{q}$ involves complex elements. Since the characteristic vectors are complex conjugates, they are linearly independent and $\mathbf{q}^{-1}$ exists. We find $\mathbf{q}^{-1}$, using the definition equation for the inverse (Equation (1–50)):

$$\mathbf{q}^{-1} = \frac{1}{|\mathbf{q}|}\text{Adj}\,\mathbf{q} = \sqrt{3/2}\begin{bmatrix}\dfrac{1-i}{2} & i\\\dfrac{1+i}{2} & -i\end{bmatrix}$$

One can easily verify that

$$\mathbf{q}^{-1}\mathbf{a}\mathbf{q} = \begin{bmatrix}+i & 0\\0 & -i\end{bmatrix} = \boldsymbol{\lambda}$$

2–4. THE *n*th-ORDER SYMMETRICAL CHARACTERISTIC-VALUE PROBLEM

The nth order symmetrical characteristic-value problem involves determining the characteristic values and corresponding nontrivial solutions for

$$\begin{aligned}a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= \lambda x_1\\a_{12}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= \lambda x_2\\\vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \quad &\quad\ \vdots\\a_{1n}x_1 + a_{2n}x_2 + \cdots + a_{nn}x_n &= \lambda x_n\end{aligned} \tag{2–22}$$

We can write (2–22) as

$$\begin{aligned}\mathbf{a}\mathbf{x} &= \lambda\mathbf{x}\\(\mathbf{a} - \lambda\mathbf{I}_n)\mathbf{x} &= \mathbf{0}\end{aligned} \tag{2–23}$$

In what follows, we suppose $\mathbf{a}$ is *real*.

For (2–23) to have a nontrivial solution, the coefficient matrix must be singular.

$$|\mathbf{a} - \lambda\mathbf{I}_n| = 0 \tag{2–24}$$

The expansion of the determinant is

$$(-1)^n(\lambda^n - \beta_1\lambda^{n-1} + \beta_2\lambda^{n-2} - \cdots + (-1)^n\beta_n) = 0$$

where

$$\begin{aligned}\beta_1 &= a_{11} + a_{22} + \cdots + a_{nn}\\\beta_n &= |\mathbf{a}|\end{aligned} \tag{2–25}$$

and β_j is the sum of all the jth order minors that can be formed on the diagonal.* Letting $\lambda_1, \lambda_2, \ldots, \lambda_n$ denote the roots, and expressing the characteristic equation in factored form, we see that

$$\begin{aligned}\beta_1 &= \lambda_1 + \lambda_2 + \cdots + \lambda_n\\\beta_2 &= \lambda_1\lambda_2 + \lambda_1\lambda_3 + \cdots + \lambda_{n-1}\lambda_n\\&\ \vdots\\\beta_n &= \lambda_1\lambda_2\cdots\lambda_n\end{aligned} \tag{2–26}$$

We summarize below the theoretical results for the *real symmetrical* case. The proofs are too detailed to be included here (see References 1 and 9):

1. The characteristic values $\lambda_1, \lambda_2, \ldots, \lambda_n$, are all *real*.
2. The normalized characteristic vectors $\mathbf{Q}_1, \mathbf{Q}_2, \ldots, \mathbf{Q}_n$, are *orthogonal*:

$$\mathbf{Q}_i^T\mathbf{Q}_j = \delta_{ij} \qquad i, j = 1, 2, \ldots, n$$

* Minors having a diagonal pivot (e.g., delete the kth row and column). They are generally called principal minors.

3. **a** is *diagonalized* by the orthogonal transformation involving the normalized modal matrix.

$$\mathbf{q}^T\mathbf{a}\mathbf{q} = \boldsymbol{\lambda}$$

where

$$\mathbf{q} = [\mathbf{Q}_1\mathbf{Q}_2 \cdots \mathbf{Q}_n]$$
$$\boldsymbol{\lambda} = [\lambda_i\delta_{ij}]$$

Example 2–4

(1)

$$\mathbf{a} = \begin{bmatrix} 5 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Since **a** is symmetrical, its characteristic values are all real. We first determine $\beta_1, \beta_2, \beta_3$, using (2–25):

$$\beta_1 = 5 + 3 + 1 = +9$$
$$\beta_2 = +11 + 5 + 2 = +18$$
$$\beta_3 = 5(2) - (-2)(-2) = +6$$

The characteristic equation is

$$f(\lambda) = \lambda^3 - 9\lambda^2 + 18\lambda - 6 = 0$$

and the approximate roots are

$$\lambda_1 \approx +0.42$$
$$\lambda_2 \approx +2.30$$
$$\lambda_3 \approx +6.28$$

To determine the characteristic solutions, we expand $\mathbf{ax} = \lambda\mathbf{x}$,

$$\begin{aligned} (5-\lambda)x_1 &= 2x_2 \\ (-2)x_1 - x_3 &= -(3-\lambda)x_2 \\ (1-\lambda)x_3 &= x_2 \end{aligned}$$

Solving the first and third equations for x_1 and x_3 in terms of x_2, the general solution is

$$x_1^{(j)} = \frac{2}{5-\lambda_j}x_2^{(j)} \qquad j = 1, 2, 3$$
$$x_3^{(j)} = \frac{1}{1-\lambda_j}x_2^{(j)}$$

Finally, the modal matrix (to 2-place accuracy) is

$$\mathbf{q} = [\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3] = \begin{bmatrix} +0.22 & +0.51 & -0.84 \\ +0.50 & +0.68 & +0.54 \\ +0.85 & -0.52 & -0.10 \end{bmatrix}$$

(2)

$$\mathbf{a} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

The expansion of $|\mathbf{a} - \lambda\mathbf{I}_3| = 0$ is

$$(3-\lambda)((1-\lambda)^2 - 4) = 0$$

and the roots are

$$\lambda_1 = 3 \qquad \lambda_2 = 3 \qquad \lambda_3 = -1$$

Writing out $\mathbf{ax} = \lambda\mathbf{x}$, we have

$$\begin{aligned} (1-\lambda)x_1 + 2x_2 &= 0 \\ 2x_1 + (1-\lambda)x_2 &= 0 \\ (3-\lambda)x_3 &= 0 \end{aligned} \qquad \text{(a)}$$

When $\lambda = 3$, (a) reduces to

$$\begin{aligned} -2x_1 + 2x_2 &= 0 \\ 2x_1 - 2x_2 &= 0 \\ (0)x_3 &= 0 \end{aligned} \qquad \text{(b)}$$

We see from (b) that $(\mathbf{a} - \lambda\mathbf{I}_3)$ is of rank 1 when $\lambda = 3$. The general solution of (b) is

$$x_1 = c_1 \qquad x_2 = c_1 \qquad x_3 = c_2$$

By specializing the constants, we can obtain two linearly independent solutions for the repeated root. Finally, the characteristic vectors for $\lambda_1 = \lambda_2 = 3$ are

$$\mathbf{Q}_1 = \left\{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right\}$$
$$\mathbf{Q}_2 = \{0, 0, 1\}$$

When $\lambda = \lambda_3 = -1$, (a) reduces to

$$\begin{aligned} 2x_1 + 2x_2 &= 0 \\ 2x_1 + 2x_2 &= 0 \\ 4x_3 &= 0 \end{aligned}$$

The general solution and characteristic vector for λ_3 are

$$x_1^{(3)} = -x_2^{(3)} \quad \text{and} \quad x_3^{(3)} = 0$$
$$\mathbf{Q}_3 = \left\{\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right\}$$

This example illustrates the case of a symmetrical matrix having two equal characteristic values. The characteristic vectors corresponding to the repeated roots are linearly independent. This follows from the fact that $\mathbf{a} - \lambda\mathbf{I}_3$ is of rank 1 for the repeated roots.

2–5. QUADRATIC FORMS

The homogeneous second-degree function

$$F = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2$$

is called a quadratic form in x_1, x_2. Using matrix notation, we can express F as

$$F = [x_1x_2]\begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \mathbf{x}^T\mathbf{a}\mathbf{x}$$

In general, the function

$$F = \sum_{k=1}^{n} \sum_{j=1}^{n} a_{jk} x_j x_k = \mathbf{x}^T \mathbf{a} \mathbf{x} \tag{2-27}$$

where $a_{jk} = a_{kj}$, for $j \neq k$, is said to be a quadratic form in $x_1, x_2, \ldots, x_n$.

If $F = \mathbf{x}^T\mathbf{a}\mathbf{x}$ is nonnegative ($\geqslant 0$) for all $\mathbf{x}$ and zero *only* when $\mathbf{x} = \mathbf{0}$, we call F a *positive definite* quadratic form. Also, we say that $\mathbf{a}$ is a positive definite matrix. If $F \geqslant 0$ for all $\mathbf{x}$ but is zero for some $\mathbf{x} \neq \mathbf{0}$, we say that F is *positive semidefinite*. We define negative definite and negative semidefinite quadratic forms in a similar manner. A quadratic form is *negative* definite if $F \leqslant 0$ for all $\mathbf{x}$ and $F = 0$ only when $\mathbf{x} = \mathbf{0}$. The question as to whether a quadratic form is positive definite is quite important. For example, we will show that an equilibrium position for a discrete system is stable when a certain quadratic form is positive definite.

Consider the quadratic form

$$F = b_1 x_1^2 + b_2 x_2^2 + \cdots + b_n x_n^2$$

$$= [x_1 x_2 \cdots x_n] \begin{bmatrix} b_1 & 0 & \cdots & 0 \\ 0 & b_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & b_n \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} \tag{2-28}$$

When F involves only squares of the variables, it is said to be in *canonical* form. According to the definition introduced above, F is positive definite when

$$b_1 > 0 \qquad b_2 > 0, \cdots b_n > 0$$

It is positive semidefinite when

$$b_1 \quad 0 \qquad b_2 \geqslant 0 \cdots b_n \geqslant 0$$

and at least one of the elements is zero.

Now, to establish whether $\mathbf{x}^T\mathbf{a}\mathbf{x}$ is positive definite, we first reduce $\mathbf{a}$ to a diagonal matrix by applying the transformation, $\mathbf{q}^{-1}(\ )\mathbf{q}$, where $\mathbf{q}$ is the orthogonal normalized modal matrix for $\mathbf{a}$. We write

$$\begin{aligned} \mathbf{x}^T\mathbf{a}\mathbf{x} &= (\mathbf{x}^T\mathbf{q})(\mathbf{q}^{-1}\mathbf{a}\mathbf{q})(\mathbf{q}^{-1}\mathbf{x}) \\ &= (\mathbf{x}^T\mathbf{q})[\lambda_i \delta_{ij}](\mathbf{q}^T\mathbf{x}) \end{aligned} \tag{a}$$

Then, letting

$$\mathbf{y} = \mathbf{q}^T\mathbf{x} \qquad \mathbf{x} = \mathbf{q}\mathbf{y} \tag{2-29}$$

(a) reduces to a canonical form in $\mathbf{y}$:

$$F = \mathbf{x}^T\mathbf{a}\mathbf{x} = \mathbf{y}^T[\lambda_i \delta_{ij}]\mathbf{y} \tag{2-30}$$

It follows that F is positive definite with respect to $\mathbf{y}$ when *all* the characteristic values of $\mathbf{a}$ are positive. But $\mathbf{y}$ is uniquely related to $\mathbf{x}$ and $\mathbf{y} = \mathbf{0}$ only when $\mathbf{x} = \mathbf{0}$. Therefore, F is also positive definite with respect to $\mathbf{x}$. The problem of establishing whether $\mathbf{x}^T\mathbf{a}\mathbf{x}$ is positive definite consists in determining whether *all* the characteristic values of $\mathbf{a}$ are *positive.*

We consider first the second-order symmetric matrix

$$a = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}$$

Using (2-26), the characteristic values are related by

$$\begin{aligned} \lambda_1 + \lambda_2 &= \beta_1 = a_{11} + a_{22} \\ \lambda_1\lambda_2 &= \beta_2 = a_{11}a_{22} - a_{12}^2 = |\mathbf{a}| \end{aligned} \tag{a}$$

We see from (a) that the conditions

$$\beta_1 > 0 \qquad \beta_2 > 0 \tag{b}$$

are equivalent to

$$\lambda_1 > 0 \qquad \lambda_2 > 0 \tag{c}$$

Suppose we specify that

$$\begin{aligned} a_{11} &> 0 \\ |\mathbf{a}| = a_{11}a_{22} - a_{12}^2 &> 0 \end{aligned} \tag{d}$$

Since $a_{11} > 0$, it follows from the second requirement in (d) that $a_{22} > 0$. Therefore, (d) is equivalent to (b). We let

$$\begin{aligned} \Delta_1 &= |a_{11}| = a_{11} \\ \Delta_2 &= \begin{vmatrix} a_{11}a_{12} \\ a_{12}a_{12} \end{vmatrix} = |\mathbf{a}| \end{aligned} \tag{2-31}$$

Then, $\mathbf{a}$ is positive definite when

$$\beta_1 > 0 \qquad \beta_2 > 0$$

or

$$\Delta_1 > 0 \qquad \Delta_2 > 0 \tag{2-32}$$

The quantities β_j and Δ_j are called the *invariants* and *discriminants* of $\mathbf{a}$.

The above criteria also apply for the nth-order case. That is, one can show that $\mathbf{a}$ is positive definite when *all* its invariants are greater than zero.

$$\beta_1 > 0 \quad \beta_2 > 0 \quad \cdots \quad \beta_n > 0 \tag{2-33}$$

where β_j is the sum of all the jth-order principal minors. Equivalent conditions can be expressed in terms of the discriminants. Let Δ_j represent the determinant of the array consisting of the first j rows and columns.

$$\Delta_j = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1j} \\ a_{12} & a_{22} & \cdots & a_{2j} \\ \vdots & \vdots & & \vdots \\ a_{1j} & a_{2j} & \cdots & a_{jj} \end{vmatrix} \tag{2-34}$$

The conditions,

$$\Delta_1 > 0 \quad \Delta_2 > 0 \quad \cdots \quad \Delta_n > 0 \tag{2-35}$$

are sufficient for $\mathbf{a}$ to be positive definite.*

* See Ref. 1 for a detailed proof. Also see Prob. 2-15.

Example 2–5

(1)

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

The discriminants are

$$\Delta_1 = +1$$
$$\Delta_2 = 2 - 1 = +1$$
$$\Delta_3 = 1(6-4) - 1(3-2) + 1(2-2) = +1$$

Since all the discriminants are positive, this matrix is positive definite. The corresponding invariants are

$$\beta_1 = 1 + 2 + 3 = +6$$
$$\beta_2 = (2-1) + (3-1) + (6-4) = +5$$
$$\beta_3 = \Delta_3 = +1$$

(2)

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

Since Δ_2 is negative ($\Delta_2 = -3$), this matrix is *not* positive definite.

Suppose **b** is obtained from **a** by an orthogonal transformation:

$$\mathbf{b} = \mathbf{p}^T\mathbf{a}\mathbf{p} = \mathbf{p}^{-1}\mathbf{a}\mathbf{p} \tag{2–36}$$

If **a** is symmetrical, **b** is also symmetrical:

$$\mathbf{b}^T = \mathbf{p}^T\mathbf{a}^T\mathbf{p} = \mathbf{p}^T\mathbf{a}\mathbf{p} \tag{2–37}$$

Now, **b** and **a** have the *same* characteristic values.* This follows from

$$|\mathbf{b} - \lambda\mathbf{I}_n| = |\mathbf{p}^{-1}(\mathbf{a} - \lambda\mathbf{I}_n)\mathbf{p}| = |\mathbf{a} - \lambda\mathbf{I}_n| \tag{2–38}$$

Then, if **a** is positive definite, **b** is also positive definite. In general, the positive definite character of a matrix is preserved under an orthogonal transformation.

REFERENCES

1. HILDEBRAND, F. B.: *Methods of Applied Mathematics*, Prentice-Hall, New York, 1952.
2. BODEWIG, E.: *Matrix Calculus*, Interscience Publishers, New York, 1956.
3. SMIRNOV, V. I.: *Linear Algebra*, Addison-Wesley Publishing Co., Reading, Mass., 1964.
4. TURNBULL, H. W., and A. C. AITKEN: *An Introduction to the Theory of Canonical Matrices*, Dover Publications, New York.
5. HADLEY, G.: *Linear Algebra*, Addison-Wesley Publishing Co., Reading, Mass., 1961.

* See Prob. 2–5.

6. CRANDALL, S. H.: *Engineering Analysis*, McGraw-Hill, New York, 1956.
7. NOBLE, B.: *Applied Linear Algebra*, Prentice-Hall, New York, 1969.
8. FRAZER, R. A., W. J. DUNCAN and A. R. COLLAR: *Elementary Matrices*, Cambridge University Press, London, 1963.
9. WILKINSON, J. H.: *The Algebraic Eigenvalue Problem*, Oxford University Press, London, 1965.
10. FADDEVA, V. N.: *Computational Methods of Linear Algebra*, Dover Publications, New York, 1953.
11. RALSTON, A., and H. S. WILF: *Mathematical Methods for Digital Computers*, Vol. 2, Wiley, New York, 1967.
12. FORSYTHE, G. E., and C. B. MALER: *Computer Solution of Linear Algebraic Systems*, Prentice-Hall, New York, 1967.
13. PETERS, G., and J. H. WILKINSON: "Eigenvalues of $\mathbf{AX} = \lambda\mathbf{BX}$ with Band Symmetric **A** and **B**," *Comput. J.*, **12**, 398–404, 1969.

PROBLEMS

2–1. Consider the system

$$\mathbf{A}\mathbf{y} = \lambda\mathbf{B}\mathbf{y} \tag{a}$$

where **A** and **B** are symmetrical nth-order matrices and λ is a scalar. Suppose **B** can be expressed as (see Prob. 1–25)

$$\mathbf{B} = \mathbf{b}^T\mathbf{b} \tag{b}$$

where **b** is nonsingular. Reduce (a) to the form

$$\mathbf{a}\mathbf{x} = \lambda\mathbf{x}$$

where $\mathbf{x} = \mathbf{b}\mathbf{y}$. Determine the expression for **a** in terms of **A** and **b**.

2–2. Let $\mathbf{x}_1$, $\mathbf{x}_2$ be two nth-order column matrices or column vectors and let c_1, c_2 be arbitrary scalars. If

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 = \mathbf{0}$$

only when $c_1 = c_2 = 0$, $\mathbf{x}_1$ and $\mathbf{x}_2$ are said to be linearly independent. It follows that $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly dependent when one is a scalar multiple of the other. Using (2–10) and (2–13), show that $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are linearly independent when $\lambda_1 \neq \lambda_2$.

2–3. Determine the characteristic values and the modal matrix for

(a)
$$\begin{bmatrix} 3 & 2 \\ 2 & 7 \end{bmatrix}$$

(b)
$$\begin{bmatrix} 2 & 0 & 3 \\ 0 & 5 & 0 \\ 3 & 0 & 2 \end{bmatrix}$$

2–4. Following the procedure outlined in Prob. 2–1, determine the characteristic values and modal matrix for

$$12y_1 + 12y_2 = 4\lambda y_1$$
$$12y_1 + 63y_2 = 9\lambda y_2$$

2–5. Suppose that $\mathbf{b}$ is derived from $\mathbf{a}$ by a similarity transformation.

$$\mathbf{b} = \mathbf{p}^{-1}\mathbf{a}\mathbf{p}$$

Then,

$$|\mathbf{b} - \lambda\mathbf{I}_n| = |\mathbf{a} - \lambda\mathbf{I}_n|$$

and it follows that $\mathbf{b}$ and $\mathbf{a}$ have the *same* characteristic equation.

(a) Deduce that

$$\begin{aligned}\lambda_k^{(b)} &= \lambda_k^{(a)} \\ \beta_k^{(b)} &= \beta_k^{(a)}\end{aligned} \qquad k = 1, 2, \ldots, n$$

Demonstrate for

$$\mathbf{a} = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} \qquad \mathbf{p} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$$

The fact that $\beta_1, \beta_2, \ldots, \beta_n$ are *invariant* under a similarity transformation is quite useful.

(b) Show that

$$\mathbf{Q}_k^{(b)} = \mathbf{p}^{-1}\mathbf{Q}_k^{(a)}$$

2–6. When $\mathbf{a}$ is symmetrical, we can write

$$\mathbf{q}^T\mathbf{a}\mathbf{q} = \boldsymbol{\lambda}$$

Express $\mathbf{a}^{-1}$ in terms of $\mathbf{q}$ and $\boldsymbol{\lambda}^{-1}$. Use this result to find the inverse of

$$\mathbf{a} = \begin{bmatrix} 3 & 2 \\ 2 & 7 \end{bmatrix}.$$

2–7. Positive integral powers of a square matrix, say $\mathbf{a}$, are defined as

$$\begin{aligned}\mathbf{a}^2 &= \mathbf{a}\mathbf{a} \\ \mathbf{a}^3 &= \mathbf{a}\mathbf{a}^2 \\ &\vdots \\ \mathbf{a}^r &= \mathbf{a}\mathbf{a}^{r-1}\end{aligned}$$

If $|\mathbf{a}| \neq 0$, $\mathbf{a}^{-1}$ exists, and it follows from the definition that

$$\mathbf{a}^{-1}\mathbf{a}^r = \mathbf{a}^{r-1}$$

(a) Show that $\mathbf{a}^r$ is symmetrical when $\mathbf{a}$ is symmetrical.

(b) Let λ_i be a characteristic value of $\mathbf{a}$. Show that λ_i^r is a characteristic value of $\mathbf{a}^r$ and $\mathbf{Q}_i$ is the corresponding characteristic vector.

$$\mathbf{a}^r\mathbf{Q}_i = \lambda_i^r\mathbf{Q}_i$$

Hint: Start with $\mathbf{a}\mathbf{Q}_i = \lambda_i\mathbf{Q}_i$ and premultiply by $\mathbf{a}$.

2–8. A linear combination of nonnegative integral powers of $\mathbf{a}$ is called a polynomial function of $\mathbf{a}$ and written as $P(\mathbf{a})$. For example, the third order polynomial has the form

$$P(\mathbf{a}) = c_0\mathbf{a}^3 + c_1\mathbf{a}^2 + c_2\mathbf{a} + c_3\mathbf{I}_n$$

Note that $P(\mathbf{a})$ is symmetrical when $\mathbf{a}$ is symmetrical.

Let $F(\lambda) = 0$ be the characteristic equation for $\mathbf{a}$. When the characteristic values of $\mathbf{a}$ are *distinct*, one can show that (see Ref. 1)

$$F(\mathbf{a}) = \mathbf{0}$$

where $\mathbf{0}$ is an nth-order null matrix. That is, $\mathbf{a}$ satisfies its own characteristic equation. This result is known as the Cayley-Hamilton Theorem.

(a) Verify this theorem for

$$\mathbf{a} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Note: $F(\mathbf{a}) = \mathbf{a}^2 - \beta_1\mathbf{a} + \beta_2\mathbf{I}_2$.

(b) Show that

$$\mathbf{a}^{-1} = \frac{1}{\beta_3}(\mathbf{a}^2 - \beta_1\mathbf{a} + \beta_2\mathbf{I}_3) \qquad \text{for } n = 3$$

(c) Establish a general expression for $\mathbf{a}^{-1}$ using (2–25).

2–9. Determine whether the following quadratic forms are positive definite.

(a) $$F = 2x_1^2 + 4x_1x_2 + 3x_2^2$$

(b) $$F = 3x_1^2 + 5x_2^2 + 6x_3^2 - 4x_1x_2 + 6x_1x_3 - 8x_2x_3$$

2–10. Show that a necessary but not sufficient condition for $\mathbf{a}$ to be positive definite is

$$a_{11} > 0, a_{22} > 0, \ldots, a_{nn} > 0$$

(*Hint*: Take $x_i \neq 0$ and $x_j = 0$ for $j \neq i, j = 1, 2, \ldots, n$)

2–11. If $|\mathbf{a}| = 0$, $\mathbf{a}\mathbf{x} = \mathbf{0}$ has a nontrivial solution, say $\mathbf{x}_1$. What is the value of $\mathbf{x}_1^T\mathbf{a}\mathbf{x}_1$? Note that $\lambda = 0$ is a characteristic value of $\mathbf{a}$ when $\mathbf{a}$ is singular.

2–12. Let $\mathbf{C}$ be a square matrix. Show that $\mathbf{C}^T\mathbf{C}$ is positive definite when $|\mathbf{C}| \neq 0$ and positive semidefinite when $|\mathbf{C}| = 0$.
(*Hint*: Start with $F = x^T(\mathbf{C}^T\mathbf{C})\mathbf{x}$ and let $\mathbf{y} = \mathbf{C}\mathbf{x}$. By definition, F can equal zero only when $\mathbf{x} = \mathbf{0}$ in order for the form to be positive definite.)

2–13. Consider the product $\mathbf{C}^T\mathbf{a}\mathbf{C}$, where $\mathbf{a}$ is positive definite and $\mathbf{C}$ is *square*. Show that $\mathbf{C}^T\mathbf{a}\mathbf{C}$ is positive definite when $|\mathbf{C}| \neq 0$ and positive semidefinite when $|\mathbf{C}| = 0$. Generalize this result for the multiple product,

$$\mathbf{C}_n^T\mathbf{C}_{n-1}^T \cdots \mathbf{C}_1^T\mathbf{a}\mathbf{C}_1 \cdots \mathbf{C}_{n-1}\mathbf{C}_n$$

2–14. Let $\mathbf{a}$ be an mth-order *positive definite* matrix and let $\mathbf{C}$ be of order $m \times n$. Consider the product,

$$\mathbf{b} = \mathbf{C}^T\mathbf{a}\mathbf{C}$$

Show that $\mathbf{b}$ is positive definite *only when* the rank of $\mathbf{C}$ is equal to n. What can we say about $\mathbf{b}$ when $r(\mathbf{C}) < n$?

2–15. Consider the quadratic form

$$F = [x_1x_2 \cdots x_n]\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix}$$

We partition **a** symmetrically,

$$F = [\mathbf{X}_1^T \mathbf{X}_2^T] \underset{}{\begin{bmatrix} \underset{(p\times p)}{\mathbf{A}_{11}} & \underset{(p\times q)}{\mathbf{A}_{12}} \\ \underset{(q\times p)}{\mathbf{A}_{12}^T} & \underset{(q\times q)}{\mathbf{A}_{22}} \end{bmatrix}} \begin{Bmatrix} \underset{(p\times 1)}{\mathbf{X}_1} \\ \underset{(q\times 1)}{\mathbf{X}_2} \end{Bmatrix}$$

where $q = n - p$. The expansion of $F = \mathbf{X}^T\mathbf{a}\mathbf{X}$ has the form

$$F = \mathbf{X}_1^T\mathbf{A}_{11}\mathbf{X}_1 + 2\mathbf{X}_1^T\mathbf{A}_{12}\mathbf{A}_2 + \mathbf{X}_2^T\mathbf{A}_{22}\mathbf{X}_2$$

Now, we take $\mathbf{X}_2 = \mathbf{0}$ and denote the result by F_p:

$$F_p = \mathbf{X}_1^T\mathbf{A}_{11}\mathbf{X}_1$$

For $F_p > 0$ for arbitrary $\mathbf{X}_1$, $\mathbf{A}_{11}$ must be positive definite. Since $|\mathbf{A}_{11}|$ is equal to the product of the characteristic values of $\mathbf{A}_{11}$, it follows that $|\mathbf{A}_{11}|$ must be positive.

(a) By taking $p = 1, 2, \ldots, n$, deduce that

$$\Delta_p = |\mathbf{A}_{11}| > 0 \qquad p = 1, 2, \ldots, n$$

are necessary conditions for **a** to be positive definite. Note that it remains to show that they are also sufficient conditions.

(b) Discuss the case where $\Delta_p = 0$.

2–16. Refer to Prob. 1–25. Consider **a** to be symmetrical.

(a) Deduce that one can always express **a** as the product of nonsingular lower and upper triangular matrices when **a** is positive definite.

(b) Suppose we take

$$b_{11} = b_{22} = \cdots = b_{nn} = +1$$

Show that **a** is positive definite when

$$g_{jj} > 0 \qquad j = 1, 2, \ldots, n$$

and positive semi-definite when

$$g_{jj} \geqslant 0 \qquad j = 1, 2, \ldots, n$$

and at least *one* of the diagonal elements of **g** is zero.

(c) Suppose we take $\mathbf{g} = \mathbf{b}^T$. Then,

$$|\mathbf{G}_{11}| = |\mathbf{B}_{11}|$$

and

$$\Delta_p = |\mathbf{A}_{11}| = b_{11}^2 b_{22}^2 \cdots b_{pp}^2$$

Show that the diagonal elements of **b** will always be real when **a** is positive definite.

2–17. If a quasi-diagonal matrix, say **a**, is symmetrically partitioned, the submatrix $\mathbf{A}_{11}$ is also a quasi-diagonal matrix. Establish that

$$\mathbf{a} = [\mathbf{A}_i\delta_{ij}] \qquad i, j = 1, 2, \ldots, N$$

is positive definite only when $\mathbf{A}_i$ $(i = 1, 2, \ldots, N)$ are positive definite.

Hint: Use the result of Prob. 1–23. Verify for

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 5 & 2 \end{bmatrix}$$

2–18. Suppose we express **a** as the product of two quasi-triangular matrices, for example,

$$\underset{(n\times n)}{\mathbf{a}} = \begin{bmatrix} \underset{(p\times p)}{\mathbf{G}_{11}} & \underset{(p\times q)}{\mathbf{0}} \\ \underset{(q\times p)}{\mathbf{G}_{21}} & \underset{(q\times q)}{\mathbf{G}_{22}} \end{bmatrix} \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{0} & \mathbf{B}_{22} \end{bmatrix}$$

where $p + q = n$. We take

$$\mathbf{B}_{11} = \mathbf{I}_p \qquad \mathbf{B}_{22} = \mathbf{I}_q$$

Show that the diagonal submatrices of **g** are nonsingular for arbitrary **p** when **a** is positive definite.