17 General Formulation—Linear System

17–1. INTRODUCTION

We consider a system comprising m linear elastic members interconnected at j joints. We suppose there are i degrees of freedom per joint (i.e., the joint displacement and force matrices are of order $i \times 1$) and the geometry and joint quantities are referred to a global frame. Also, we neglect geometry change due to deformation. In the previous chapter, we applied the direct stiffness method, which is actually a displacement method, to this system. Now, in this chapter, we first develop the governing matrix equations and then deduce the equations corresponding to the force and displacement solution procedures. We also establish variational principles for the force and displacement methods. Finally, we discuss how one can introduce member deformation constraints in the displacement method. Since the basic steps involved in the member system formulation are the same as for the ideal truss formulation, we recommend that the reader review Chapters 6 through 9 before starting this chapter.

Let r be the number of prescribed joint displacements. Then, the total number of joint displacement unknowns, n_d, is

$$n_d = ij - r \tag{17–1}$$

The total number of force unknowns, n_f, is equal to r (the reactions corresponding to the prescribed displacements) plus q_T, the total number of *member force unknowns*:

$$n_f = r + q_T = r + (q_1 + q_2 + \cdots + q_m) \tag{17–2}$$

where q_n represents the number of force unknowns for member n. By definition, q_n is equal to the number of force quantities that have to be specified in order to be able to determine the total internal force matrix ($\mathscr{F}$) at an arbitrary point. If the member is fully restrained at each end, $q_n = i$. For partial restraint, q_n is equal to i minus the number of independent force releases. Note that when the member is pinned at both ends, $q_n = 1$ since there are only five independent moment releases.

There are q_T equations relating the member forces and the joint displacements. Also, there are ij equilibrium equations relating the external joint forces and the member forces. The formulation is consistent, i.e., the number of equations is equal to the number of unknowns. If $n_f = ij$, the system is said to be *statically determinate* since the force unknowns can be determined using only the equilibrium equations. The difference, $n_f - ij$, is generally called the degree of static indeterminacy, and represents the order of the final system of equations for the force method. For the displacement method, the final system of equations are of order n_d. In what follows, we first establish the member force–joint displacement relations by generalizing the results of Sec. 15–12. Then, we assemble the joint force-equilibrium equations. Finally, we introduce the joint displacement restraints.

17–2. MEMBER EQUATIONS

The reduced member equations were developed in Sec. 15–12. For convenience, we summarize the notation and equations below (see (15–100)):

$$\begin{aligned}
\mathbf{Z} &= \text{member force matrix } (q_n \times 1) \\
\bar{\mathscr{F}}^n_b &= \mathbf{EZ} + \mathbf{G} \qquad (i \times 1) \\
\bar{\mathscr{F}}^n_A &= -\mathscr{F}^n_{A,o} - \mathscr{X}^n_{BA}\mathscr{F}^n_B \\
&= -\mathscr{F}^n_{A,o} - \mathscr{X}^n_{BA}\mathbf{G} - \mathscr{X}^n_{BA}\mathbf{EZ} \\
\mathbf{f}^n &= \text{member flexibility matrix } (i \times i) \\
\mathbf{f}_r &= \text{reduced member flexibility matrix } (q_n \times q_n) = \mathbf{E}^T\mathbf{f}^n\mathbf{E} \\
\mathscr{V}^n &= \text{member deformation matrix } (i \times i) = \mathscr{U}^n_B - \mathscr{X}^{n,T}_{BA}\mathscr{U}^n_A \\
\mathscr{V}^n_{o,z} &= \text{initial member deformation matrix } (i \times i) = \mathscr{V}^n_o + \mathbf{f}^n\mathbf{G} \\
\mathbf{f}_r\mathbf{Z} &= \mathbf{E}^T(\mathscr{V}^n - \mathscr{V}^n_{o,z})
\end{aligned} \tag{a}$$

These equations include the effect of partial end restraint, internal force releases, and reductions due to symmetry or antisymmetry. We can also use (a) for complete end restraint by setting $\mathbf{E} = \mathbf{I}_i$ and $\mathbf{G} = \mathbf{0}$.

Now, we introduce new notation which is more convenient. First, we note that $\mathbf{G}$ contains the end forces at B due to the external member loads acting on the primary structure defined by $\mathbf{Z} = \mathbf{0}$. Also, $-\mathscr{F}^n_{A,o} - \mathscr{X}^n_{BA}\mathbf{G}$ are the end forces at A. Then we write

$$\begin{aligned}
\bar{\mathscr{F}}^n_{B,o} &= \mathbf{G} \\
\bar{\mathscr{F}}^n_{A,o} &= -\mathscr{F}^n_{A,o} - \mathscr{X}^n_{BA}\mathbf{G}
\end{aligned} \tag{17–3}$$

Next, we note that the equation relating $\mathbf{Z}$ and $\mathscr{V}^n$, $\mathscr{V}^n_{o,z}$, is a compatibility requirement. The term $\mathbf{f}_r\mathbf{Z} + \mathbf{E}^T\mathscr{X}^n_{o,z}$ is the relative deformation in the *positive* sense of $\mathbf{Z}$ due to the member loads and the member redundants, $\mathbf{Z}$, whereas $\mathbf{E}^T\mathscr{V}^n$ is the relative deformation in the *negative* sense of $\mathbf{Z}$ due to support (joint) movement. The net relative deformation must be zero for continuity.

Then, we define

$$\begin{aligned}\mathscr{V}_r &= \text{reduced member deformation matrix } (q_n \times 1)\\ &= \mathbf{E}^T\mathscr{V}^n = \mathbf{E}^T(\mathscr{U}^n_B - \mathscr{X}^{n,T}_{BA}\mathscr{U}^n_A) \qquad (17\text{–}4)\\ \mathscr{V}_{ro} &= \text{reduced initial member deformation matrix } (q_n \times 1)\\ &= \mathbf{E}^T\mathscr{V}_{o,z} = \mathbf{E}^T(\mathscr{V}^n_o + \mathbf{f}^n\mathbf{G})\end{aligned}$$

With this notation, the member equations take the form

$$\begin{aligned}\mathscr{F}^n_B &= \mathscr{F}^n_{B,o} + \mathbf{EZ}\\ \mathscr{F}^n_A &= \mathscr{F}^n_{A,o} - \mathscr{X}^n_{BA}\mathbf{EZ} \qquad (17\text{–}5)\\ \mathscr{V}_r &= \mathscr{V}_{ro} + \mathbf{f}_r\mathbf{Z} = \mathbf{E}^T(\mathscr{U}^n_B - \mathscr{X}^{n,T}_{BA}\mathscr{U}^n_A)\end{aligned}$$

We generalize the relations for member n by setting

$$\begin{aligned}&B = n_+ \qquad A = n_-\\ &\mathbf{E} = \mathbf{E}_n \qquad \mathbf{Z} = \mathbf{Z}_n \qquad (17\text{–}6)\\ &\mathscr{X}^n_{BA} = \mathscr{X}^n_{n_+n_-} = \mathscr{X}^n_n \qquad \mathscr{V}_r = \mathscr{V}_{r,n} \qquad \mathscr{V}_{ro} = \mathscr{V}_{ro,n} \qquad \mathbf{f}_r = \mathbf{f}_{r,n}\end{aligned}$$

Since the joint quantities are referred to the global frame, we must transform the end forces and displacements from the member frame (frame n) to the global frame (frame o), using

$$\begin{aligned}\mathscr{U}^n &= \mathscr{R}^{on}\mathscr{U}^o \qquad (b)\\ \mathscr{F}^o &= \mathscr{R}^{on,T}\mathscr{F}^n\end{aligned}$$

The final equations follow.

Member Forces—End Forces

$$\begin{aligned}\mathscr{F}^o_{n_+} &= (\mathscr{R}^{on,T}\mathbf{E}_n)\mathbf{Z}_n + \mathscr{F}^o_{n_+,o} \qquad (17\text{–}7)\\ \mathscr{F}^o_{n_-} &= -(\mathscr{R}^{on,T}\mathscr{X}^n_n\mathbf{E}_n)\mathbf{Z}_n + \mathscr{F}^o_{n_-,o}\end{aligned}$$

Member Forces—Joint Displacements (q_n Equations)

$$\begin{aligned}\mathscr{V}_{r,n} &= \mathscr{V}_{ro,n} + \mathbf{f}_{r,n}\mathbf{Z}_n \qquad (17\text{–}8)\\ &= (\mathbf{E}^T_n\mathscr{R}^{on})\mathscr{U}^o_{n_+} - (\mathbf{E}^T_n\mathscr{X}^{n,T}_n\mathscr{R}^{on})\mathscr{U}^o_{n_-}\end{aligned}$$

The force translation transformation matrix, $\mathscr{X}$, is a second-order tensor, i.e., it transforms according to†

$$\mathscr{X}^q = \mathscr{R}^{qp,T}\mathscr{X}^p\mathscr{R}^{qp} \qquad (a)$$

where p and q are arbitrary orthogonal frames. Then,

$$\mathscr{X}^o_n = \mathscr{X}^o_{n_+n_-} = \mathscr{R}^{on,T}\mathscr{X}^n_n\mathscr{R}^{on} \qquad (17\text{–}9)$$

and it follows that

$$\begin{aligned}\mathscr{R}^{on,T}\mathscr{X}^n_n &= \mathscr{X}^o_n\mathscr{R}^{on,T} \qquad (17\text{–}10)\\ \mathscr{X}^{n,T}_n\mathscr{R}^{on} &= \mathscr{R}^{on}\mathscr{X}^{on,T}_n\end{aligned}$$

† See Sec. 10–2.

Using (17–10), we can express $\mathscr{F}^o_{n_-}$ and $\mathscr{V}_{r,n}$ as

$$\begin{aligned}\mathscr{F}^o_{n_-} &= \mathscr{F}^o_{n_-,o} - \mathscr{X}^o_n(\mathscr{R}^{on,T}\mathbf{E}_n)\mathbf{Z}_n \qquad (17\text{–}11)\\ \mathscr{V}_{r,n} &= (\mathbf{E}^T_n\mathscr{R}^{on})(\mathscr{U}^o_{n_+} - \mathscr{X}^{o,T}_n\mathscr{U}^o_{n_-})\end{aligned}$$

We prefer to work with $\mathscr{X}^n_n$ since it is a natural property of the member whereas $\mathscr{X}^o_n$ depends on the selection of the global frame.

17–3. SYSTEM FORCE-DISPLACEMENT RELATIONS

Equation (17–8) represents the q_n force-deformation relations for member n. By defining general flexibility and deformation matrices, we can express the complete set of q_T member force-deformation relations as a single matrix equation. We let

$$\begin{aligned}\mathbf{Z} &= \text{total member force matrix } (q_T \times 1)\\ &= \{\mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_m\}\\ \mathscr{V} &= \text{total reduced member deformation matrix } (q_T \times 1)\\ &= \{\mathscr{V}_{r,1}, \mathscr{V}_{r,2}, \ldots, \mathscr{V}_{r,m}\}\\ \mathscr{V}_o &= \text{total reduced initial member deformation matrix } (q_T \times 1)\\ &= \{\mathscr{V}_{ro,1}, \mathscr{V}_{ro,2}, \ldots, \mathscr{V}_{ro,m}\} \qquad (17\text{–}12)\\ \mathbf{f} &= \text{total reduced member flexibility matrix } (q_T \times q_T)\\ &= \begin{bmatrix}\mathbf{f}_{r,1} & & & \\ & \mathbf{f}_{r,2} & & \\ & & \ddots & \\ & & & \mathbf{f}_{r,m}\end{bmatrix}\end{aligned}$$

Note that $\mathbf{f}$ is quasi-diagonal, symmetrical, and *positive definite.* With this notation, the q_T force-deformation relations are given by

$$\mathscr{V} = \mathscr{V}_o + \mathbf{fZ} \qquad (17\text{–}13)$$

It remains to generalize the deformation-displacement relations.

We define $\mathscr{U}$ as the total joint displacement matrix referred to the global frame.

$$\underset{(ij\times 1)}{\mathscr{U}} = \{\mathscr{U}^o_1, \mathscr{U}^o_2, \ldots, \mathscr{U}^o_j\} \qquad (17\text{–}14)$$

and express $\mathscr{V}$ as

$$\mathscr{V} = \mathscr{A}\mathscr{U} \qquad (17\text{–}15)$$

The partitioned form is

$$\begin{Bmatrix}\mathscr{V}_{r,1}\\ \mathscr{V}_{r,2}\\ \vdots\\ \mathscr{V}_{r,m}\end{Bmatrix} = \begin{bmatrix}\mathscr{A}_{11} & \mathscr{A}_{12} & \cdots & \mathscr{A}_{1j}\\ \mathscr{A}_{21} & \mathscr{A}_{22} & & \mathscr{A}_{2j}\\ \vdots & \vdots & & \\ \mathscr{A}_{m1} & \mathscr{A}_{m2} & & \mathscr{A}_{m2}\end{bmatrix}\begin{Bmatrix}\mathscr{U}^o_1\\ \mathscr{U}^o_2\\ \vdots\\ \mathscr{U}^o_j\end{Bmatrix} \qquad (17\text{–}16)$$

Row n of $\mathscr{A}$ corresponds to member n. The submatrices in row n are of order $q_n \times i$. Now, we see from (17–8) that there are only two non-zero elements in row n and they are at columns n_+, n_-. The assembly of $\mathscr{A}$ is defined by

$$\begin{aligned} \mathscr{A}_{nn_+} &= \mathbf{E}_n^T \mathscr{R}^{on} \\ \mathscr{A}_{nn_-} &= -\mathbf{E}_n^T \mathscr{X}_n^{n,T} \mathscr{R}^{on} \\ &= -\mathbf{E}_n^T \mathscr{R}^{on} \mathscr{X}_n^{o,T} \\ \mathscr{A}_{ns} &= \mathbf{0} \\ s &\neq n_+, n_- \\ s &= 1, 2, \dots, j \\ n &= 1, 2, \dots, m \end{aligned} \tag{17–17}$$

It is of interest to express $\mathscr{A}$ in factored form. First, we define the following matrices:

$$\mathscr{U}_+ = \{\mathscr{U}^o_{1_+}, \mathscr{U}^o_{2_+}, \dots, \mathscr{U}^o_{m_+}\} \qquad (im \times 1)$$

$$\mathscr{U}_- = \{\mathscr{U}^o_{1_-}, \mathscr{U}^o_{2_-}, \dots, \mathscr{U}_{m_-}\} \qquad (im \times 1)$$

$$\mathscr{R} = \begin{bmatrix} \mathscr{R}^{o1} & & & \\ & \mathscr{R}^{o2} & & \\ & & \ddots & \\ & & & \mathscr{R}^{om} \end{bmatrix} \qquad (im \times im)$$

$$\mathbf{E} = \begin{bmatrix} \mathbf{E}_1 & & & \\ & \mathbf{E}_2 & & \\ & & \ddots & \\ & & & \mathbf{E}_m \end{bmatrix} \qquad (im \times q_T) \tag{17–18}$$

$$\mathscr{X} = \begin{bmatrix} \mathscr{X}_1^1 & & & \\ & \mathscr{X}_2^2 & & \\ & & \ddots & \\ & & & \mathscr{X}_m^m \end{bmatrix} \qquad (im \times im)$$

$$\mathscr{X}^o = \begin{bmatrix} \mathscr{X}_1^o & & & \\ & \mathscr{X}_2^o & & \\ & & \ddots & \\ & & & \mathscr{X}_m^o \end{bmatrix} \qquad (im \times im)$$

Using this notation, the expression for $\mathscr{V}$ takes the form

$$\begin{aligned} \mathscr{V} &= \mathbf{E}^T \mathscr{R} \mathscr{U}_+ - \mathbf{E}^T \mathscr{X}^T \mathscr{R} \mathscr{U}_- \\ &= \mathbf{E}^T \mathscr{R} (\mathscr{U}_+ - \mathscr{X}^{o,T} \mathscr{U}_-) \end{aligned} \tag{a}$$

Next, we relate $\mathscr{U}_+$, $\mathscr{U}_-$ to $\mathscr{U}$, using member-joint connectivity matrices for the positive ($\mathbf{C}_+$) and negative ($\mathbf{C}_-$) ends:

$$\begin{aligned} \mathscr{U}_+ &= \mathbf{C}_+ \mathscr{U} \\ \mathscr{U}_- &= \mathbf{C}_- \mathscr{U} \end{aligned} \tag{17–19}$$

Note that rows n of $\mathbf{C}_+$, $\mathbf{C}_-$ correspond to member n. There is only one nonzero element in a row. For row n, we enter $+\mathbf{I}_i$ in column n_+ of $\mathbf{C}_+$ and column n_- of $\mathbf{C}_-$. Finally, combining (a) and (17–19), we have

$$\begin{aligned} \mathscr{V} &= (\mathbf{E}^T \mathscr{R} \mathbf{C}_+ - \mathbf{E}^T \mathscr{X}^T \mathscr{R} \mathbf{C}_-)\mathscr{U} \\ &= (\mathbf{E}^T \mathscr{R})(\mathbf{C}_+ - \mathscr{X}^{o,T} \mathbf{C}_-)\mathscr{U} \end{aligned} \tag{17–20}$$

and it follows that

$$\begin{aligned} \mathscr{A} &= \mathbf{E}^T \mathscr{R} \mathbf{C}_+ - \mathbf{E}^T \mathscr{X}^T \mathscr{R} \mathbf{C}_- \\ &= (\mathbf{E}^T \mathscr{R})(\mathbf{C}_+ - \mathscr{X}^{o,T} \mathbf{C}_-) \end{aligned} \tag{17–21}$$

For an ideal truss, (17–21) reduces to (see Equation 6–28)

$$\mathscr{A} = \boldsymbol{\alpha}(\mathbf{C}_+ - \mathbf{C}_-) \tag{b}$$

where $\boldsymbol{\alpha}$ contains the direction cosines for the bars.

17–4. SYSTEM EQUILIBRIUM EQUATIONS

We have used the member force-equilibrium equations in developing the member force-displacement relations, so it remains only to satisfy equilibrium of the joints. There are i equations for each joint, and a total of ij equations. The expressions for the end forces in terms of the member forces are given by (17–7). Assembling the joint force-equilibrium equations involves only summing at each joint the end forces incident on the joint.

We define $\mathscr{P}$ as the total external joint force matrix referred to the global frame:

$$\underset{(ij \times 1)}{\mathscr{P}} = \{\mathscr{P}^o_1, \mathscr{P}^o_2, \dots, \mathscr{P}^o_j\} \tag{17–22}$$

and $\mathscr{P}_I$ as the initial ($\mathbf{Z} = \mathbf{0}$) joint force matrix:

$$\underset{(ij \times 1)}{\mathscr{P}_I} = \{\mathscr{P}^o_{I,1}, \mathscr{P}^o_{I,2}, \dots, \mathscr{P}^o_{I,j}\} \tag{17–23}$$

The elements of $\mathscr{P}_I$ are the joint forces due to external forces acting on the members with $\mathbf{Z} = \mathbf{0}$. We express the complete set of equations as

$$\mathscr{P} = \mathscr{P}_I + \mathscr{B}\mathbf{Z}$$

$$\begin{Bmatrix} \mathscr{P}^o_1 \\ \mathscr{P}^o_2 \\ \vdots \\ \mathscr{P}^o_j \end{Bmatrix} = \begin{Bmatrix} \mathscr{P}^o_{I,1} \\ \mathscr{P}^o_{I,2} \\ \vdots \\ \mathscr{P}^o_{I,j} \end{Bmatrix} + \begin{bmatrix} \mathscr{B}_{11} & \mathscr{B}_{12} & \cdots & \mathscr{B}_{1m} \\ \mathscr{B}_{21} & \mathscr{B}_{22} & \cdots & \mathscr{B}_{2m} \\ \vdots & \vdots & & \vdots \\ \mathscr{B}_{j1} & \mathscr{B}_{j2} & \cdots & \mathscr{B}_{jm} \end{bmatrix} \begin{Bmatrix} \mathbf{Z}_1 \\ \mathbf{Z}_2 \\ \vdots \\ \mathbf{Z}_m \end{Bmatrix} \tag{17–24}$$

We assemble $\mathscr{P}_I$ and $\mathscr{B}$, working with successive members. The contribution of member n follows from (17–7):

In $\mathscr{P}_I$:

$$\begin{aligned} &\mathscr{F}^o_{n_+,o} \quad \text{in row } n_+ \\ &\mathscr{F}^o_{n_-,o} \quad \text{in row } n_- \end{aligned} \tag{17–25}$$

Column n of $\mathscr{B}$:

$$\begin{aligned}\mathscr{B}_{n_+n} &= \mathscr{R}^{on,\,T}\mathbf{E}_n\\ \mathscr{B}_{n_-n} &= -\mathscr{R}^{on,\,T}\mathscr{X}_n^n\mathbf{E}_n\\ &= -\mathscr{X}_n^o\mathscr{R}^{on,\,T}\mathbf{E}_n\\ \mathscr{B}_{sn} &= \mathbf{0}\\ s &\neq n_+, n_-\\ s &= 1, 2, \ldots, j\end{aligned} \tag{17-26}$$

Comparing (17–26) with (17–17), we see that

$$\mathscr{B} = \mathscr{A}^T \tag{17-27}$$

We let

$$\begin{aligned}\bar{\mathscr{F}}_{+,o} &= \{\bar{\mathscr{F}}^1_{1_+}, \bar{\mathscr{F}}^2_{2_+}, \ldots, \bar{\mathscr{F}}^m_{m_+}\}\\ \bar{\mathscr{F}}_{-,o} &= \{\bar{\mathscr{F}}^1_{1_-}, \bar{\mathscr{F}}^2_{2_-}, \ldots, \bar{\mathscr{F}}^m_{m_-}\}\end{aligned} \tag{17-28}$$

Then, we can express $\mathscr{P}_I$ as

$$\mathscr{P}_I = \mathbf{C}_+^T\mathscr{R}^T\bar{\mathscr{F}}_{+,o} + \mathbf{C}_-^T\mathscr{R}^T\bar{\mathscr{F}}_{-,o} \tag{17-29}$$

17–5. INTRODUCTION OF JOINT DISPLACEMENT RESTRAINTS; GOVERNING EQUATIONS

The governing equations for the unrestrained system are

$$\begin{aligned}\mathscr{P} &= \mathscr{P}_I + \mathscr{B}\mathbf{Z} = \mathscr{P}_I + \mathscr{A}^T\mathbf{Z}\\ \mathscr{V} &= \mathscr{V}_o + \mathbf{f}\mathbf{Z} = \mathscr{A}\mathscr{U}\end{aligned} \tag{17-30}$$

Now, we suppose r joint displacements are prescribed. We rearrange $\mathscr{U}$ so that the prescribed displacements are last. We also rearrange $\mathscr{P}$ and $\mathscr{P}_I$:

$$\begin{aligned}\mathscr{U} \to \mathbf{U} &= \begin{Bmatrix}\mathbf{U}_1\\ \bar{\mathbf{U}}_2\end{Bmatrix} \begin{matrix}(n_d \times 1)\\ (r \times 1)\end{matrix}\\ \mathscr{P} \to \mathbf{P} &= \begin{Bmatrix}\bar{\mathbf{P}}_1\\ \mathbf{P}_2\end{Bmatrix} \begin{matrix}(n_d \times 1)\\ (r \times 1)\end{matrix}\\ \mathscr{P}_I \to \mathbf{P}_I &= \begin{Bmatrix}\bar{\mathbf{P}}_{I,1}\\ \bar{\mathbf{P}}_{I,2}\end{Bmatrix} \begin{matrix}(n_d \times 1)\\ (r \times 1)\end{matrix}\end{aligned} \tag{17-31}$$

where $\bar{\mathbf{U}}_2$, $\bar{\mathbf{P}}_1$, and $\mathbf{P}_I$ are prescribed. We use $\mathbf{B}$, $\mathbf{A}$ to represent the rearranged forms of $\mathscr{B}$, $\mathscr{A}$:

$$\begin{aligned}\mathscr{A} \to \mathbf{A} &= \overset{(q_T \times n_d)\ (q_T \times r)}{[\mathbf{A}_1 \mid \mathbf{A}_2]}\\ \mathscr{B} \to \mathbf{B} &= \begin{bmatrix}\mathbf{B}_1\\ \hline \mathbf{B}_2\end{bmatrix} = \begin{bmatrix}\mathbf{A}_1^T\\ \hline \mathbf{A}_2^T\end{bmatrix} \begin{matrix}(n_d \times q_T)\\ (r \times q_T)\end{matrix}\end{aligned} \tag{17-32}$$

Finally, we write the equations for the *restrained* system as

$$\bar{\mathbf{P}}_1 = \bar{\mathbf{P}}_{I,1} + \mathbf{B}_1\mathbf{Z} = \mathbf{P}_{I,1} + \mathbf{A}_1^T\mathbf{Z} \quad (n_d \text{ eqs.}) \tag{17-33}$$

$$\mathbf{P}_2 = \bar{\mathbf{P}}_{I,2} + \mathbf{B}_2\mathbf{Z} = \mathbf{P}_{I,2} + \mathbf{A}_2^T\mathbf{Z} \quad (r \text{ eqs.}) \tag{17-34}$$

$$\begin{aligned}\mathscr{V} &= \mathbf{f}\mathbf{Z} + \mathscr{V}_o = \mathbf{A}_1\mathbf{U}_1 + \mathbf{A}_2\bar{\mathbf{U}}_2 \quad (q_T \text{ eqs.})\\ &= \mathbf{B}_1^T\mathbf{U}_1 + \mathbf{B}_2^T\bar{\mathbf{U}}_2\end{aligned} \tag{17-35}$$

The unknowns are the q_T member forces ($\mathbf{Z}$), the n_d displacements ($\mathbf{U}_1$), and the r reactions ($\mathbf{P}_2$).

If the restraints are parallel to the directions of the global frame, the transformation of $\mathscr{A}$ to $\mathbf{A}$ (or $\mathscr{B}$ to $\mathbf{B}$) involve only a permutation of the columns of $\mathscr{A}$ (rows of $\mathscr{B}$). The same permutation is applied to the rows of $\mathscr{P}_I$.

Suppose joint q is partially restrained and the restraint directions do not coincide with the global frame directions. We first transform the force and displacement matrices for joint q from the global frame to the restraint frame, using

$$\begin{aligned}\mathscr{U}_q^o &= \mathscr{R}^{oq,\,T}\mathscr{U}_q^q\\ \mathscr{P}_q^q &= \mathscr{R}^{oq}\mathscr{P}_q^o\end{aligned} \tag{17-36}$$

This step involves postmultiplying column q of $\mathscr{A}$ by $\mathscr{R}^{oq,\,T}$ and premultiplying row q of $\mathscr{P}_I$, $\mathscr{B}$ by $\mathscr{R}^{oq}$. We write the transformed equations as

$$\begin{aligned}\mathscr{P}^J &= \mathscr{P}_I^J + \mathscr{B}^J\mathbf{Z}\\ \mathscr{V} &= \mathscr{V}_o + \mathbf{f}\mathbf{Z} = \mathscr{A}^J\mathscr{U}^J\end{aligned} \tag{17-37}$$

where the superscript J indicates that joint forces and displacements are referred to local restraint frames. The final equations are obtained by permuting the columns of $\mathscr{A}^J$ (rows of $\mathscr{B}^J$), the rows of $\mathscr{P}_I^J$, and then partitioning.

The transformation of $\mathscr{U}$ to $\mathbf{U}$ can be expressed as a matrix product,

$$\mathbf{U} = \mathbf{D}\mathscr{U} = \mathbf{\Pi}\mathscr{R}^{oJ}\mathscr{U} \tag{17-38}$$

where $\mathscr{R}^{oJ}$ contains the rotation matrices for the joint restraint frames,

$$\mathscr{R}^{oJ} = \begin{bmatrix}\mathscr{R}^{o1} & & & \\ & \mathscr{R}^{o2} & & \\ & & \ddots & \\ & & & \mathscr{R}^{oj}\end{bmatrix} \tag{17-39}$$

and $\mathbf{\Pi}$ is the row permutation matrix. One can generate $\mathbf{\Pi}$ by starting with $\mathbf{I}$ and permuting the rows according to the new listing of the joint displacements, i.e., with the prescribed displacements last. Now, $\mathbf{D}$ is an orthogonal matrix,

$$\mathbf{D}^{-1} = \mathbf{D}^T \tag{17-40}$$

Then,

$$\begin{aligned}\mathscr{U} &= \mathbf{D}^T\mathbf{U}\\ \mathbf{P} &= \mathbf{D}\mathscr{P}\end{aligned} \tag{a}$$

and it follows that

$$\begin{aligned} \mathbf{P}_I &= \mathbf{D}\mathscr{P}_I \\ \mathbf{A} &= \mathscr{A}\mathbf{D}^T \\ \mathbf{B} &= \mathbf{A}^T = \mathbf{D}\mathscr{B} \end{aligned} \tag{17-41}$$

The partitioned forms are obtained by partitioning $\mathbf{D}$:

$$\mathbf{D} = \begin{bmatrix} \mathbf{D}_1 \\ \hline \mathbf{D}_2 \end{bmatrix} \begin{matrix} (n_d \times ij) \\ (r \times ij) \end{matrix} \tag{17-42}$$

Finally, we can write

$$\begin{aligned} \mathbf{A}_1 &= \mathscr{A}\mathbf{D}_1^T = \mathbf{B}_1^T \\ \mathbf{A}_2 &= \mathscr{A}\mathbf{D}_2^T = \mathbf{B}_2^T \\ \mathbf{P}_{I,1} &= \mathbf{D}_1\mathscr{P}_I \\ \mathbf{P}_{I,2} &= \mathbf{D}_2\mathscr{P}_I \end{aligned} \tag{17-43}$$

To determine the requirement for initial stability, we consider (17–33),

$$\mathbf{B}_1\mathbf{Z} = \bar{\mathbf{P}}_1 - \bar{\mathbf{P}}_{I,1} \tag{a}$$

which represents n_d equations in q_T unknowns. For the equations to be consistent for an *arbitrary* loading, the rank of $\mathbf{B}_1$ must equal n_d. Therefore, the stability requirement for the system is

$$r(\mathbf{B}_1) = r(\mathbf{A}_1) = n_d \tag{17-44}$$

Since $\mathbf{B}_1$ is of order $n_d \times q_T$, a necessary but not sufficient condition for stability is

$$q_T > n_d = ij - r \tag{17-45}$$

Equation (17–44) is the stability requirement for a geometrically linear system. It is also the *initial* stability requirement for a geometrically nonlinear system. In the next chapter, we develop the stability criteria for a geometrically nonlinear system subjected to a finite loading.

17–6. NETWORK FORMULATION

In the formulation presented in the previous articles, we worked with the *actual* joint displacements and external joint forces referred to the global frame. The governing equations are given by (17–30), which we list below for convenience:

$$\begin{aligned} \mathscr{P} &= \mathscr{P}_I + \mathscr{A}^T\mathbf{Z} \\ \mathscr{V} &= \mathscr{V}_o + \mathbf{f}\mathbf{Z} = \mathscr{A}\mathscr{U} \end{aligned} \tag{a}$$

where

$$\begin{aligned} \mathscr{A} &= (\mathbf{E}^T\mathscr{R})(\mathbf{C}_+ - \mathscr{X}^{o,T}\mathbf{C}_-) \\ \mathscr{P}_I &= \mathbf{C}_+^T\mathscr{R}^T\bar{\mathscr{F}}_{+,o} + \mathbf{C}_-^T\mathscr{R}^T\bar{\mathscr{F}}_{-,o} \end{aligned} \tag{b}$$

One assembles $\mathscr{A}, \mathscr{P}_I$, using (17–17), (17–25), which are actually the expansions of (b). By introducing new joint variables, we can express $\mathscr{A}$ in terms of only one

connectivity matrix, $\mathbf{C}_+ - \mathbf{C}_- = \mathbf{C}$. The rule (17–17) for assembling $\mathscr{A}$ still applies except that now $\mathscr{A}_{nn_-} = -\mathscr{A}_{nn_+}$.

Let Y denote some arbitrary point. Suppose we express the *actual* force and displacement matrices for joint k in terms of their *equivalents* at point Y. We define

$$\begin{aligned} \mathscr{P}^o_{Y,k} &= \text{statically equivalent force at } Y \text{ due to } \mathscr{P}^o_k, \text{ the actual force matrix at joint } k. \\ \mathscr{U}^o_{Y,k} &= \text{displacement at } Y \text{ due to rigid body motion about joint } k. \end{aligned} \tag{17-46}$$

The *actual* and *equivalent* quantities are related by

$$\begin{aligned} \mathscr{P}^o_{Y,k} &= \mathscr{X}^o_{kY}\mathscr{P}^o_k \\ \mathscr{U}^o_k &= \mathscr{X}^{o,T}_{kY}\mathscr{U}^o_{Y,k} \end{aligned} \tag{17-47}$$

where

$$\mathscr{X}^o_{kY} = \begin{bmatrix} \mathbf{I}_\alpha & \mathbf{0} \\ \mathbf{X}^o_{kY} & \mathbf{I}_\beta \end{bmatrix}$$

$$X^o_{kY} = \left|\begin{array}{c:c|c} 0 & -(x^o_{k3} - x^o_{Y3}) & (x^o_{k2} - x^o_{Y2}) \\ \hdashline (x^o_{k3} - x^o_{Y3}) & 0 & -(x^o_{k1} - x^o_{Y1}) \\ \hline -(x^o_{k2} - x^o_{Y2}) & (x^o_{k1} - x^o_{Y1}) & 0 \end{array}\right| \leftarrow \text{out-of-plane} \tag{17-48}$$

↑ planar

We could operate on (b), but it is more convenient to start with (17–11):

$$\mathscr{V}_{r,n} = (\mathbf{E}_n^T\mathscr{R}^{on})(\mathscr{U}^o_{n+} - \mathscr{X}^{o,T}_n\mathscr{U}^o_{n-}) \tag{c}$$

Now, by definition,

$$\mathscr{X}^o_n = \mathscr{X}^o_{n_+n_-} \tag{d}$$

Substituting for $\mathscr{U}^o_{()}$, using (17–47), and noting that

$$\mathscr{X}^{o,T}_{n_+n_-}\mathscr{X}^{o,T}_{n_-Y} = \mathscr{X}^{o,T}_{n_+Y} \tag{e}$$

we obtain

$$\mathscr{V}_{r,n} = (\mathbf{E}_n^T\mathscr{R}^{on}\mathscr{X}^{o,T}_{n_+Y})(\mathscr{U}^o_{Y,n_+} - \mathscr{U}^o_{Y,n_-}) \tag{17-48}$$

The remaining steps are the same as followed previously. We let

$$\mathscr{U}_Y = \{\mathscr{U}^o_{Y,1}, \mathscr{U}^o_{Y,2}, \ldots, \mathscr{U}^o_{Y,j}\} \tag{17-49}$$

and write

$$\mathscr{V} = \mathscr{A}_Y\mathscr{U}_Y \tag{17-50}$$

The generation of $\mathscr{A}_Y$ follows from (17–48). For now n,

$$\begin{aligned} \mathscr{A}_{Y,nn_+} &= -\mathscr{A}_{Y,nn_-} = \mathbf{E}_n^T\mathscr{R}^{on}\mathscr{X}^{o,T}_{n_+Y} \\ \mathscr{A}_{Y,ns} &= \mathbf{0} \qquad (q_n \times 1) \\ s &\neq n_+, n_- \\ s &= 1, 2, \ldots, j \end{aligned} \tag{17-51}$$

To express $\mathcal{A}_Y$ in factored form, we let

$$\mathcal{X}^o_+ = \begin{bmatrix} \mathcal{X}^o_{1+Y} & & & \\ & \mathcal{X}^o_{2+Y} & & \\ & & \ddots & \\ & & & \mathcal{X}^o_{m+Y} \end{bmatrix} \quad (im \times im) \tag{17–52}$$

Then,

$$\begin{aligned} \mathcal{A}_Y &= \mathbf{E}^T \mathcal{R} \mathcal{X}^{o,T}_{+}(\mathbf{C}_+ - \mathbf{C}_-) \\ &= \mathbf{E}^T \mathcal{R} \mathcal{X}^{o,T}_{+} \mathbf{C} \end{aligned} \tag{17–53}$$

We transform the joint forces, using (17–47), and write the resulting equations as

$$\begin{aligned} \mathcal{P}_Y &= \mathcal{P}_{YI} + \mathcal{A}_Y^T \mathbf{Z} \\ \mathcal{V} &= \mathcal{V}_o + \mathbf{f}\mathbf{Z} = \mathcal{A}_Y \mathcal{U}_Y \end{aligned} \tag{17–54}$$

To relate corresponding terms in (a) and (17–54), we generalize (17–47):

$$\begin{aligned} \mathcal{P}_Y &= \mathcal{X}^o_{JY} \mathcal{P} \\ \mathcal{U} &= \mathcal{X}^{o,T}_{JY} \mathcal{U}_Y \end{aligned} \tag{17–55}$$

$$\mathcal{X}^o_{JY} = \begin{bmatrix} \mathcal{X}^o_{1Y} & & \\ & \mathcal{X}^o_{2Y} & \\ & & \ddots \\ & & & \mathcal{X}^o_{jY} \end{bmatrix} \quad (ij \times ij)$$

It follows that

$$\begin{aligned} \mathcal{A}_Y &= \mathcal{A} \mathcal{X}^{o,T}_{JY} \\ \mathcal{P}_{YI} &= \mathcal{X}^o_{JY} \mathcal{P}_I \end{aligned} \tag{17–56}$$

The expression for $\mathcal{A}_Y$ reduces to (17–53) when (d) and (e) are introduced.

The formulation developed above can be interpreted as a *network* formulation since the connectivity term appears seperately in the factored form of $\mathcal{A}$. A simplified version which does not allow for member force releases has been presented by Fenves and Branin (see Ref. 1). The only operational advantage of not working with the actual joint quantities is in the generation of $\mathcal{A}_{nn+}$ and $\mathcal{A}_{nn-}$. This advantage is trivial compared to the additional operations required to generate $\mathcal{P}_Y$, $\mathcal{P}_{YI}$, to introduce the displacement restraints, and finally, to transform $\mathcal{U}_Y$ to $\mathcal{U}$ once the solution is obtained. Another serious disadvantage is that the equations tend to become *ill-conditioned.*

Fenves and Branin's primary objective was to show that the governing equations for a member system can be cast in a form such that geometrical and topological effects are separated, i.e., a network formulation. DiMaggio and Spillars (Ref. 2) have also presented a network formulation for a rigid jointed member system. Actually their formulation is a special case of our first formulation. It is not, strictly speaking, a true network formulation since connectivity is not completely separated from geometry (see (17–21)). The only way that one can separate connectivity from geometry is to redefine the joint variables. Note that the ideal truss is an exception. Connectivity and geometry are *naturally* uncoupled for this system.

Whether one interprets the governing equations for a member system from a network viewpoint is of academic interest only. In the displacement method, the equations reduce to the equations for the direct stiffness method. The only possible advantage of the network interpretation is in the force method. There one can use certain concepts of the mesh method† to select a primary structure, provided that there are *no* member force releases or partial joint restraints. However, the selection of a primary structure for a rigid-jointed frame having fixed supports is quite simple, and even this advantage is debatable.

17–7. DISPLACEMENT METHOD

The governing equations are given by (17–33), (17–34), and (17–35). Once the member forces are known, we can find the reactions from (17–34). Now, we start by solving (17–35) for $\mathbf{Z}$ in terms of the displacements,

$$\mathbf{Z} = \mathbf{Z}_I + \mathbf{k}\mathbf{A}_1\mathbf{U}_1 + \mathbf{k}\mathbf{A}_2\bar{\mathbf{U}}_2 \tag{17–57}$$

where

$$\begin{aligned} \mathbf{Z}_I &= \text{initial member force matrix } (q_T \times 1) \\ &= -\mathbf{k}\mathcal{V}_o = \{-\mathbf{k}_{r,s}\mathcal{V}_{ro,s}\} \end{aligned} \tag{17–58}$$

$$\mathbf{k} = \mathbf{f}^{-1} = \text{reduced member stiffness matrix } (q_T \times q_T)$$

$$= \begin{bmatrix} \mathbf{f}^{-1}_{r,1} & & & \\ & \mathbf{f}^{-1}_{r,2} & & \\ & & \ddots & \\ & & & \mathbf{f}^{-1}_{r,m} \end{bmatrix} = \begin{bmatrix} \mathbf{k}_{r,1} & & & \\ & \mathbf{k}_{r,2} & & \\ & & \ddots & \\ & & & \mathbf{k}_{r,m} \end{bmatrix}$$

Note that $\mathbf{k}$ is quasi-diagonal, symmetrical, and positive definite. The matrix, $\mathbf{Z}_I$, contains the initial member forces due to external loads acting on the members and initial deformation resulting from fabrication errors or temperature changes.

We substitute for $\mathbf{Z}$ in (17–33) and write the result as

$$\bar{\mathbf{P}}_1 = \bar{\mathbf{P}}_{o,1} + \mathbf{K}_{11}\mathbf{U}_1 + \mathbf{K}_{12}\bar{\mathbf{U}}_2 \tag{17–59}$$

where

$$\begin{aligned} \mathbf{K}_{11} &= \mathbf{A}_1^T\mathbf{k}\mathbf{A}_1 && (n_d \times n_d) \\ \mathbf{K}_{12} &= \mathbf{A}_1^T\mathbf{k}\mathbf{A}_2 && (n_d \times r) \\ \bar{\mathbf{P}}_{o,1} &= \bar{\mathbf{P}}_{I,1} + \mathbf{A}_1^T\mathbf{Z}_I && (n_d \times 1) \end{aligned} \tag{17–60}$$

The elements of $\bar{\mathbf{P}}_{o,1}$ are the joint forces due to the initial end forces. Since $\mathbf{A}_1$ is of rank n_d (when the system is stable) and $\mathbf{k}$ is positive definite, it follows‡

† See Sec. 9–5.

‡ See Prob. 2–18.

that $\mathbf{K}_{11}$ is positive definite. Conversely, if $\mathbf{K}_{11}$ is not positive definite, the system is unstable. The joint displacements are determined by solving (17–59) and the member forces are obtained by back substitution in (17–57).

Operating on the restrained equations, as we have done above, is not efficient since the various coefficient matrices must be generated by matrix multiplication. By first manipulating the unrestrained equations and then introducing the displacement restraints, one can avoid any matrix multiplication. This procedure corresponds to the direct stiffness method.

Operating on (17–30), we obtain

$$\mathbf{Z} = \mathbf{Z}_I + \mathbf{k}\mathcal{A}\mathcal{U} \tag{17–61}$$

and

$$\begin{aligned}\mathcal{P} &= \mathcal{P}_I + \mathcal{A}^T\mathbf{Z}_I + \mathcal{A}^T\mathbf{k}\mathcal{A}\mathcal{U} \\ &= \mathcal{P}_o + \mathcal{K}\mathcal{U}\end{aligned} \tag{17–62}$$

Equation (17–62) is identical to (16–8). The generation of $\mathcal{P}_o$, $\mathcal{K}$ reduces to (16–9), (16–10) when we introduce the factored forms of $\mathcal{P}_I$, $\mathcal{A}$, $\mathbf{Z}_I$.

First, we review the definitions of the member stiffness matrices, $\mathbf{k}^o_{n+n+}$, $\mathbf{k}^o_{n+n-}$, $\mathbf{k}^o_{n-n-}$. The effective member stiffness matrix (see (16–104)) has been defined as

$$\mathbf{k}^n_{e,n} = \mathbf{E}_n\mathbf{k}_{r,n}\mathbf{E}_n^T \tag{a}$$

Transforming $\mathbf{k}^n_e$ to the global frame and applying (16–107) leads to

$$\begin{aligned}\mathbf{k}^o_{e,n} &= \mathcal{R}^{on,T}\mathbf{k}^n_{e,n}\mathcal{R}^{on} \\ &= (\mathbf{E}_n^T\mathcal{R}^{on})^T\mathbf{k}_{r,n}(E_n^T\mathcal{R}^{on})\end{aligned} \tag{b}$$

and

$$\begin{aligned}\mathbf{k}^o_{n+n+} &= \mathbf{k}^o_{e,n} \\ \mathbf{k}^o_{n+n-} &= -\mathbf{k}^o_{e,n}\mathcal{X}^{o,T}_n \\ \mathbf{k}^o_{n-n-} &= \mathcal{X}^o_n\mathbf{k}^o_{e,n}\mathcal{X}^{o,T}_n\end{aligned} \tag{c}$$

Now, substituting for $\mathcal{A}$ using (17–21), the expression for $\mathcal{K}$ takes the form

$$\mathcal{K} = (\mathbf{C}_+^T - \mathbf{C}_-^T\mathcal{X}^o)\mathbf{k}^o_e(\mathbf{C}_+ - \mathcal{X}^{o,T}\mathbf{C}_-) \tag{d}$$

where

$$\mathbf{k}^o_e = (\mathbf{E}^T\mathcal{R})^T\mathbf{k}(\mathbf{E}^T\mathcal{R}) \tag{17–63}$$

Finally, we expand (d):

$$\begin{aligned}\mathcal{K} &= \mathbf{C}_+^T\mathbf{k}^o_e\mathbf{C}_+ + \mathbf{C}_+^T(-\mathbf{k}^o_e\mathcal{X}^{o,T}_o)\mathbf{C}_- \\ &\quad + \mathbf{C}_-^T(-\mathbf{k}^o_e\mathcal{X}^{o,T}_o)^T\mathbf{C}_+ + \mathbf{C}_-^T(\mathcal{X}^o\mathbf{k}^o_e\mathcal{X}^{o,T})\mathbf{C}_-\end{aligned} \tag{17–64}$$

One can easily show that (17–64) reduces to (16–10) when the properties of $\mathbf{C}_+$, $\mathbf{C}_-$ are taken into account.

The initial end actions for member n are†

$$\begin{aligned}\mathcal{F}^o_{n+,i} &= \mathcal{R}^{on,T}\mathcal{F}^n_{n+,o} + (\mathcal{R}^{on,T}\mathbf{E}_n)(-\mathbf{k}_{r,n}\mathcal{V}_{ro,n}) \\ \mathcal{F}^o_{n-,i} &= \mathcal{R}^{on,T}\mathcal{F}^o_{n-,o} - \mathcal{X}^o_n\mathcal{R}^{on,T}\mathbf{E}_n(-\mathbf{k}_{r,n}\mathcal{V}_{ro,n})\end{aligned} \tag{e}$$

† See Eqs. (17–7), (17–8), and (17–11).

Using the factored forms for $\mathcal{P}_I$, $\mathcal{A}$, and $\mathbf{Z}_I$, the expression for $\mathcal{P}_o$ takes the form

$$\begin{aligned}\mathcal{P}_o &= \mathbf{C}_+^T\mathcal{R}^T\mathcal{F}_{+,o} + \mathbf{C}_-^T\mathcal{R}^T\mathcal{F}_{-,o} \\ &\quad + (\mathbf{C}_+^T - \mathbf{C}_-^T\mathcal{X}^o)\mathcal{R}^T\mathbf{E}(-\mathbf{k}\mathcal{V}_o)\end{aligned} \tag{17–65}$$

The general form of $\mathcal{P}_o$ defined according to (16–9) is

$$\mathcal{P}_o = \mathbf{C}_+^T\mathcal{F}^o_{+,i} + \mathbf{C}_-^T\mathcal{F}^o_{-,i} \tag{17–66}$$

Substituting (e) in (17–65) results in (17–66).

In Sec. 16–4, we presented a procedure for introducing joint displacement restraints and represented the modified equations as

$$\mathcal{K}^*\mathcal{U}^J = \mathcal{P}^*_N \qquad (ij \text{ eqs.}) \tag{f}$$

Now, (f) consists of (17–59) plus r relations for the prescribed displacements. We obtain (f) by starting with

$$\left[\begin{array}{c|c}\mathbf{K}_{11} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{I}_r\end{array}\right]\begin{Bmatrix}\mathbf{U}_1 \\ \mathbf{U}_2\end{Bmatrix} = \begin{Bmatrix}\bar{\mathbf{P}}_1 - \bar{\mathbf{P}}_{o,1} - \mathbf{K}_{12}\bar{\mathbf{U}}_2 \\ \bar{\mathbf{U}}_2\end{Bmatrix} \tag{g}$$

and permuting the rows and columns. This operation can be represented in terms of the permutation matrix, $\mathbf{\Pi}$, defined by

$$\begin{aligned}\mathbf{U} &= \mathbf{\Pi}\mathcal{U}^J \\ \mathcal{P}^J &= \mathbf{\Pi}^T\mathbf{P}\end{aligned} \tag{h}$$

Then,

$$\begin{aligned}\mathcal{K}^* &= \mathbf{\Pi}^T\left[\begin{array}{c|c}\mathbf{K}_{11} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{I}_r\end{array}\right]\mathbf{\Pi} \\ \mathcal{P}^*_N &= \mathbf{\Pi}^T\begin{Bmatrix}\bar{\mathbf{P}}_1 - \bar{\mathbf{P}}_{o,1} - \mathbf{K}_{12}\bar{\mathbf{U}}_2 \\ \bar{\mathbf{U}}_2\end{Bmatrix}\end{aligned} \tag{i}$$

It follows that $\mathcal{K}^*$ is positive definite when $\mathbf{K}_{11}$ is positive definite, i.e., when the system is stable.

17–8. FORCE METHOD

We start with the governing equations for the restrained system:

$$\begin{aligned}\mathbf{B}_1\mathbf{Z} &= \bar{\mathbf{P}}_1 - \mathbf{P}_{I,1} && (n_d \text{ eqs.}) && \text{(a)} \\ \mathbf{B}_1^T\mathbf{U}_1 + \mathbf{B}_2^T\bar{\mathbf{U}}_2 &= \mathcal{V} = \mathcal{V}_o + \mathbf{f}\mathbf{Z} && (q_T \text{ eqs.}) && \text{(b)} \\ \mathbf{P}_2 &= \mathbf{P}_{I,2} + \mathbf{B}_2\mathbf{Z} && (r \text{ eqs.}) && \text{(c)}\end{aligned}$$

Equation (a) represents n_d equations in q_T unknowns where $q_T \geq n_d$. Also, $\mathbf{B}_1$ is of rank n_d. The system is statically determinate when $q_T = n_d$. We let q_R be the degree of static indeterminacy, i.e., the number of member force redundants:

$$q_R = q_T - n_d \tag{17–67}$$

Since $\mathbf{B}_1$ is of rank n_d, we can solve (a) for n_d member forces in terms of the net prescribed joint forces $(\bar{\mathbf{P}}_1 - \bar{\mathbf{P}}_{I,1})$ and r member forces. The compatibility equations for the member force redundants are obtained by eliminating $\mathbf{U}_1$ from (b). This is possible since (b) represents q_T equations whereas $\mathbf{U}_1$ is only of order $n_d \times 1$. In the next section, we specialize the principle of virtual forces for a member system and utilize it to establish the compatibility equations.

We suppose the first n_d columns of $\mathbf{B}_1$ are linearly independent. If the system is initially stable, the member force matrix $\mathbf{Z}$ can always be rearranged so that this condition is satisfied. We partition $\mathbf{Z}$ after row n_d:

$$\underset{(q_T \times 1)}{\mathbf{Z}} = \begin{Bmatrix} \mathbf{Z}_P \\ \hline \mathbf{Z}_R \end{Bmatrix} \begin{matrix} (nd \times ij) \\ (q_R \times 1) \end{matrix} \tag{17–68}$$

The elements of $\mathbf{Z}_R$ are the force redundants for the system. We refer to the system obtained by setting $\mathbf{Z}_R = \mathbf{0}$ as the primary system. Continuing, we partition $\mathbf{B}_1$ and $\mathbf{B}_2$ consistent with (17–68):

$$\begin{aligned} \overset{(n_d \times q_T)}{\mathbf{B}_1} &= [\, \overset{(n_d \times n_d)}{\mathbf{B}_{1P}} \mid \overset{(n_d \times q_R)}{\mathbf{B}_{1R}} \,] \\ \underset{(r \times q_T)}{\mathbf{B}_2} &= [\, \underset{(r \times n_d)}{\mathbf{B}_{2P}} \mid \underset{(r \times q_R)}{\mathbf{B}_{2R}} \,] \end{aligned} \tag{17–69}$$

The equilibrium equations take the form

$$\mathbf{B}_{1P}\mathbf{Z}_P = \bar{\mathbf{P}}_1 - \bar{\mathbf{P}}_{I,1} - \mathbf{B}_{1R}\mathbf{Z}_R \tag{17–70}$$

$$\mathbf{P}_2 = \mathbf{P}_{I,2} + \mathbf{B}_{2P}\mathbf{Z}_P + \mathbf{B}_{2R}\mathbf{Z}_R \tag{17–71}$$

We write the solution of (17–70) as

$$\mathbf{Z}_P = \mathbf{Z}_{P,o} + \mathbf{Z}_{P,R}\mathbf{Z}_R \tag{17–72}$$

The force influence matrices can be expressed as

$$\begin{aligned} \overset{(n_d \times 1)}{\mathbf{Z}_{P,o}} &= (\mathbf{B}_{1P})^{-1}(\bar{\mathbf{P}}_1 - \bar{\mathbf{P}}_{I,1}) \\ \overset{(n_d \times q_R)}{\mathbf{Z}_{P,R}} &= -(\mathbf{B}_{1P})^{-1}\mathbf{B}_{1R} \end{aligned} \tag{17–73}$$

but it is not necessary to determine $(\mathbf{B}_{1P})^{-1}$. Actually, the solution procedure can be completely automated.† The complete solution for $\mathbf{Z}$ is

$$\mathbf{Z} = \begin{Bmatrix} \mathbf{Z}_{P,o} \\ \hline \mathbf{0} \end{Bmatrix} + \overset{(q_T \times q_R)}{\begin{bmatrix} \mathbf{Z}_{P,R} \\ \hline \mathbf{I}_{q_R} \end{bmatrix}} \mathbf{Z}_R \tag{17–74}$$

Note that the member forces due to $\mathbf{Z}_R$ are self-equilibrating, i.e., they satisfy $\mathbf{B}_1\mathbf{Z} = \mathbf{0}$. Finally, we substitute for $\mathbf{Z}_P$ in the expression for $\mathbf{P}_2$ and write the result as

$$\mathbf{P}_2 = \mathbf{P}_{2,o} + \mathbf{P}_{2,R}\mathbf{Z}_R \tag{17–75}$$

† See Sec. 9–2.

where

$$\begin{aligned} \overset{(r \times 1)}{\mathbf{P}_{2,o}} &= \mathbf{P}_{I,2} + \mathbf{B}_{2P}\mathbf{Z}_{P,o} \\ \underset{(r \times q_R)}{\mathbf{P}_{2,R}} &= \mathbf{B}_{2R} + \mathbf{B}_{2P}\mathbf{Z}_{P,R} \end{aligned} \tag{17–76}$$

It remains to determine $\mathbf{Z}_R$.

Equation (b) represents q_T equations in n_d unknowns, $\mathbf{U}_1$. Since $q_T = n_d + q_R$, there are q_R excess equations. We partition (b) consistent with the partioning of $\mathbf{Z}$,

$$\begin{bmatrix} \mathbf{B}_{1P}^T \\ \hline \mathbf{B}_{1R}^T \end{bmatrix} \mathbf{U}_1 + \begin{bmatrix} \mathbf{B}_{2P}^T \\ \hline \mathbf{B}_{2R}^T \end{bmatrix} \bar{\mathbf{U}}_2 = \begin{Bmatrix} \mathscr{V}_P \\ \hline \mathscr{V}_R \end{Bmatrix} = \begin{Bmatrix} \mathscr{V}_{o,P} \\ \hline \mathscr{V}_{o,R} \end{Bmatrix} + \left[\begin{array}{c|c} \mathbf{f}_{PP} & \mathbf{f}_{PR} \\ \hline \mathbf{f}_{PR}^T & \mathbf{f}_{RR} \end{array}\right] \begin{Bmatrix} \mathbf{Z}_P \\ \hline \mathbf{Z}_R \end{Bmatrix} \tag{c}$$

and obtain the following two sets of equations relating to $\mathbf{U}_1$ and $\mathbf{Z}_R$:

$$\mathbf{B}_{1P}^T\mathbf{U}_1 + \mathbf{B}_{2P}^T\bar{\mathbf{U}}_2 = \mathscr{V}_P = \mathscr{V}_{o,P} + \mathbf{f}_{PP}\mathbf{Z}_P + \mathbf{f}_{PR}\mathbf{Z}_R \quad (n_d \text{ eqs.}) \tag{17–77}$$

$$\mathbf{B}_{1R}^T\mathbf{U}_1 + \mathbf{B}_{2R}^T\bar{\mathbf{U}}_2 = \mathscr{V}_R = \mathscr{V}_{o,R} + \mathbf{f}_{PR}^T\mathbf{Z}_P + \mathbf{f}_{RR}\mathbf{Z}_R \quad (q_R \text{ eqs.}) \tag{d}$$

The joint displacements can be determined from (17–77) once $\mathbf{Z}_R$ is known. Eliminating $\mathbf{U}_1$ from (d) leads to

$$\begin{aligned} \mathbf{P}_{2,R}^T\bar{\mathbf{U}}_2 &= \mathscr{V}_R + \mathbf{Z}_{P,R}^T\mathscr{V}_P \\ &= \mathscr{V}_{o,R} + \mathbf{f}_{PR}^T\mathbf{Z}_P + \mathbf{f}_{RR}\mathbf{Z}_R + \mathbf{Z}_{P,R}^T(\mathscr{V}_{o,P} + \mathbf{f}_{PP}\mathbf{Z}_P + \mathbf{f}_{PR}\mathbf{Z}_R) \end{aligned} \tag{17–78}$$

Equation (17–78) represents the compatibility equations for the force redundants. Finally, we substitute for $\mathbf{Z}_P$ using (17–72) and write the resulting equations as

$$\mathbf{f}_{Z_R}\mathbf{Z}_R = \mathbf{\Delta} \tag{17–79}$$

where

$$\begin{aligned} \overset{(q_R \times q_R)}{\mathbf{f}_{Z_R}} &= \mathbf{f}_{RR} + \mathbf{Z}_{P,R}^T\mathbf{f}_{PP}\mathbf{Z}_{P,R} + \mathbf{Z}_{P,R}^T\mathbf{f}_{PR} + (\mathbf{Z}_{P,R}^T\mathbf{f}_{PR})^T \\ \overset{(q_R \times 1)}{\mathbf{\Delta}} &= \mathbf{P}_{2,R}^T\bar{\mathbf{U}}_2 - (\mathscr{V}_{o,R} + \mathbf{f}_{PR}^T\mathbf{Z}_{P,o}) - \mathbf{Z}_{P,R}^T(\mathscr{V}_{o,P} + \mathbf{f}_{PP}\mathbf{Z}_{P,o}) \end{aligned} \tag{17–80}$$

These equations are similar in form to the corresponding equations for the ideal truss developed in Sec. 9–2.

The flexibility matrix, $\mathbf{f}_{Z_R}$, can be expressed as

$$\begin{aligned} \mathbf{f}_{Z_R} &= \begin{bmatrix} \mathbf{Z}_{P,R} \\ \hline \mathbf{I}_{q_R} \end{bmatrix}^T \left[\begin{array}{c|c} \mathbf{f}_{PP} & \mathbf{f}_{PR} \\ \hline \mathbf{f}_{PR}^T & \mathbf{f}_{RR} \end{array}\right] \begin{bmatrix} \mathbf{Z}_{P,R} \\ \hline \mathbf{I}_{q_R} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{Z}_{P,R} \\ \hline \mathbf{I}_{q_R} \end{bmatrix}^T \mathbf{f} \begin{bmatrix} \mathbf{Z}_{P,R} \\ \hline \mathbf{I}_{q_R} \end{bmatrix} \end{aligned} \tag{17–81}$$

Now, $\mathbf{f}$ is positive definite for a deformable system. Then, it follows that $\mathbf{f}_{Z_R}$ is also positive definite. In a later article, we consider the case where certain member deformations may be prescribed.

Once the preliminary force analyses have been carried out, the remaining steps are straightforward. We generate $\mathbf{f}_{Z_R}$, $\mathbf{\Delta}$, solve for $\mathbf{Z}_R$, and then determine

$\mathbf{Z}_P$, $\mathbf{P}_2$ by back substitution. If the displacements are also desired, they can be determined by solving (17–77).

The final number of equations for the force method is usually smaller than for the displacement method (q_R vs. n_d). However, the force method requires considerably more operations to generate the equations. The force method can be completely automated, but not as conveniently as the direct stiffness method. Also, automating the preliminary force analyses requires solving an *additional* set of n_d equations. Another disadvantage of the force method is that the compatibility equations tend to be ill-conditioned unless one is careful in selecting force redundants.

17–9. VARIATIONAL PRINCIPLES

In Chapter 7, we developed variational principles for the displacement and force formulations for an ideal truss. Now, in this section, we develop the corresponding variational principles for a member system. The extension is quite straightforward since the governing equations are almost identical in form.

We start with the force-equilibrium equations,†

$$\mathbf{P} = \bar{\mathbf{P}}_I + \mathbf{A}^T\mathbf{Z} \tag{a}$$

The partitioned form is

$$\bar{\mathbf{P}}_1 = \bar{\mathbf{P}}_{I,1} + \mathbf{A}_1^T\mathbf{Z} \tag{b}$$
$$\bar{\mathbf{P}}_2 = \bar{\mathbf{P}}_{I,2} + \mathbf{A}_2^T\mathbf{Z} \tag{c}$$

To interpret (a) as a stationary requirement, we consider the deformation-displacement relation,

$$\mathscr{V} = \mathbf{AU} = A_1U_1 + A_2U_2 \tag{d}$$

The first differential of $\mathscr{V}$ due to an increment in $\mathbf{U}$ is

$$d\mathscr{V} = \mathbf{A}\,\Delta\mathbf{U} = \mathbf{A}_1\,\Delta\mathbf{U}_1 + A_2\,\Delta U_2 \tag{17–82}$$

Then, the requirement that

$$\mathbf{P}^T\,\Delta\mathbf{U} = \bar{\mathbf{P}}_I^T\,\Delta\mathbf{U} + \mathbf{Z}^T\,d\mathscr{V} \tag{17–83}$$

be satisfied for arbitrary $\Delta\mathbf{U}$ is equivalent to (a). If we consider $\mathbf{U}_2$ to be prescribed, (17–83) results in only (b). We refer to (17–83) as the principle of virtual displacements for a member system.

In the displacement method, we substitute for $\mathbf{Z}$ in the joint force-equilibrium equations, using

$$\mathbf{Z} = \mathbf{k}(\mathscr{V} - \mathscr{V}_o) = (\mathbf{AU} - \mathscr{V}_o) \tag{e}$$

The form of (17–83) suggests that we define a scalar quantity, $V = V(\mathbf{U})$, having the property

$$dV = \mathbf{Z}^T d\mathscr{V} = dV(\mathbf{U}) \tag{17–84}$$

† We work with the governing equations for the *restrained* system. See (17–33), (17–34), (17–35).

One can interpret V as the strain energy function for the members. For the linear case, V can be expressed as

$$\begin{aligned} V &= \tfrac{1}{2}(\mathscr{V} - \mathscr{V}_o)^T\mathbf{k}(\mathscr{V} - \mathscr{V}_o) \\ &= \tfrac{1}{2}(\mathbf{AU} - \mathscr{V}_o)^T\mathbf{k}(\mathbf{AU} - \mathscr{V}_o) \end{aligned} \tag{17–85}$$

Continuing, we define the potential energy function, Π_p, as

$$\Pi_p = V + \mathbf{P}_I^T\mathbf{U} - \mathbf{P}^T\mathbf{U} \tag{17–86}$$

The Euler equations for Π_p are the unpartitioned joint force-equilibrium equations expressed in terms of $\mathbf{U}$. Finally, we introduce the joint displacement constraint condition, $\mathbf{U}_2 = \bar{\mathbf{U}}_2$, by writing (17–86) as

$$\Pi_p = V + \bar{\mathbf{P}}_{I,1}\mathbf{U}_1 + \bar{\mathbf{P}}_{I,2}^T\mathbf{U}_2 - \bar{\mathbf{P}}_1^T\mathbf{U}_1 - \mathbf{P}_2^T(\mathbf{U}_2 - \bar{\mathbf{U}}_2) \tag{17–87}$$

where $\mathbf{U}_1$, $\mathbf{U}_2$, and $\mathbf{P}_2$ are variables. The Euler equations for (17–87) are the partitioned equilibrium equations (Equations (b), (c)) expressed in terms of the displacements with $\mathbf{U}_2$ set equal to $\bar{\mathbf{U}}_2$, i.e., they are the governing equations for the displacement formulation presented in Sec. 17–7.

If only the equations for $\bar{\mathbf{P}}_1$ are desired, we set $\mathbf{U}_2 = \bar{\mathbf{U}}_2$ in (17–87),

$$\Pi_p = V + \bar{\mathbf{P}}_{I,1}^T\mathbf{U}_1 - \bar{\mathbf{P}}_1^T\mathbf{U}_1 \tag{17–88}$$

where

$$V = \tfrac{1}{2}(\mathbf{A}_1\mathbf{U}_1 + \mathbf{A}_2\bar{\mathbf{U}}_2 - \mathscr{V}_o)^T\mathbf{k}(\mathbf{A}_1\mathbf{U}_1 + \mathbf{A}_2\bar{\mathbf{U}}_2 - \mathscr{V}_o) \tag{17–89}$$

The Euler equation for (17–88) is (17–59), and the second differential has the form

$$\begin{aligned} d^2\Pi_p &= \Delta\mathbf{U}_1^T(\mathbf{A}_1^T\mathbf{k}\mathbf{A}_1)\Delta\mathbf{U}_1 \\ &= \Delta\mathbf{U}_1^T\mathbf{K}_{11}\,\Delta\mathbf{U}_1 \end{aligned} \tag{17–90}$$

Since $\mathbf{K}_{11}$ is positive definite, we can state that the displacements defining the equilibrium position correspond to a minimum value of Π_p defined by (17–88) or (17–87).

We consider next the force-method formulation. We let $\Delta\mathbf{P}$, $\Delta\mathbf{Z}$ be a statically permissible virtual-force system. By definition,

$$\Delta\mathbf{P} = \mathbf{A}^T\,\Delta\mathbf{Z} = \mathbf{B}\,\Delta\mathbf{Z} \tag{17–91}$$

Premultiplying both sides of (d) with $\Delta\mathbf{Z}^T$ and introducing (17–91) leads to the principle of virtual forces,

$$\Delta P^T\mathbf{U} = \Delta\mathbf{Z}^T\mathscr{V} \tag{17–92}$$

Note that (17–92) is valid only for a *statically* permissible virtual-force system, i.e., one which satisfies (17–91).

The compatibility equations follow directly from the principle of virtual forces by requiring the virtual-force system to be self-equilibrating. If $\Delta\mathbf{Z}$ satisfies

$$\Delta\mathbf{P}_1 = \mathbf{B}_1\,\Delta\mathbf{Z} = \mathbf{0} \tag{17–93}$$

then (17–92) reduces to

$$\Delta\mathbf{P}_2^T\bar{\mathbf{U}}_2 = \Delta\mathbf{Z}^T\mathscr{V} \tag{17–94}$$

This result is valid for an arbitrary self-equilibrating virtual-force system. The formulation presented in the previous section corresponds to taking

$$\Delta\mathbf{Z} = \begin{bmatrix} \mathbf{Z}_{P,R} \\ \mathbf{I}_{q_r} \end{bmatrix} \Delta\mathbf{Z}_R \qquad (17\text{–}95)$$

$$\Delta\mathbf{P}_2 = \mathbf{P}_{2,R}\,\Delta\mathbf{Z}_R$$

We define the member complementary energy function, $V^* = V^*(\mathbf{Z})$, such that

$$dV^* = \Delta\mathbf{Z}^T\mathscr{V}(\mathbf{Z}) \qquad (17\text{–}96)$$

For the linear case,

$$\mathscr{V} = \mathscr{V}_o + \mathbf{f}\mathbf{Z} \qquad \text{(f)}$$

and

$$V^* = \tfrac{1}{2}\mathbf{Z}^T\mathbf{f}\mathbf{Z} + \mathbf{Z}^T\mathscr{V}_o \qquad (17\text{–}97)$$

We also define the total complementary energy function, Π_c, as

$$\Pi_c = V^* - P_2^T\bar{\mathbf{U}}_2 = \Pi_c(\mathbf{Z}, \mathbf{P}_2) \qquad (17\text{–}98)$$

The deformation compatibility equations, (17–94), can be interpreted as the stationary requirement for Π_c subject to the following constraints on $\mathbf{Z}$, $\mathbf{P}_2$:

$$\begin{aligned} \mathbf{B}_1\mathbf{Z} &= \bar{\mathbf{P}}_1 - \bar{\mathbf{P}}_{I,1} \\ \mathbf{P}_2 &= \bar{\mathbf{P}}_{I,2} + \mathbf{B}_2\mathbf{Z} \end{aligned} \qquad \text{(g)}$$

The constraint conditions are the joint force-equilibrium equations. Operating on (g), and noting that $\bar{\mathbf{P}}_1$, $\bar{\mathbf{P}}_{I,1}$, $\bar{\mathbf{P}}_{I,2}$ are prescribed, lead to the constraint conditions on the force variations

$$\begin{aligned} \mathbf{B}_1\,\Delta\mathbf{Z} &= \mathbf{0} \\ \Delta\mathbf{P}_2 &= \mathbf{B}_2\,\Delta\mathbf{Z} \end{aligned} \qquad \text{(h)}$$

Note that (h) require the virtual-force system to be statically permissible *and* self-equilibrating.

In the previous section, we expressed $\mathbf{Z}$, $\mathbf{P}_2$ as

$$\mathbf{Z} = \begin{Bmatrix} \mathbf{Z}_{P,o} \\ \mathbf{0} \end{Bmatrix} + \begin{bmatrix} \mathbf{Z}_{P,R} \\ \mathbf{I}_{q_R} \end{bmatrix} \mathbf{Z}_R \qquad \text{(i)}$$

$$\mathbf{P}_2 = \mathbf{P}_{2,o} + \mathbf{P}_{2,R}\mathbf{Z}_R$$

This representation satisfies (g) and (h) identically for arbitrary $\Delta\mathbf{Z}_R$. Substituting for $\mathbf{Z}$, $\mathbf{P}_2$ in (17–98) and expanding V^* using (17–97), we obtain

$$\Pi_c = \mathbf{Z}_R^T\,[\mathbf{Z}_{P,R}^T \mid \mathbf{I}_{q_R}] \left[\mathscr{V}_o + \mathbf{f}\begin{Bmatrix} \mathbf{Z}_{P,o} \\ \mathbf{0} \end{Bmatrix} + \tfrac{1}{2}\mathbf{f}\begin{bmatrix} \mathbf{Z}_{P,R} \\ \mathbf{I}_q \end{bmatrix}\mathbf{Z}_R\right] \qquad (17\text{–}99)$$
$$-\mathbf{Z}_R^T\mathbf{P}_{2,R}^T\bar{\mathbf{U}}_2 + \text{const}$$

The Euler equations for (17–99) are (17–79), and the second differential has the form

$$d^2\Pi_c = \Delta\mathbf{Z}_R^T\mathbf{f}_{Z_R}\,\Delta\mathbf{Z}_R \qquad (17\text{–}100)$$

Since $\mathbf{f}_{Z_R}$ is positive definite, it follows that the true forces, i.e., the forces that satisfy compatibility as well as equilibrium, correspond to a minimum value of Π_c.

Instead of developing separate principles for the displacements and force redundants, we could have started with a general variational principle whose Euler equations are the complete set of governing equations. One can easily show that the stationary requirement for

$$\begin{aligned} \Pi_R = {} & \mathbf{Z}^T(\mathbf{B}_1^T\mathbf{U}_1 + \mathbf{B}_2^T\mathbf{U}_2) - V^* - \bar{\mathbf{P}}_1^T\mathbf{U}_1 \\ & + \bar{\mathbf{P}}_{I,1}^T\mathbf{U}_1 + \bar{\mathbf{P}}_{I,2}^T\mathbf{U}_2 - \mathbf{P}_2^T(\mathbf{U}_2 - \bar{\mathbf{U}}_2) \end{aligned} \qquad (17\text{–}101)$$

considering $\mathbf{Z}$, $\mathbf{U}_1$, $\mathbf{U}_2$, and $\mathbf{P}_2$ as variables, lead to the partitioned joint force-equilibrium equations and the member force-joint displacement relations. This principle is a specialized form of Reissner's principle.

We obtain (17–87) from (17–101) by introducing the force-displacement relations as a constraint condition on $\mathbf{Z}$,

$$\begin{aligned} \mathbf{Z} &= \mathbf{k}(\mathbf{B}_1^T\mathbf{U}_1 + \mathbf{B}_2^T\mathbf{U}_2 - \mathscr{V}_o) \\ &= \mathbf{k}(\mathscr{V} - \mathscr{V}_o) \end{aligned} \qquad \text{(j)}$$

and noting that, by definition,

$$\mathbf{Z}^T(\mathbf{B}_1^T\mathbf{U}_1 + \mathbf{B}_2^T\mathbf{U}_2) - V^* = V \qquad \text{(k)}$$

Introducing the joint force-equilibrium equations as constraint conditions reduces Π_R to $-\Pi_c$ as defined by (17–98).

17–10. INTRODUCTION OF MEMBER DEFORMATION CONSTRAINTS

Suppose a member is assumed to be either completely or partially restrained with respect to deformation due to force. The rigidity assumption is introduced by setting the corresponding deformation parameters equal to zero in the *local* flexibility matrix, $\mathbf{g}$. For example, if axial extension is to be neglected, we set $1/AE = 0$. For complete rigidity, we set $\mathbf{g} = \mathbf{0}_i$. Now, in what follows, we discuss the case where neglecting member deformation parameters causes the member flexibility matrix $\mathbf{f}_r$ to be *singular*. This happens, for example, when axial extension is neglected for a *straight* member. The rank is decreased† by 1 and the axial force-deformation relation degenerates to

$$v_1^n = u_{B1}^n - u_{A1}^n = v_{o,1} + \frac{L}{AE}\bar{F}_{B1}^n \Rightarrow \bar{v}_{o,1}^n \qquad \text{(a)}$$

where $\bar{v}_{o,1}^n$ is the initial axial deformation due to temperature and fabrication error. Note that now the axial force has to be determined from the equilibrium equations. For complete rigidity, $\mathbf{f}_r = \mathbf{0}$, and the force-displacement relations (see (17–5)) degenerate to

$$\mathscr{V}_r = \bar{\mathscr{V}}_{ro} = \mathbf{E}^T\mathscr{U}_B^n - \mathbf{E}^T\mathscr{X}_{BA}^{n,T}\mathscr{U}_A^n \qquad \text{(b)}$$

† See (16–75).

One can interpret (a), (b) as either member deformation constraints or as constraint conditions on the joint displacements. In general, the decrease in rank of the system flexibility matrix **f** is equal to the number of constraint conditions.

We consider first the force method. The governing equations are given by

$$\mathbf{f}_{Z_r}\mathbf{Z}_R = \boldsymbol{\Delta} \qquad (q_R \text{ eqs.}) \tag{c}$$

where

$$\mathbf{f}_{Z_r} = \begin{bmatrix} \mathbf{Z}_{P,R} \\ \hline \mathbf{I}_{q_R} \end{bmatrix}^T \mathbf{f} \begin{bmatrix} \mathbf{Z}_{P,R} \\ \hline \mathbf{I}_{q_R} \end{bmatrix} \tag{d}$$

Suppose these are c deformation constraints. Then, **f** is of rank $q_T - c$. In order to solve (c), $\mathbf{f}_{Z_r}$ must be nonsingular, i.e., it must be of rank q_R. This requires

$$q_T - c \quad q_R \tag{17-102}$$

That is, there must be at least q_R unconstrained member deformations. This condition is necessary but not sufficient as we will illustrate below. Aside from insuring that the flexibility matrix is of rank q_R, there is no difficulty involved in introducing member deformation constraints in the force formulation.

Example 17–1

Consider the ideal truss shown. For this system,

$$q_T = 4$$
$$q_R = 2$$

We take the forces in bars 3, 4 as the redundants:

$$\mathbf{Z}_P = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \qquad Z_R = \begin{Bmatrix} F_3 \\ F_4 \end{Bmatrix}$$

Then,

$$\mathbf{Z}_{P,R} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

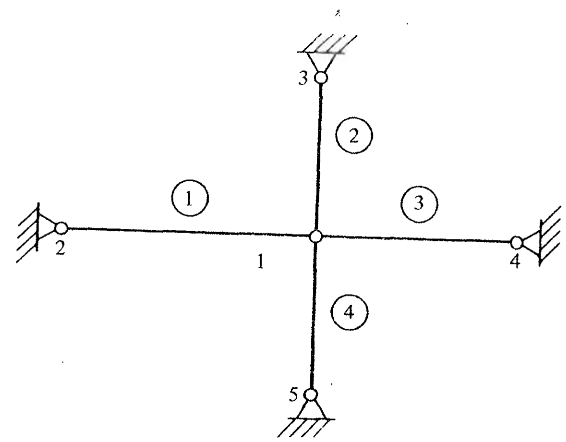
and

$$\mathbf{f}_{Z_R} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}^T \begin{bmatrix} f_1 & & & \\ & f_2 & & \\ & & f_3 & \\ & & & f_4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} f_1 + f_3 & 0 \\ 0 & f_2 + f_4 \end{bmatrix}$$

We can specify that, at most, two bars are rigid. No difficulty is encountered if only one bar is rigid. However, we cannot specify that bars 1, 3 or 2, 4 are rigid.

We consider next the displacement formulation. Since **f** is singular, **k** does not exist and, therefore, we cannot invert the complete set of force-displacement

Fig. E17–1



relations, i.e., (17–57) are not applicable. In what follows, we first develop the appropriate equations by manipulating the original set of governing equations. We then show how the equations can be deduced from the variational principle for displacements.

The governing equations are

$$\bar{\mathbf{P}}_1 = \bar{\mathbf{P}}_{I,1} + \mathbf{A}_1^T\mathbf{Z} \qquad (n_d \text{ eqs.}) \tag{a}$$

$$\mathscr{V} = \mathscr{V}_o + \mathbf{f}\mathbf{Z} = \mathbf{A}_1\mathbf{U}_1 + \mathbf{A}_2\bar{\mathbf{U}}_2 \qquad (q_T \text{ eqs.}) \tag{b}$$

Now, we suppose there are c deformation constraints and the elements of $\mathscr{V}$ are listed such that the last c elements are the prescribed deformations. We partition $\mathscr{V}$ and **Z** as follows:

$$\mathscr{V} = \begin{Bmatrix} \mathscr{V}_u \\ \hline \mathscr{V}_c \end{Bmatrix} \begin{matrix} (q_T - c) \times 1 \\ (c \times 1) \end{matrix} \qquad \mathbf{Z} = \begin{Bmatrix} \mathbf{Z}_u \\ \hline \mathbf{Z}_c \end{Bmatrix} \tag{17-103}$$

where $\mathscr{V}_c$ contains the constrained member deformations and $\mathbf{Z}_c$ the corresponding member forces. We use subscripts c, u to indicate quantities associated with the constrained and unconstrained deformations. Continuing, we partition $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathscr{V}_o$, and **f** consistent with (17–103):

$$\underset{(q_T \times n_d)}{\mathbf{A}_1} = \begin{bmatrix} \mathbf{A}_{1u} \\ \hline \mathbf{A}_{1c} \end{bmatrix} \begin{matrix} (q_T - c) \times n_d \\ (c \times n_d) \end{matrix}$$
$$\underset{(q_T \times r)}{\mathbf{A}_2} = \begin{bmatrix} \mathbf{A}_{2u} \\ \hline \mathbf{A}_{2c} \end{bmatrix} \begin{matrix} (q_T - c) \times r \\ (c \times r) \end{matrix}$$
$$\underset{(q_T \times 1)}{\mathscr{V}_o} = \begin{Bmatrix} \mathscr{V}_{u,o} \\ \hline \mathscr{V}_{c,o} \end{Bmatrix} \begin{matrix} (q_T - c) \times 1 \\ (c \times 1) \end{matrix}$$
$$\underset{(q_T \times q_T)}{\mathbf{f}} = \mathbf{f}^T = \begin{bmatrix} \mathbf{f}_u & \mathbf{0} \\ \mathbf{0} & \underset{(c \times c)}{\mathbf{f}_c} \end{bmatrix} \tag{17-104}$$

The deformation constraints are introduced by setting $\mathbf{f}_c = \mathbf{0}$. Note that, in order for $\mathbf{f}$ to be singular, there must be no coupling between $\mathscr{V}_u$ and $\mathbf{Z}_c$, i.e., $\mathbf{f}$ must have the form shown above. Using this notation, the governing equations take the form

$$\bar{\mathbf{P}}_1 = \bar{\mathbf{P}}_{I,1} + \mathbf{A}_{1u}^T\mathbf{Z}_u + \mathbf{A}_{1c}^T\mathbf{Z}_c \tag{17–105}$$

$$\mathscr{V}_u = \mathscr{V}_{u,o} + \mathbf{f}_u\mathbf{Z}_u = \mathbf{A}_{1u}\mathbf{U}_1 + \mathbf{A}_{2u}\bar{\mathbf{U}}_2 \tag{17–106}$$

$$\mathscr{V}_c = \mathscr{V}_{c,o} = \mathbf{A}_{1c}\mathbf{U}_1 + \mathbf{A}_{2c}\bar{\mathbf{U}}_2 \tag{17–107}$$

Equation (17–107) represents c constraint conditions on the unknown joint displacements, $\mathbf{U}_1$. The rank of $\mathbf{A}_{1c}$ is equal to the number of independent constraint equations. One can easily demonstrate that c independent constraint conditions are required in order to be able to analyze the system for an arbitrary loading.

Example 17–2

Suppose we specify that bars 1, 3 are rigid for the system considered in Example 17–1. The constraint equations are (we take $\mathscr{V}_c = \{e_1, e_3\}$)

$$\begin{aligned} e_1 &= e_{1,o} = u_{11} - \bar{u}_{21} \\ e_3 &= e_{3,o} = -u_{11} + \bar{u}_{41} \end{aligned} \tag{a}$$

For (a) to be consistent, we must have

$$e_{1,o} + e_{3,o} = -\bar{u}_{21} + \bar{u}_{41} \tag{b}$$

Even if (b) is satisfied, we cannot find the forces in bars 1, 3 due to p_{11}.

In what follows, we assume $\mathbf{A}_{1c}$ is of rank c. We solve (17–106) for $\mathbf{Z}_u$ and substitute in (17–105). This is permissible since $\mathbf{f}_u$ is nonsingular. The resulting equations are

$$\begin{aligned} \mathbf{Z}_u &= \mathbf{k}_u(\mathscr{V}_u - \mathscr{V}_{u,o}) \\ &= \mathbf{k}_u\mathbf{A}_{1u}\mathbf{U}_1 + \mathbf{k}_u(\mathbf{A}_{2u}\bar{\mathbf{U}}_2 - \mathscr{V}_{u,o}) \end{aligned} \tag{17–108}$$

and

$$(\mathbf{A}_{1u}^T\mathbf{k}_u\mathbf{A}_{1u})\mathbf{U}_1 + \mathbf{A}_{1c}^T\mathbf{Z}_c = \mathbf{P}_1 - \mathbf{P}_{I,1} - \mathbf{A}_{1u}^T\mathbf{k}_u(\mathbf{A}_{2u}\bar{\mathbf{U}}_2 - \mathscr{V}_{u,o}) \tag{17–109}$$

$$\mathbf{A}_{1c}\mathbf{U}_1 = \mathscr{V}_{c,o} - \mathbf{A}_{2c}\bar{\mathbf{U}}_2 \tag{17–110}$$

Now, the coefficient matrix, $\mathbf{A}_{1u}^T\mathbf{k}_u\mathbf{A}_{1u}$, is nonsingular only when the structure obtained by deleting the restraint forces ($\mathbf{Z}_c$) is *stable*. By suitably redefining $\mathbf{Z}_c$, we can transform (17–109) such that the coefficient matrix is always nonsingular. Suppose we write

$$\begin{aligned} \mathbf{Z}_c &= \mathbf{Z}'_c + \mathbf{k}'_c(\mathscr{V}_c - \mathscr{V}_{c,o}) \\ &= \mathbf{Z}'_c + \mathbf{k}'_c(\mathbf{A}_{1c}\mathbf{U}_1 + \mathbf{A}_{2c}\bar{\mathbf{U}}_2 - \mathscr{V}_{c,o}) \end{aligned} \tag{17–111}$$

where $\mathbf{Z}'_c$ represents the new force variable and $\mathbf{k}'_c$ is an arbitrary symmetrical

positive definite matrix of order c. Substituting for $\mathbf{Z}_c$ in (17–109), we obtain

$$\begin{aligned} \bar{\mathbf{P}}_1 = \bar{\mathbf{P}}_{I,1} &+ \mathbf{A}_{1c}^T\mathbf{Z}'_c \\ &+ [\mathbf{A}_{1u}^T \mid \mathbf{A}_{1c}^T]\left[\begin{array}{c|c} \mathbf{k}_u & \\ \hline & \mathbf{k}'_c \end{array}\right]\left(\left[\begin{array}{c|c} \mathbf{A}_{1u} & \mathbf{A}_{2u} \\ \hline \mathbf{A}_{1c} & \mathbf{A}_{2c} \end{array}\right]\left\{\begin{array}{c} \mathbf{U}_1 \\ \hline \bar{\mathbf{U}}_2 \end{array}\right\} - \left\{\begin{array}{c} \mathscr{V}_{u,o} \\ \hline \mathscr{V}_{c,o} \end{array}\right\}\right) \end{aligned} \tag{a}$$

By defining

$$\mathbf{k}' = \left[\begin{array}{c|c} \mathbf{k}_u & \\ \hline & \mathbf{k}'_c \end{array}\right] = \left[\begin{array}{c|c} \mathbf{f}_u & \\ \hline & \mathbf{f}'_c \end{array}\right]^{-1} \tag{17–112}$$

and noting (17–104), we can write (a) as

$$\bar{\mathbf{P}}_1 = \bar{\mathbf{P}}_{I,1} + \mathbf{A}_1^T\mathbf{k}'(\mathbf{A}_2\bar{\mathbf{U}}_2 - \mathscr{V}_o) + (\mathbf{A}_1^T\mathbf{k}'\mathbf{A}_1)\mathbf{U}_1 + \mathbf{A}_{1c}^T\mathbf{Z}'_c \tag{b}$$

Using the notation introduction in Sec. 17–7 (see (17–60)), we let

$$\begin{aligned} \mathbf{K}_{rs} &= \mathbf{A}_r^T\mathbf{k}'\mathbf{A}_s \\ \mathbf{Z}_I &= -\mathbf{k}'\mathscr{V}_o \\ \bar{\mathbf{P}}_{o,1} &= \bar{\mathbf{P}}_{I,1} + \mathbf{A}_1^T\mathbf{Z}_I \end{aligned} \tag{17–113}$$

Finally, the governing equations take the form

$$\mathbf{Z} = \left\{\begin{array}{c} \mathbf{Z}_u \\ \hline \mathbf{Z}_c \end{array}\right\} = \left\{\begin{array}{c} \mathbf{0} \\ \hline \mathbf{Z}'_c \end{array}\right\} + \mathbf{k}'(\mathscr{V} - \mathscr{V}_o) \tag{17–114}$$

$$\mathbf{K}_{11}\mathbf{U}_1 + \mathbf{A}_{1c}^T\mathbf{Z}'_c = \bar{\mathbf{P}}_1 - \bar{\mathbf{P}}_{o,1} - \mathbf{K}_{12}\bar{\mathbf{U}}_2 = \mathbf{H}_1 \tag{17–115}$$

$$\mathbf{A}_{1c}\mathbf{U}_1 = \mathscr{V}_{c,o} - \mathbf{A}_{2c}\bar{\mathbf{U}}_2 = \mathbf{H}_2 \tag{17–116}$$

Since $\mathbf{A}_1$ must be of rank n_d for stability and we have required $\mathbf{k}'_c$ to be positive definite, it follows that $\mathbf{K}_{11}$ is *positive definite*. Also, the solution for $\mathbf{U}_1$ must satisfy (17–116) and we see from (17–111) that $\mathbf{Z}'_c$ is equal to $\mathbf{Z}_c$, the actual constraint force matrix, for *arbitrary* $\mathbf{k}'$.

The expressions for $\mathbf{Z}$ and $\bar{\mathbf{P}}_1$, with $\mathbf{Z}_c$ deleted, have the same form as the unconstrained expressions (17–57) and (17–59). Now, it is not necessary to rearrange $\mathbf{Z}$ such that the constraint forces are last. One can work with the *natural* member force listing,

$$\mathbf{Z} = \{\mathbf{Z}_1, \ldots, \mathbf{Z}_m\} \tag{a}$$

and take *arbitrary* values for the member deformation parameters that are to be *negelected*. We obtain $\mathbf{K}_{11}$ and $\mathbf{H}_1$ by first generating $\mathscr{K}^*$, $\mathscr{P}^*_N$, using the direct stiffness method and then deleting the rows and columns corresponding to the prescribed displacements. The constrained deformations, $\mathscr{V}_c$, can be listed *arbitrarily*. It is only necessary to specify the locations of the constraint forces (elements of $\mathbf{Z}'_c$) in the natural member force listing. Once the displacements and constraint forces are known, we can determine the force matrix for member n by first evaluating (see (17–8) and (17–11))

$$\begin{aligned} \mathbf{Z}_n &= \mathbf{k}'_{r,n}(\mathscr{V}_{r,n} - \mathscr{V}_{ro,n}) \\ \mathscr{V}_{r,n} &= (\mathbf{E}_n^T\mathscr{R}^{on})(\mathscr{U}^o_{n+} - \mathscr{X}_n^{o.T}\mathscr{U}^o_{n-}) \end{aligned} \tag{17–117}$$

where $\mathbf{k}'_{r,n}$ is the modified stiffness matrix, and then adding the constraint forces in the appropriate locations. In what follows, we describe two procedures for solving (17–115) and (17–116).

In the first method, we solve (17–115) for $\mathbf{U}_1$,

$$\mathbf{U}_1 = \mathbf{K}_{11}^{-1}\mathbf{H}_1 - \mathbf{K}_{11}^{-1}\mathbf{A}_{1c}^T\mathbf{Z}'_c \tag{17–118}$$

and then substitute in (17–116),

$$(\mathbf{A}_{1c}\mathbf{K}_{11}^{-1}\mathbf{A}_{1c}^T)\mathbf{Z}'_c = \mathbf{A}_{1c}\mathbf{K}_{11}^{-1}\mathbf{H}_1 - \mathbf{H}_2 \tag{17–119}$$

The coefficient matrix for $\mathbf{Z}'_c$ is positive definite since $\mathbf{K}_{11}$ is positive definite and $\mathbf{A}_{1c}$ is of rank c. Note that, with this procedure, we must invert an n_dth order matrix and also solve a set of c equations. For the unconstrained case, we have to solve only n_d equations.

Example 17–3

We suppose bar 2 (Fig. E17–3) is rigid. The constraint equation is

$$e_2 = u_2 = e_{2,0} \tag{a}$$

To simplify the example, we consider only the effect of joint forces. Using the notation introduced above, the various matrices for this example are

$$\begin{aligned} \mathbf{U}_1 &= \{u_1, u_2\} \\ \bar{\mathbf{P}}_1 &= \{p_1, p_2\} \end{aligned}$$

$$\begin{array}{lll} \mathscr{V}_u = e_1 & \mathbf{Z}_u = F_1 & \mathbf{k}_u = k_1 \\ \mathscr{V}_c = e_2 & \mathbf{Z}_c = F_2 & \mathbf{k}'_c = k'_2 \\ q_T = 2 & n_d = 2 & c = 1 \end{array} \tag{b}$$

($\mathbf{P}_{I,1}$, $\bar{\mathbf{U}}_2$, $\mathscr{V}_o$ are null matrices.)

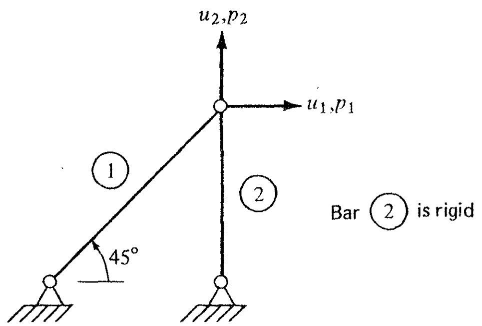


Fig. E17–3

We start by assembling $\mathbf{A}_1$,

$$\mathscr{V} = \mathbf{A}_1\mathbf{U}_1$$

$$\begin{Bmatrix} e_1 \\ e_2 \end{Bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \tag{c}$$

and then partition according to (17–104):

$$\begin{aligned} \mathbf{A}_{1u} &= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \\ \mathbf{A}_{1c} &= [0 \quad 1] \end{aligned} \tag{d}$$

Note that we cannot invert (17–109), since $\mathbf{A}_{1u}^T\mathbf{k}_u\mathbf{A}_{1u}$ is singular.

Now, we assume an arbitrary value for the stiffness of bar 2,

$$k'_2 = ak_1 \tag{e}$$

where a is an arbitrary positive constant, and assemble $\mathbf{K}_{11}$:

$$\mathbf{k}' = \begin{bmatrix} \mathbf{k}_u & \\ & \mathbf{k}'_c \end{bmatrix} = k_1\begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix} \tag{f}$$

$$\mathbf{K}_{11} = \mathbf{A}_1^T\mathbf{k}'\mathbf{A}_1 = \frac{k_1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 + 2a \end{bmatrix} \tag{g}$$

The governing equations (17–114), (17–115), and (17–116) reduce to

$$\begin{Bmatrix} F_1 \\ F'_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ F'_2 \end{Bmatrix} + k_1 \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & a \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \tag{h}$$

$$\mathbf{K}_{11}\mathbf{U}_1 + \mathbf{A}_{1c}^T F'_2 = \bar{\mathbf{P}}_1 \tag{i}$$

$$\mathbf{A}_{1c}\mathbf{U}_1 = \mathbf{0} \tag{j}$$

The solution follows from (17–118), (17–119). We just have to take

$$\mathbf{H}_1 = \bar{\mathbf{P}}_1 \qquad \mathbf{H}_2 = \mathbf{0} \qquad \mathbf{Z}'_c = F'_2 \tag{k}$$

The inverse of $\mathbf{K}_{11}$ is

$$\mathbf{K}_{11}^{-1} = \frac{1}{ak_1}\begin{bmatrix} 1 + 2a & -1 \\ -1 & +1 \end{bmatrix} \tag{l}$$

Then,

$$\begin{aligned} \mathbf{A}_{1c}\mathbf{K}_{11}^{-1} &= \frac{1}{ak_1}[-1 \quad +1] \\ \mathbf{A}_{1c}\mathbf{K}_{11}^{-1}\mathbf{A}_{1c}^T &= \frac{1}{ak_1} \end{aligned} \tag{m}$$

and (17–119) reduces to

$$\begin{aligned} \left(\frac{1}{ak_1}\right)F'_2 &= \frac{1}{ak_1}(p_2 - p_1) \\ F'_2 &= p_2 - p_1 \end{aligned} \tag{n}$$

Substituting for F'_2 in (17–118), we obtain

$$\begin{aligned} u_1 &= \frac{2p_1}{k_1} \\ u_2 &= 0 \end{aligned} \tag{o}$$

Finally, we substitute for F'_2, u_1, u_2 in (h):

$$\begin{aligned} F_1 &= \sqrt{2}p_1 \\ F_2 &= F'_2 = p_2 - p_1 \end{aligned} \tag{p}$$

Instead of first solving (17–115) for $\mathbf{U}_1$ in terms of $\mathbf{Z}'_c$, one can start with (17–116),

$$\mathbf{A}_{1c}\mathbf{U}_1 = \mathcal{V}_{c,o} - \mathbf{A}_{2c}\bar{\mathbf{U}}_2 = \mathbf{H}_2 \qquad \text{(a)}$$

which represents c relations between the n_d displacements. Since $\mathbf{A}_{1c}$ is of rank c, we can express c displacements in terms of $n_d - c$ displacements, i.e., there are only $n_d - c$ independent displacements.

We suppose the first c columns of $\mathbf{A}_{1c}$ are linearly independent. Since $\mathbf{A}_{1c}$ is of rank c, we can always permute the columns such that this requirement is satisfied. We let

$$n = n_d - c \qquad (17\text{–}120)$$

and partition $\mathbf{A}_{1c}$, $\mathbf{U}_1$:

$$\mathbf{U}_1 \rightarrow \begin{Bmatrix} \mathbf{U}_c \\ \hline \mathbf{U} \end{Bmatrix} \begin{matrix} (c \times 1) \\ (n \times 1) \end{matrix}$$
$$\underset{(c \times n_d)}{\mathbf{A}_{1c}} \rightarrow [\underset{(c \times c)}{\mathbf{A}_{1c,1}} \mid \underset{(c \times n)}{\mathbf{A}_{1c,2}}] \qquad (17\text{–}121)$$

The elements of $\mathbf{U}$ are the *independent* displacements. By definition, $\mathbf{A}_{1c,1}$ is nonsingular. Then, solving (a) for $\mathbf{U}_c$, the constrained displacements, we have

$$\mathbf{U}_c = \mathbf{A}_{1c,1}^{-1}\mathbf{H}_2 - \mathbf{A}_{1c,1}^{-1}\mathbf{A}_{1c,2}\mathbf{U} \qquad (17\text{–}122)$$

Finally, we express $\mathbf{U}_1$ as

$$\mathbf{U}_1 = \mathbf{B}\mathbf{U} + \mathbf{H}_3 \qquad (17\text{–}123)$$

where

$$\underset{(n_d \times n)}{\mathbf{B}} = \begin{bmatrix} -\mathbf{A}_{1c,1}^{-1}\mathbf{A}_{1c,2} \\ \hline \mathbf{I}_n \end{bmatrix} \begin{matrix} (c \times n) \\ (n \times n) \end{matrix}$$
$$\mathbf{H}_3 = \begin{Bmatrix} \mathbf{A}_{1c,1}^{-1}\mathbf{H}_2 \\ \hline \mathbf{0} \end{Bmatrix} \begin{matrix} (c \times 1) \\ (n \times 1) \end{matrix} \qquad (17\text{–}124)$$

Note that $\mathbf{B}$ is of rank n and

$$\mathbf{A}_{1c}\mathbf{B} = \underset{(c \times n)}{\mathbf{0}} \qquad (17\text{–}125)$$

The generation of $\mathbf{B}$, $\mathbf{H}_3$ from $\mathbf{A}_{1c}$, $\mathbf{H}_2$ can be completely automated using the same procedure as employed in the force method to select the primary structure.

We consider next the joint force-equilibrium equations, (17–115),

$$\mathbf{K}_{11}\mathbf{U}_1 + \mathbf{A}_{1c}^T\mathbf{Z}'_c = \mathbf{H}_1 \qquad (n_d \text{ eqs.}) \qquad \text{(a)}$$

Substituting for $\mathbf{U}_1$ leads to

$$(\mathbf{K}_{11}\mathbf{B})\mathbf{U} + \mathbf{A}_{1c}^T\mathbf{Z}'_c = \mathbf{H}_1 - \mathbf{K}_{11}\mathbf{H}_3 = \mathbf{H}_4 \qquad \text{(b)}$$

We eliminate $\mathbf{Z}'_c$ from (b) by premultiplying by $\mathbf{B}^T$ and noting (17–125). The resulting system of n equations for $\mathbf{U}$ is

$$(\mathbf{B}^T\mathbf{K}_{11}\mathbf{B})\mathbf{U} = \mathbf{B}^T\mathbf{H}_4 \qquad (n \text{ eqs.}) \qquad (17\text{–}126)$$

Since $\mathbf{B}$ is of rank n, the coefficient matrix is positive definite. One can interpret

$\mathbf{B}^T\mathbf{K}_{11}\mathbf{B}$ as the reduced system stiffness matrix. We solve for $\mathbf{U}$ and then evaluate $\mathbf{U}_1$ from (17–123). It remains to determine the restraint forces, $\mathbf{Z}'_c$.

We consider again Eq. (a). Assuming $\mathbf{U}_1$ is known, we can write

$$\mathbf{A}_{1c}^T\mathbf{Z}'_c = \mathbf{H}_1 - \mathbf{K}_{11}\mathbf{U}_1 = \mathbf{H}_5 \qquad (n_d \text{ eqs.}) \qquad (17\text{–}127)$$

The matrix, $\mathbf{H}_5$, is the difference between the external applied force, $\bar{\mathbf{P}}_1$, and the joint force due to member force with the constraint forces *deleted*, i.e.,

$$\mathbf{H}_5 = \bar{\mathbf{P}}_1 - \mathbf{P}_{I,1} - \mathbf{A}_1^T\mathbf{Z} \qquad \text{(c)}$$

where (see (17–114))

$$\begin{aligned} \mathbf{Z} &= \mathbf{k}'(\mathcal{V} - \mathcal{V}_o) \\ &= \mathbf{k}'(\mathbf{A}_1\mathbf{U}_1 + \mathbf{A}_2\bar{\mathbf{U}}_2 - \mathcal{V}_o) \end{aligned} \qquad \text{(d)}$$

We determine $\mathbf{Z}$ using the member force-displacement relations and assemble $\mathbf{P}_{I,1} + \mathbf{A}_1^T\mathbf{Z}$ by the direct stiffness method. Now $\mathbf{A}_{1c}^T$ has c independent rows. In determining $\mathbf{B}$, we permuted the columns of $\mathbf{A}_{1c}$ such that the first c columns are linearly independent. We apply the same permutation to (17–127) and partition after row c:

$$\mathbf{A}_{1c}^T \rightarrow \begin{bmatrix} \mathbf{A}_{1c,1}^T \\ \hline \mathbf{A}_{1c,2}^T \end{bmatrix}$$
$$\mathbf{H}_5 \rightarrow \begin{bmatrix} \mathbf{H}_{5,1} \\ \hline \mathbf{H}_{5,2} \end{bmatrix} \qquad (17\text{–}128)$$

Considering the first c equations, we have

$$\mathbf{A}_{1c,1}^T\mathbf{Z}'_c = \mathbf{H}_{5,1} \qquad (17\text{–}129)$$

Since $\mathbf{A}_{1c,1}$ is nonsingular, we can solve (17–129) for $\mathbf{Z}'_c$. We obtain the final member forces by adding the elements of $\mathbf{Z}'_c$ in the appropriate locations of $\mathbf{Z}$ defined by (17–114) and (d).

In this approach, we have to invert a matrix of order c and solve a system of $n_d - c$ equations. Although the final number of equations is less than in the first approach, there is more preliminary computation (generation of $\mathbf{B}$) and the procedure cannot be automated as easily.

Example 17–4

For this example (Fig. E17–4),

$$n_d = 5 \qquad c = 4 \qquad n = 1 \qquad \text{(a)}$$

The constraint conditions are

$$\mathcal{V}_c = \begin{Bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{Bmatrix} = \begin{Bmatrix} u_{12} - \bar{u}_{42} \\ u_{21} - u_{11} \\ u_{22} - \bar{u}_{32} \\ u_{31} - \bar{u}_{41} \end{Bmatrix} = \begin{Bmatrix} e_{1,o} \\ e_{2,o} \\ e_{3,o} \\ e_{4,o} \end{Bmatrix} = \mathcal{V}_{c,o} \qquad \text{(b)}$$

Note that (b) corresponds to (17–107). The form corresponding to (17–116) is

$$\underbrace{\begin{bmatrix} & +1 & & & \\ -1 & & +1 & & \\ & & & +1 & \\ & & & & +1 \end{bmatrix}}_{\mathbf{A}_{1c}} \underbrace{\begin{Bmatrix} u_{11} \\ u_{12} \\ u_{21} \\ u_{22} \\ u_{31} \end{Bmatrix}}_{\mathbf{U}_1} = \underbrace{\begin{Bmatrix} e_{1,o} + \bar{u}_{42} \\ e_{2,o} \\ e_{3,o} + \bar{u}_{32} \\ e_{4,o} + \bar{u}_{41} \end{Bmatrix}}_{\mathbf{H}_2} \qquad \text{(c)}$$

Columns 2, 4, 5, and either 1 or 3 comprise a linearly independent set. Then, we can take either u_{11} or u_{21} as $\mathbf{U}$. It is convenient to take $\mathbf{U} = u_{11}$. We permute the columns according to

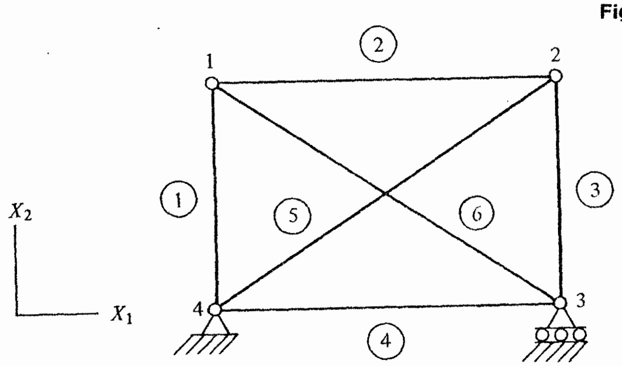
$$\begin{aligned} 1 &\to 5 \\ 2 &\to 1 \\ 3 &\to 2 \\ 4 &\to 3 \\ 5 &\to 4 \end{aligned} \qquad \text{(d)}$$

The rearranged form of $\mathbf{U}_1$ is

$$\begin{aligned} \mathbf{U}_1 &= \{u_{12}, u_{21}, u_{22} u_{31} \mid u_{11}\} \\ &= \{\mathbf{U}_c \mid \mathbf{U}\} \end{aligned} \qquad \text{(e)}$$

We determine $\mathbf{U}_c$ by applying (17–122). This step is simple for this example since $\mathbf{A}_{1c,1}^{-1} = \mathbf{I}$. Finally, we assemble $\mathbf{U}_1$ defined by (e) and then permute the rows to obtain the initial listing of $\mathbf{U}_1$. The final result is

Fig. E17–4



Bars 1, 2, 3, 4 are rigid

$$\begin{Bmatrix} u_{11} \\ u_{12} \\ u_{21} \\ u_{22} \\ u_{31} \end{Bmatrix} = \underbrace{\begin{bmatrix} +1 \\ 0 \\ +1 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{B}} \{u_{11}\} + \underbrace{\begin{Bmatrix} 0 \\ e_{1,o} + \bar{u}_{42} \\ e_{2,o} \\ e_{3,o} + \bar{u}_{32} \\ e_{4,o} + \bar{u}_{41} \end{Bmatrix}}_{\mathbf{H}_3} \qquad \text{(f)}$$

The constraint forces are determined from (17–127), which for this example has the form

$$\underbrace{\begin{bmatrix} & -1 & & \\ +1 & & & \\ & +1 & & \\ & & +1 & \\ & & & +1 \end{bmatrix}}_{\mathbf{A}_{1c}^T} \underbrace{\begin{Bmatrix} F'_1 \\ F'_2 \\ F'_3 \\ F'_4 \end{Bmatrix}}_{\mathbf{Z}'_c} = \underbrace{\begin{Bmatrix} H_{5,1} \\ H_{5,2} \\ H_{5,3} \\ H_{5,4} \\ H_{5,5} \end{Bmatrix}}_{\mathbf{H}_5} \qquad \text{(g)}$$

We permute the *rows* of (g) according to (d) and consider only the first four equations. The resulting equations correspond to (17–129).

It is of interest to derive the equations for the constrained case by suitably specializing the variational principle for displacements. We start with the unconstrained form of Π_p developed in Sec. 17–9,

$$\Pi_p = V + \bar{\mathbf{P}}_{I,1}^T \mathbf{U}_1 - \bar{\mathbf{P}}_1^T \mathbf{U}_1 \qquad \text{(a)}$$

where

$$\begin{aligned} V &= \tfrac{1}{2}(\mathscr{V} - \mathscr{V}_o)^T \mathbf{k} (\mathscr{V} - \mathscr{V}_o) \\ \mathscr{V} &= \mathbf{A}_1 \mathbf{U}_1 + \mathbf{A}_2 \bar{\mathbf{U}}_2 \end{aligned} \qquad \text{(b)}$$

Now, the displacements are constrained by

$$\mathscr{V}_c = \mathbf{A}_{1c} \mathbf{U}_1 + \mathbf{A}_{2c} \bar{\mathbf{U}}_2 = \mathscr{V}_{c,o} \qquad \text{(c)}$$

Then, V reduces to

$$\begin{aligned} V &= \tfrac{1}{2}(\mathscr{V}_u - \mathscr{V}_{u,o})^T \mathbf{k}_u (\mathscr{V}_u - \mathscr{V}_{u,o}) \\ \mathscr{V}_u &= \mathbf{A}_{1u} \mathbf{U}_1 + \mathbf{A}_{2u} \bar{\mathbf{U}}_2 \end{aligned} \qquad \text{(d)}$$

We obtain the appropriate form of Π_p by substituting for V using (d) and introducing the constraint condition, $\mathscr{V}_c - \mathscr{V}_{c,o} = \mathbf{0}$:

$$\Pi_p = V + \bar{\mathbf{P}}_{I,1}^T \mathbf{U}_1 - \bar{\mathbf{P}}_1^T \mathbf{U}_1 + \mathbf{Z}_c^T (\mathscr{V}_c - \mathscr{V}_{c,o}) \qquad \text{(17–130)}$$

The elements of $\mathbf{Z}_c$ are Lagrange multipliers. One can easily show that the stationary requirement for (17–130) considering $\mathbf{U}_1$ and $\mathbf{Z}_c$ as independent variables leads to (17–109) and (17–110).

Since $\mathscr{V}_c = \mathscr{V}_{c,o}$, we can add the term

$$\tfrac{1}{2}(\mathscr{V}_c - \mathscr{V}_{c,o})^T \mathbf{k}'_c (\mathscr{V}_c - \mathscr{V}_{c,o}) \qquad \text{(e)}$$

to (d). Taking

$$V = \tfrac{1}{2}(\mathscr{V} - \mathscr{V}_o)^T \mathbf{k}' (\mathscr{V} - \mathscr{V}_o) \qquad \text{(17–131)}$$

in (17–130) leads to (17–115) and (17–116).

In the second approach, we substitute

$$\mathbf{U}_1 = \mathbf{B}\mathbf{U} + \mathbf{H}_3 \qquad \text{(f)}$$

in (a) and (17–131):

$$\begin{aligned} \Pi_p &= V + (\bar{\mathbf{P}}_{I,1}^T - \bar{\mathbf{P}}_1^T)(\mathbf{B}\mathbf{U} + \mathbf{H}_3) \\ V &= \tfrac{1}{2}(\mathscr{V} - \mathscr{V}_o)^T \mathbf{k}'(\mathscr{V} - \mathscr{V}_o) \\ \mathscr{V} &= \mathbf{A}_1\mathbf{B}\mathbf{U} + \mathbf{A}_1\mathbf{H}_3 + \mathbf{A}_2\bar{\mathbf{U}}_2 \end{aligned} \tag{17–132}$$

The variation of Π_p considering $\mathbf{U}$ as the independent variable is

$$\begin{aligned} d\Pi_p &= \Delta\mathbf{U}^T[\mathbf{B}^T(\bar{\mathbf{P}}_{I,1} - \bar{\mathbf{P}}_1) + (\mathbf{B}^T\mathbf{A}_1^T\mathbf{k}'\mathbf{A}_1\mathbf{B})\mathbf{U} \\ &\quad + \mathbf{B}^T\mathbf{A}_1^T\mathbf{k}'(\mathbf{A}_1\mathbf{H}_3 + \mathbf{A}_2\bar{\mathbf{U}}_2 - \mathscr{V}_o)] \\ &= \Delta\mathbf{U}^T[(\mathbf{B}^T\mathbf{K}_{11}\mathbf{B})\mathbf{U} - \mathbf{B}^T\mathbf{H}_4] \end{aligned} \tag{g}$$

Requiring Π_p to be stationary for arbitrary $\Delta\mathbf{U}$ results in (17–126). Note that we could have used the reduced form for V, i.e., equation (d). Also, we still have to determine the constraint forces.

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18

Analysis of Geometrically Nonlinear Systems

18–1. INTRODUCTION

In this chapter, we extend the displacement formulation to include geometric nonlinearity. The derivation is restricted to *small rotation*, i.e., where squares of rotations are negligible with respect to unity. We also consider the material to be linearly elastic and the member to be *prismatic*.

The first phase involves developing appropriate member force-displacement relations by integrating the governing equations derived in Sec. 13–9. We treat first planar deformation, since the equations for this case are easily integrated and it reveals the essential *nonlinear* effects. The three-dimensional problem is more formidable and one has to introduce numerous approximations in order to generate an explicit solution. We will briefly sketch out the solution strategy and then present a *linearized* solution applicable for doubly symmetric cross-sections.

The direct stiffness method is employed to assemble the system equations. This phase is essentially the same as for the linear case. However, the governing equations are now nonlinear.

Next, we described two iterative procedures for solving a set of nonlinear algebraic equations, successive substitution and Newton-Raphson iteration. These methods are applied to the system equations and the appropriate recurrence relations are developed. Finally, we utilize the *classical* stability criterion to investigate the stability of an equilibrium position.

18–2. MEMBER EQUATIONS—PLANAR DEFORMATION

Figure 18–1 shows the initial and deformed positions of the member. The centroidal axis initially coincides with the X_1 direction and X_2 is an axis of symmetry for the cross section. We work with displacements (u_1, u_2, ω_3),