This variational statement is called Reissner's principle (see Ref. 8).

(a) Transform Π_R to Π_p by requiring the stresses to satisfy the stress displacement relations. *Hint*: Note (10–101).

(b) Transform Π_R to $-\Pi_c$ by restricting the geometry to be linear ($\sigma^k = \sigma$ and $e_{ij} = (u_{i,j} + u_{j,i})/2$) and requiring the stresses to satisfy the stress equilibrium equations and stress boundary conditions on Ω_σ. *Hint*: Integrate $\sigma_{ij}e_{ij}$ by parts, using (10–81).

10–29. Interpret (10–90) as

$$d_Q = \frac{\partial}{\partial P_Q} \Pi_c$$

where P_Q is a force applied at Q in the direction of the displacement measure, d_Q.

11

St. Venant Theory of Torsion-Flexure of Prismatic Members

11–1. INTRODUCTION AND NOTATION

A body whose cross-sectional dimensions are small in comparison with its axial dimension is called a *member*. If the centroidal axis is straight and the shape and orientation of the normal cross section are constant,† the member is said to be *prismatic*. We define the member geometry with respect to a global reference frame (X_1, X_2, X_3), as shown in Fig. 11–1. The X_1 axis is taken to coincide with the centroidal axis and X_2, X_3 are taken as the principal inertia directions. We employ the following notation for the cross-sectional properties:

$$\begin{aligned} A &= \iint dx_2\, dx_3 = \iint dA \\ I_2 &= \iint (x_3)^2\, dA \\ I_3 &= \iint (x_2)^2\, dA \end{aligned} \tag{11–1}$$

Since X_2, X_3 pass through the centroid and are principal inertia directions, the centroidal coordinates and product of inertia vanish:

$$\begin{gathered} x_{2,c} = \frac{1}{A}\iint x_2\, dA = 0 \qquad x_{3,c} = \frac{1}{A}\iint x_3\, dA = 0 \\ I_{23} = \iint x_2 x_3\, dA = 0 \end{gathered} \tag{11–2}$$

One can work with an arbitrary orientation of the reference axes, but this will complicate the derivation.

St. Venant's theory of torsion-flexure is restricted to *linear* behavior. It is an *exact* linear formulation for a prismatic member subjected to a prescribed

† The case where the cross-sectional shape is constant but the orientation varies along the centroidal axis is treated in Chapter 15.

distribution of surface forces applied on the end cross sections. Later, in Chapter 13, we modify the St. Venant theory to account for displacement restraint at the ends and for geometric nonlinearity.

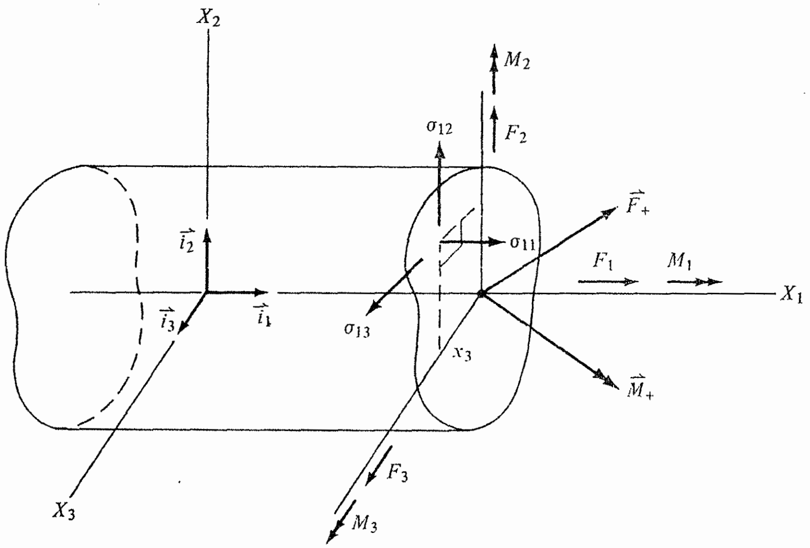


Fig. 11–1. Notation for prismatic member.

The distribution of surface forces on a cross section is specified in terms of its statically equivalent force system at the centroid. Figure 11–1 shows the stress components on a positive face. We define $\vec{F}_+$, $\vec{M}_+$ as the force and moment vectors acting at the centroid which are statically equivalent to the distribution of stresses over the section. The components of $\vec{F}_+$, $\vec{M}_+$ are called *stress resultants* and *stress couples*, respectively, and their definition equations are

$$\begin{gathered} F_1 = \iint \sigma_{11}\,dA \qquad F_2 = \iint \sigma_{12}\,dA \qquad F_3 = \iint \sigma_{13}\,dA \\ M_1 = \iint (x_2\sigma_{13} - x_3\sigma_{12})dA \\ M_2 = \iint x_3\sigma_{11}\,dA \\ M_3 = -\iint x_2\sigma_{11}\,dA \end{gathered} \tag{11-3}$$

The internal force and moment vectors acting on the negative face are denoted by $\vec{F}_-$, $\vec{M}_-$. Since

$$\vec{F}_- = -\vec{F}_+ \qquad \vec{M}_- = -\vec{M}_+ \tag{11-4}$$

it follows that the positive sense of the stress resultants and couples for the negative face is opposite to that shown in Fig. 11–1.

We discuss next the pure-torsion case, i.e., where the end forces are statically equivalent to only M_1. We then extend the formulation to account for flexure

and treat torsional-flexural coupling. Finally, we describe an approximate procedure for determining the flexural shear stress distribution in thin-walled sections.

11–2. THE PURE-TORSION PROBLEM

Consider the prismatic member shown in Fig. 11–2. There are *no* boundary forces acting on the cylindrical surface. The boundary forces acting on the end cross sections are statically equivalent to just a twisting moment M_1. Also, there is *no* restraint with respect to axial (out-of-plane) displacement at the ends. The analysis of this member presents the *pure-torsion* problem. In what follows, we establish the governing equations for pure torsion, using the approach originally suggested by St. Venant.

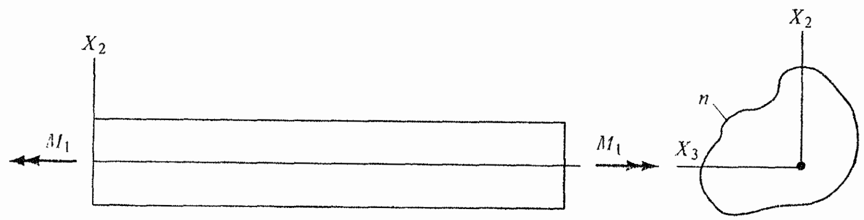


Fig. 11–2. Prismatic member in pure torsion.

Rather than attempt to solve the three-dimensional problem directly, we impose the following conditions on the behavior and then determine what problem these conditions correspond to.

1. Each cross section is rigid with respect to deformation in its plane, i.e., $\varepsilon_2 = \varepsilon_3 = \gamma_{23} = 0$.
2. Each cross section experiences a rotation ω_1 about the X_1 axis† and an out-of-plane displacement u_1.

These conditions lead to the following expansions for the in-place displacements:

$$\begin{aligned} u_2 &= -\omega_1 x_3 \\ u_3 &= +\omega_1 x_2 \end{aligned} \tag{11-5}$$

The corresponding linear strains are

$$\begin{aligned} \varepsilon_2 &= \varepsilon_3 = \gamma_{23} = 0 \\ \varepsilon_1 &= u_{1,1} \\ \gamma_{12} &= u_{1,2} + u_{2,1} = u_{1,2} - x_3\omega_{1,1} \\ \gamma_{13} &= u_{1,3} + u_{3,1} = u_{1,3} + x_2\omega_{1,1} \end{aligned} \tag{11-6}$$

† Problem 11–1 treats the general case where the cross section rotates about an arbitrary point.

Now, the strains must be independent of x_1 since each cross section is subjected to the same moment. This requires

$$\begin{aligned} \omega_{1,1} &= \text{const} = k_1 \\ u_1 &= u_1(x_2, x_3) \end{aligned} \tag{11-7}$$

We consider the left end to be fixed with respect to rotation and express ω_1, u_1 as

$$\begin{aligned} \omega_1 &= k_1 x_1 \\ u_1 &= k_1 \phi_t \end{aligned} \tag{11-8}$$

where $\phi_t = \phi_t(x_2, x_3)$ defines the out-of-plane displacement (warping) of a cross section. The strains and stresses corresponding to this postulated displacement behavior are

$$\begin{aligned} \varepsilon_1 = \varepsilon_2 = \varepsilon_3 &= \gamma_{23} = 0 \\ \gamma_{12} &= k_1(\phi_{t,2} - x_3) \\ \gamma_{13} &= k_1(\phi_{t,3} + x_2) \end{aligned} \tag{11-9}$$

and

$$\begin{aligned} \sigma_{11} = \sigma_{22} = \sigma_{33} &= \sigma_{23} = 0 \\ \sigma_{12} = G\gamma_{12} &= Gk_1(\phi_{t,2} - x_3) = \sigma_{12}(x_2, x_3) \\ \sigma_{13} = G\sigma_{13} &= Gk_1(\phi_{t,3} + x_2) = \sigma_{13}(x_2, x_3) \end{aligned} \tag{11-10}$$

We are assuming that the material is isotropic† and there are no initial strains.

One step remains, namely, to satisfy the stress-equilibrium equations and stress boundary conditions on the cylindrical surface. The complete system of linear stress-equilibrium equations, (10–49), reduces to

$$\sigma_{21,2} + \sigma_{31,3} = 0 \tag{11-11}$$

Substituting for the shearing stresses and noting that Gk_1 is constant lead to the differential equation

$$\left(\frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right)\phi_t = \nabla^2 \phi_t = 0 \tag{11-12}$$

which must be satisfied at all points in the cross section.

The exterior normal n for the cylindrical surface is *perpendicular* to the X_1 direction. Then $\alpha_{n1} = 0$, and the stress boundary conditions, (10–49), reduce to

$$p_{n1} = \alpha_{n2}\sigma_{21} + \alpha_{n3}\sigma_{31} = 0 \tag{11-13}$$

Using (11–10), the boundary condition for ϕ_t is

$$\begin{gathered} \alpha_{n2}(\phi_{t,2} - x_3) + \alpha_{n3}(\phi_{t,3} + x_2) = 0 \\ \Downarrow \\ \frac{\partial \phi_t}{\partial n} = \alpha_{n2}x_3 - \alpha_{n3}x_2 \qquad (\text{on } S) \end{gathered} \tag{11-14}$$

† Problem 11–3 treats the orthotropic case.

The pure-torsion problem involves solving $\nabla^2\phi_t = 0$ subject to (11–14). Once ϕ_t is known, we determine the distribution of transverse shearing stresses from (11–10). Note that ϕ_t depends only on the shape of the cross section.

The shearing stress distribution must lead to no shearing stress resultants:

$$\begin{aligned} F_2 &= \iint \sigma_{12}\, dA = 0 \\ F_3 &= \iint \sigma_{13}\, dA = 0 \end{aligned} \tag{a}$$

This requires

$$\iint \frac{\partial \phi_t}{\partial x_2} dA = \iint \frac{\partial \phi_t}{\partial x_3} dA = 0 \tag{b}$$

To proceed further, we need certain integration formulas. We start with

$$\iint \frac{\partial f}{\partial x_j} dA = \oint f\alpha_{nj}\, dS \tag{11-15}$$

which is just a special case of (10–81). Applying (11–15) to $\iint \nabla^2\psi\, dA$ leads to Green's theorem,

$$\begin{aligned} \iint \nabla^2\psi\, dA &= \oint \left(\alpha_{n2}\frac{\partial \psi}{\partial x_2} + \alpha_{n3}\frac{\partial \psi}{\partial x_3}\right) dS \\ &= \oint \frac{\partial \psi}{\partial n}\, dS \end{aligned} \tag{11-16}$$

If ψ is a harmonic function (i.e., $\nabla^2\psi = 0$), Green's theorem requires

$$\oint \frac{\partial \psi}{\partial n}\, dS = 0 \tag{c}$$

Now, ϕ_t is a harmonic function. For the formulation to be consistent, (11–14) must satisfy (c). Using (11–15), (c) transforms to

$$\oint (\alpha_{n2}x_3 - \alpha_{n3}x_2) dS = \iint \left(\frac{\partial}{\partial x_2}x_3 - \frac{\partial}{\partial x_3}x_2\right) dA \equiv 0 \tag{d}$$

Since $\partial\phi_t/\partial n$ is specified on the boundary, we cannot apply (11–15) directly to (b). In this case, we use the fact that $\nabla^2\phi_t = 0$ and write

$$\frac{\partial \phi_t}{\partial x_j} = \frac{\partial}{\partial x_2}\left(x_j \frac{\partial \phi_t}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\left(x_j \frac{\partial \phi_t}{\partial x_3}\right) \qquad (j = 2, 3) \tag{e}$$

Integrating (e),

$$\iint \frac{\partial \phi_t}{\partial x_j} dA = \oint x_j \frac{\partial \phi_t}{\partial n}\, dS \qquad (j = 2, 3) \tag{f}$$

and then substituting for the normal derivative, verifies (b).

The constant k_1 is determined from the remaining boundary condition,

$$M_1 = \iint (x_2\sigma_{13} - x_3\sigma_{12}) dA \tag{11-17}$$

We substitute for the shearing stresses and write the result as

$$M_1 = Gk_1 J \tag{11-18}$$

where J is a cross-sectional property,

$$J = \iint \left(x_2^2 + x_3^2 + x_2 \frac{\partial \phi_t}{\partial x_3} - x_3 \frac{\partial \phi_t}{\partial x_2} \right) dA \tag{11-19}$$

At this point, we summarize the results for the pure-torsion problem.

1. ***Displacements***

$$\begin{aligned} u_1 &= k_1 \phi_t \\ u_2 &= -\omega_1 x_3 \\ u_3 &= \omega_1 x_2 \\ \omega_1 &= k_1 x_1 \\ k_1 &= \frac{M_1}{GJ} \end{aligned}$$

2. ***Stresses***

$$\begin{aligned} \sigma_{12} &= \frac{M_1}{J}\left(\frac{\partial \phi_t}{\partial x_2} - x_3\right) \\ \sigma_{13} &= \frac{M_1}{J}\left(\frac{\partial \phi_t}{\partial x_3} + x_2\right) \end{aligned} \tag{11-20}$$

3. ***Governing Equations***

in A: $\nabla^2 \phi_t = 0$

on S: $\partial \phi_t / \partial n = \alpha_{n2} x_3 - \alpha_{n3} x_2$

It is possible to obtain the exact solution for ϕ_t for simple cross sections.

The procedure outlined above is basically a *displacement* method. One can also use a *force* approach for this problem. We start by expressing the shearing stresses in terms of a stress function ψ, so that the stress-equilibrium equation (Equation 11–11) is identically satisfied. An appropriate definition is

$$\begin{aligned} \sigma_{12} &= \frac{\partial \psi}{\partial x_3} \\ \sigma_{13} &= -\frac{\partial \psi}{\partial x_2} \end{aligned} \tag{11-21}$$

The shearing stresses for the λ, ν directions, shown in Fig. 11–3, follow directly from the definition equation

$$\begin{aligned} \sigma_{1\lambda} &= \frac{\partial \psi}{\partial \nu} \\ \sigma_{1\nu} &= -\frac{\partial \psi}{\partial \lambda} \end{aligned} \tag{11-22}$$

Taking S 90° counterclockwise from the exterior normal direction, and noting that the stress boundary condition is $\sigma_{1n} = 0$, lead to the boundary condition for ψ,

$$\psi = \text{const on } S \tag{11-23}$$

We establish the differential equation for ψ by requiring the warping function ϕ_t to be continuous. First, we equate the expressions for σ in terms of ψ and ϕ_t:

$$\begin{aligned} \sigma_{12} &= \psi_{,3} = \frac{M_1}{J}(\phi_{t,2} - x_3) \\ \sigma_{13} &= -\psi_{,2} = \frac{M_1}{J}(\phi_{t,3} + x_2) \end{aligned} \tag{a}$$

Now, for continuity,

$$\phi_{t,23} = \phi_{t,32} \tag{b}$$

Operating on (a), we obtain

$$\nabla^2 \psi = -2\frac{M_1}{J} \tag{c}$$

It is convenient to express ψ as

$$\psi = \frac{M_1}{J}\bar{\psi} \tag{11-24}$$

The governing equations in terms of $\bar{\psi}$ are†

$$\begin{aligned} \sigma_{12} &= \frac{M_1}{J}\frac{\partial \bar{\psi}}{\partial x_3} \\ \sigma_{13} &= -\frac{M_1}{J}\frac{\partial \bar{\psi}}{\partial x_2} \end{aligned} \tag{11-25}$$

and

$$\begin{aligned} \nabla^2 \bar{\psi} &= -2 && (\text{in } A) \\ \bar{\psi} &= C_i && (\text{on boundary } S_i) \end{aligned} \tag{11-26}$$

Substituting (11–25) in the definition equation for M_1 leads to the following expression for J:

$$J = -\iint \left(x_2 \frac{\partial \bar{\psi}}{\partial x_2} + x_3 \frac{\partial \bar{\psi}}{\partial x_3} \right) dA \tag{a}$$

Applying (10–81) to (a) and noting‡ that

$$-\oint_{S_i} x_j \alpha_{nj}\, dS = A_i = \text{area enclosed by the interior boundary curve, } S_i \tag{b}$$

$$|\bar{\psi}|_{S_i} = C_i = \text{const}$$

† Equations (11–26) can be interpreted as the governing equations for an initially stretched membrane subjected to normal pressure. This interpretation is called the "membrane analogy." See Ref. 3.

‡ The S direction is always taken such that $n - S$ has the same sense as $X_2 - X_3$. Then, the $+S$ direction for an interior boundary is opposite to the $+S$ direction for an exterior boundary since the direction for n is *reversed*. This is the reason for the negative sign on the boundary integral.

we can write

$$J = 2\iint \bar{\psi}\, dA + 2\sum A_i C_i \tag{11-27}$$

where $\bar{\psi} = 0$ on the *exterior* boundary.

To determine the constants C_i for the multiply connected case, we use the fact that ϕ_t is continuous. This requires

$$\oint_S \frac{\partial \phi_t}{\partial S}\, dS = 0 \tag{11-28}$$

for an arbitrary closed curve in the cross section.

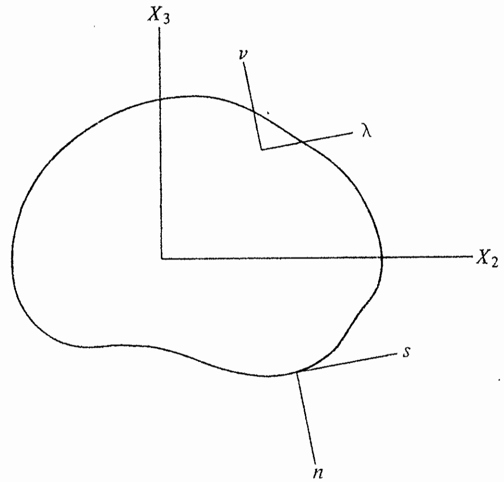


Fig. 11–3. Definition of n-s and λ-ν directions.

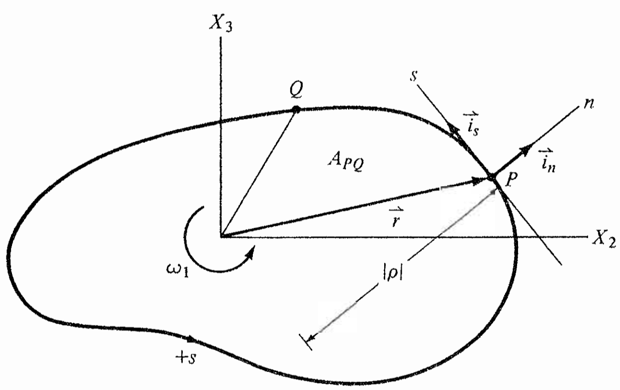


Fig. 11–4. Graphical representation of sector area.

Consider the closed curve shown in Fig. 11–4. The shearing strain γ_{1S} is given by

$$\gamma_{1S} = \alpha_{S2}\gamma_{12} + \alpha_{S3}\gamma_{13} \tag{a}$$

Using (11–9), we can write (a) as

$$\begin{aligned}\gamma_{1S} &= k_1(\alpha_{S2}\phi_{t,2} + \alpha_{S3}\phi_{t,3} - x_3\alpha_{S2} + x_2\alpha_{S3}) \\ &= k_1\left(\frac{\partial \phi_t}{\partial S} + \rho\right)\end{aligned} \tag{11-29}$$

where ρ is the projection of the radius vector on the *outward* normal.† The magnitude of ρ is equal to the perpendicular distance from the origin to the tangent. Integrating between points P, Q, we obtain

$$\int_{S_P}^{S_Q} \gamma_{1S}\, dS = k_1(\phi_{t,Q} - \phi_{t,P} + 2A_{PQ}) \tag{11-30}$$

where

$$A_{PQ} = \frac{1}{2}\int_{S_P}^{S_Q} \rho\, dS = \text{sector area enclosed by the arc } PQ \text{ and the radius vectors to } P \text{ and } Q.$$

Finally, taking $P = Q$,‡

$$\oint \gamma_{1S}\, dS = 2k_1 A_S \tag{11-31}$$

where A_S denotes the area enclosed by the curve. Since $\sigma_{1j} = G\gamma_{1j}$, we can write

$$\oint \sigma_{1S}\, dS = 2Gk_1 A_S = 2\frac{M_1}{J} A_S \tag{11-32}$$

Note that the $+S$ direction for (11–32) is from X_2 toward X_3. Also, this result is *independent* of the location of the origin.

Instead of using (11–9), we could have started with the fact that the cross section rotates about the centroid. The displacement in the $+S$ direction follows from Fig. 11–4:§

$$u_S = \omega_1(\vec{i}_1 \times \vec{r} \cdot \vec{i}_S) = \omega_1\rho = k_1 x_1 \rho \tag{11-33}$$

Substituting for u_S in

$$\gamma_{1S} = u_{S,1} + u_{1,S} \tag{11-34}$$

and noting that $u_1 = k_1\phi_t$ lead to (11–29).

Using (11–22), we can write

$$\sigma_{1S} = -\frac{\partial \psi}{\partial n} = -\frac{M_1}{J}\frac{\partial \bar{\psi}}{\partial n} \tag{11-35}$$

† This interpretation of ρ is valid only when S is directed from X_2 to X_3, i.e., counterclockwise for this case.

‡ See Prob. 11–14 for an alternate derivation.

§ This development applies for arbitrary choice of the $+S$ direction. The sign of ρ is positive if a rotation about X_1 produces a translation in the $+S$ direction. Equation (11–29) is used to determine the warping distribution once the shearing stress distribution is known. See Prob. 11–4.

Then, substituting for σ_{1S} in (11–32), we obtain

$$\oint_S \frac{\partial \bar{\psi}}{\partial n}\, dS = -2A_S \tag{11–36}$$

where n is the *outward* normal, A_S is the area enclosed by S, and the $+S$ sense is from X_2 to X_3. This result is valid for an arbitrary closed curve in the cross section. We employ (11–36) to determine the values of $\bar{\psi}$ at the interior boundaries of a multiply connected cross section.

It is of interest to determine the energy functions associated with pure torsion. When the material is linearly elastic and there are no initial strains, the strain and complementary energy densities are equal, i.e., $V = V^* = \frac{1}{2}\boldsymbol{\sigma}^T\boldsymbol{\varepsilon}$. We let

$$\bar{V} = \iint_A V\, dA = \text{strain energy per unit length} \tag{11–37}$$

The strain energy density is given by

$$V = \frac{G}{2}(\gamma_{12}^2 + \gamma_{13}^2) \tag{a}$$

Substituting for γ_{12}, γ_{13},

$$V = \frac{Gk_1^2}{2}[(\phi_{t,2} - x_3)^2 + (\phi_{t,3} + x_2)^2] \tag{b}$$

and integrating (b) over the cross section, we obtain

$$\bar{V} = \tfrac{1}{2}GJk_1^2 \tag{11–38}$$

Since $\bar{V}^* = \bar{V}$, and $M_1 = GJk_1$, it follows that

$$\bar{V}^* = \frac{1}{2G}\iint(\sigma_{12}^2 + \sigma_{13}^2)dA = \frac{M_1^2}{2GJ} \tag{11–39}$$

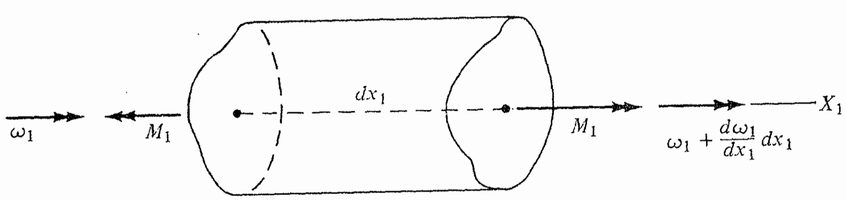


Fig. 11–5. Differential element for determination of the rotational work.

Instead of integrating the strain-energy density, we could have determined the work done by the moments acting on a differential element. Consider the element shown in Fig. 11–5. The boundary forces acting on a face are statically equivalent to just a torsional moment. Also, the cross sections are *rigid* in their plane and rotate about X_1. The relative rotation of the faces is

$$\left(\omega_1 + \frac{d\omega_1}{dx_1}dx_1\right) - \omega_1 = k_1\, dx_1 \tag{a}$$

and the first-order work done by the external forces due to an increment in ω_1 reduces to

$$\delta W_E = \oiint \mathbf{p}^T\, \Delta\mathbf{u}\, dS = M_1\, \Delta k_1\, dx_1 \tag{b}$$

Now,

$$\delta W_E = \delta V_T = \iiint \delta V\, dx_1\, dx_2\, dx_3 = \delta\bar{V}\, dx_1 \tag{c}$$

for an elastic body. Then, expanding $\delta\bar{V}$,

$$\delta\bar{V} = \frac{d\bar{V}}{dk_1}\Delta k_1 = M_1\, \Delta k_1 \tag{d}$$

and it follows that

$$\begin{aligned}\frac{d\bar{V}}{dk_1} &= M_1 = GJk_1\\ \bar{V} &= \tfrac{1}{2}GJk_1^2\end{aligned} \tag{11–40}$$

11–3. APPROXIMATE SOLUTION OF THE TORSION PROBLEM FOR THIN-WALLED OPEN CROSS SECTIONS

We consider first the rectangular cross section shown in Fig. 11–6. The exact solution for this problem is contained in numerous texts (e.g., see Art. 5–3 of Ref. 1) and therefore we will only summarize the results obtained.

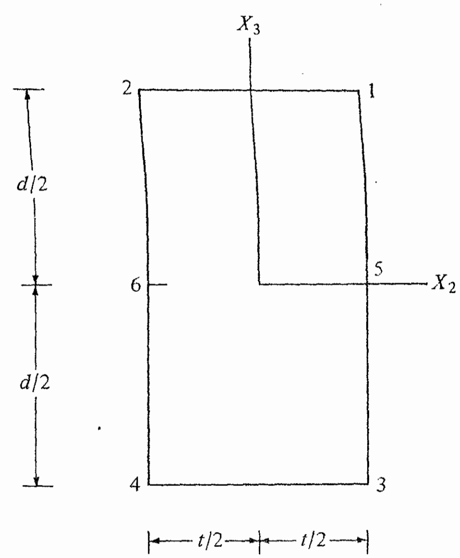


Fig. 11–6. Notation for rectangular section.

When $t \leqslant d$, the maximum shearing stress σ_{max} occurs at $x_2 = \pm t/2, x_3 = 0$ (points 5, 6). The exact expressions are

$$J = K_1 \frac{dt^3}{3}$$
$$|\sigma_{max}| = \frac{M_1}{J} K_2 t \tag{11-41}$$

where

$$K_1 = 1 - \frac{192}{\pi^5}\left(\frac{t}{d}\right) \sum_{n=0,1,\ldots} \frac{1}{(2n+1)^5} \tanh \lambda_n$$
$$K_2 = 1 - \frac{8}{\pi^2} \sum_{n=0,1} \frac{1}{(2n+1)^2} \frac{1}{\cosh \lambda_n}$$
$$\lambda_n = \frac{2n+1}{2} \pi \left(\frac{d}{t}\right)$$

Values of K_1, K_2 for d/t ranging from 1 to 10 are tabulated below:

d/t	K_1	K_2
1	0.422	0.675
2	.687	.930
3	.789	.985
4	.843	.997
5	.873	.999
10	0.936	1.000

If $t \ll d$, we say the cross section is *thin*. The approximate solution for a thin rectangle is

$$J \approx \tfrac{1}{3} dt^3$$
$$\sigma_{13} \approx 2\frac{M_1}{J} x_2 = 2Gk_1 x_2$$
$$\phi_t \approx x_2 x_3 \tag{11-42}$$
$$\bar{\psi} \approx \left(\frac{t}{2}\right)^2 - x_2^2$$

(We take $d/t = \infty$ in the exact solution.) The shearing stress σ_{13} varies linearly across the thickness and

$$|\sigma_{max}| \approx \frac{M_1}{J} t \approx \frac{3M_1}{dt^2}$$

A view of the warped cross section is shown in Fig. 11–7.

Since the stress function approach is quite convenient for the analysis of *thin*-walled cross sections, we illustrate its application to a thin rectangular

cross section. Later, we shall extend the results obtained for this case to an arbitrary thin walled open cross section. The governing equations for a simply connected cross section are summarized below for convenience (see (11–26), (11–27)):

$$\nabla^2 \bar{\psi} = -2 \quad \text{(in } A\text{)}$$
$$\bar{\psi} = 0 \quad \text{(on the boundary)}$$
$$\sigma_{1n} = \frac{M_1}{J} \frac{\partial \bar{\psi}}{\partial S} \tag{a}$$
$$J = 2\iint \bar{\psi}\, dA$$

where the S direction is 90° counterclockwise from the n direction.† Since t is small and σ_{12}, the shearing stress component in the *thickness* direction, must

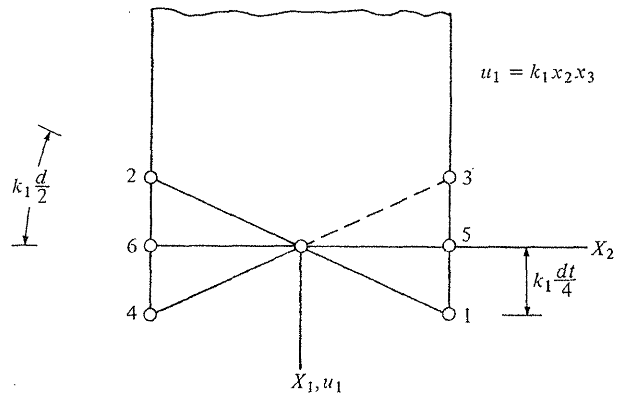


Fig. 11–7. Warping function for a rectangular cross section.

vanish on the boundary faces, it is reasonable to assume $\sigma_{12} = 0$ at all points in the cross section. This corresponds to taking $\bar{\psi}$ independent of x_3. The equations reduce to

$$\frac{d^2}{dx_2^2} \bar{\psi} = -2$$
$$\bar{\psi} = 0 \quad \text{at } x_2 = \pm \frac{t}{2} \tag{b}$$

Solving (b), we obtain

$$\bar{\psi} = -x_2^2 + \frac{t^2}{4}$$
$$J = 2d \int_{-t/2}^{t/2} \bar{\psi}\, dx_2 = \tfrac{1}{3} dt^3 \tag{c}$$
$$\sigma_{13} = -\frac{M_1}{J} \frac{\partial \bar{\psi}}{\partial x_2} = 2\frac{M_1}{J} x_2$$

† This applies for X_3 counterclockwise from X_2. The general requirement is the $n - S$ sense must coincide with the X_2-X_3 sense.

The expression for $\bar{\psi}$ developed above must be corrected near the ends ($x_3 = \pm d/2$) since it does not satisfy the boundary condition,

$$\bar{\psi} = 0 \qquad \text{at } x_3 = \pm\frac{d}{2} \tag{d}$$

This will lead to $\sigma_{12} \neq 0$ near the ends, but will have a negligible effect on J and σ_{max}. Actually, the moment due to the approximate linear expansion for σ_{13} is equal to only one half the applied moment:

$$(M_1)_{\sigma_{13}} = d\int_{-t/2}^{t/2} x_2\sigma_{13}\,dx_2 = \frac{M_1}{J}\left(\frac{1}{6}dt^3\right) = \frac{1}{2}M_1 \tag{e}$$

The corrective stress system (σ_{12}) carries $M_1/2$. This is reasonable since, even though σ_{12} is small in comparison to σ_{max}, its moment arm is large.

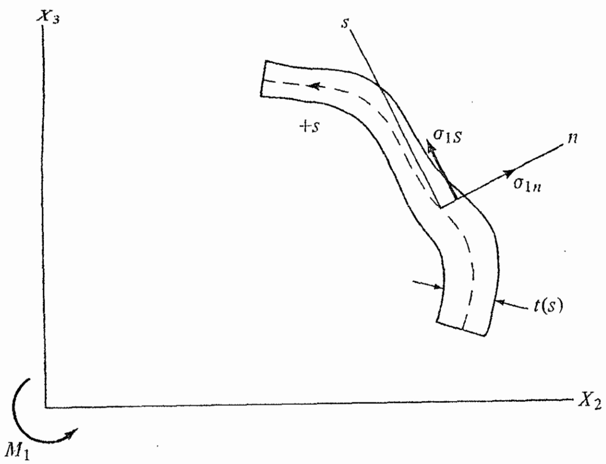


Fig. 11–8. Notation for thin-walled open cross section.

We consider next the arbitrary thin-walled open cross section shown in Fig. 11–8. The S curve defines the centerline (bisects the thickness) and the n direction is normal to S. We assume $\sigma_{1n} = 0$ and take $\bar{\psi} = -n^2 + t^2/4$. This corresponds to using the solution for the thin rectangle and is reasonable when S is a *smooth* curve. The resulting expressions for J and σ_{1S} are

$$\begin{aligned} J &= \tfrac{1}{3}\int_S t^3\,dS \\ \sigma_{1S} &= 2\frac{M_1}{J}n \\ \sigma_{1S,\,max} &= \frac{M_1}{J}t_{max} = Gk_1t_{max} \end{aligned} \tag{11–43}$$

The results for a single thin rectangle are also applied to a cross section consisting of thin rectangular elements. Let d_i, t_i denote the length and thickness of element i. We take J as

$$J = \tfrac{1}{3}\sum_i d_it_i^3 \tag{11–44}$$

As an illustration, consider the symmetrical section shown in Fig. 11–9. Applying (11–44), we obtain

$$J = \tfrac{1}{3}(2b_ft_f^3 + d_Wt_W^3)$$

The maximum shearing stress in the center zone of an element is taken as

$$\sigma_{m,i} = \frac{M_1}{J}t_i = Gk_1t_i \tag{11–45}$$

In general, there is a stress concentration at a reentrant corner (e.g., point A in Fig. 11–9) which depends on the ratio of fillet radius to thickness. For the case

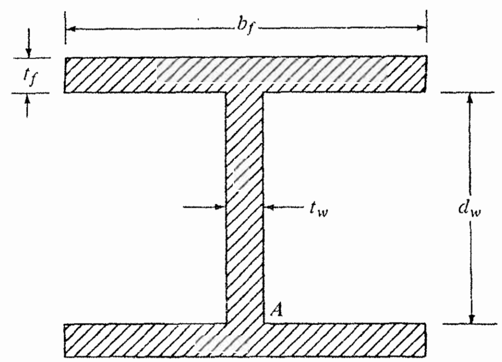


Fig. 11–9. Symmetrical wide-flange section.

of an angle having equal flange thicknesses, the formula†

$$\sigma_{fillet} = \sigma_m\left(1 + \frac{t}{4r_f}\right) \tag{11–46}$$

where r_f is the fillet radius and σ_m is given by (14–45), gives good results for $r_f/t < 0.3$. The stress increase can be significant for small values of r_f/t. For example, $\sigma_{fillet} = 3.5\sigma_m$ for $r_f = 0.1t$. Numerical procedures such as finite differences or the finite element method‡ must be resorted to in order to obtain exact solutions for irregular sections.

† See Ref. 2 and Appendix of Ref. 9.
‡ See Ref. 4.

11–4. APPROXIMATE SOLUTION OF THE TORSION PROBLEM FOR THIN-WALLED CLOSED CROSS SECTIONS

The stress function method is generally used to analyze thin-walled closed cross sections. For convenience, the governing equations are summarized below (see (11–26), (11–27), (11–36)):

$$
\begin{aligned}
\nabla^2\bar{\psi} &= -2 && \text{(in } A\text{)}\\
\bar{\psi} &= 0 && \text{(on the exterior boundary)}\\
\bar{\psi} &= C_i && \text{(on the interior boundary, } S_i\text{)}\\
J &= 2\iint\bar{\psi}\,dA + 2\textstyle\sum C_iA_i && (A_i = \text{area enclosed by } S_i)\\
\sigma_{1S} &= -\frac{M_1}{J}\frac{\partial\bar{\psi}}{\partial n} && (n \text{ is the } \textit{outward} \text{ normal for } S \text{ and } +S \text{ sense from } X_2 \text{ toward } X_3)
\end{aligned}
$$

$$\oint_S \frac{\partial\bar{\psi}}{\partial n}\,dS = -2A_S$$

We consider first the single cell shown in Fig. 11–10. The S_{cl} curve defines the centerline. Since there is an interior boundary, we have to add a term

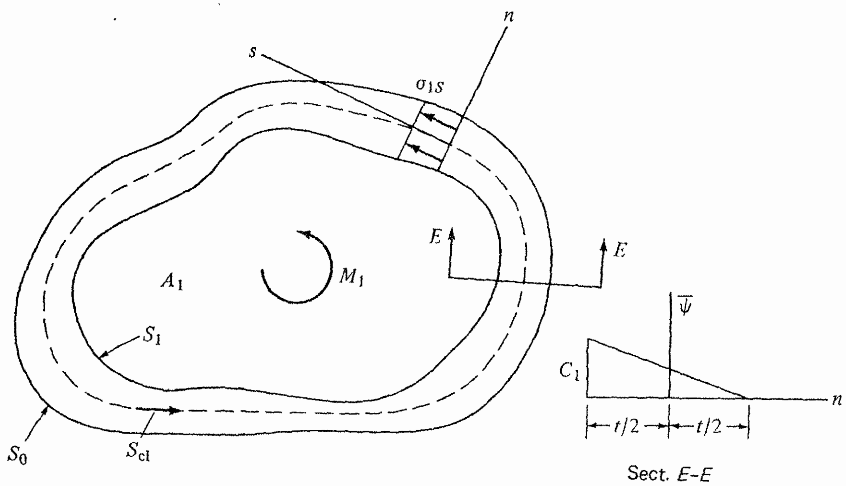


Fig. 11–10. Single closed cell.

involving C_1 to the approximate expression for $\bar{\psi}$ used for the open section. We take $\bar{\psi}$ as

$$
\begin{aligned}
\bar{\psi} &= \bar{\psi}^0 + \bar{\psi}^c\\
&= \frac{t^2}{4} - n^2 + \frac{C_1}{2}\left(1 - \frac{2n}{t}\right)
\end{aligned}
\tag{11–47}
$$

where $\bar{\psi}^c$ represents the contribution of the interior boundary. This expression

satisfies the one-dimensional compatibility equation and boundary conditions,

$$
\begin{aligned}
&\frac{\partial^2\bar{\psi}}{\partial n^2} = -2\\
&\bar{\psi} = 0 \qquad \text{at } n = +t/2\\
&\bar{\psi} = C_1 \qquad \text{at } n = -t/2
\end{aligned}
\tag{a}
$$

and is a reasonable approximation when S is a smooth curve.

Differentiating $\bar{\psi}$,

$$\frac{\partial\bar{\psi}}{\partial n} = -2n - \frac{C_1}{t} \qquad \frac{\partial\bar{\psi}}{\partial s} = 0 \tag{b}$$

and substituting (b) in the expressions for the shearing stress components lead to

$$
\begin{aligned}
\sigma_{1n} &= 0\\
\sigma_{1s} &= \frac{M_1}{J}\left(2n + \frac{C_1}{t}\right)\\
&= \sigma^0_{1s} + \sigma^c_{1s}
\end{aligned}
\tag{11–48}
$$

The tangential shearing stress varies linearly over the thickness and its average value is σ^c_{1s}. We let q be the shear stress resultant per unit length along S, positive when pointing in the $+S$ direction,

$$q = \int_{-t/2}^{t/2} \sigma_{1S}\,dn \tag{11–49}$$

and call q the *shear flow*. Substituting for σ_{1S}, we find

$$q = \frac{M_1}{J}C_1 = \sigma^c_{1S}t \tag{11–50}$$

The additional shearing stress due to the interior boundary (i.e., closed cell) corresponds to a constant shear flow around the cell. One can readily verify† that the distribution, q = const, is statically equivalent to only a torsional moment, M^c_1, given by

$$M^c_1 = 2qA_{cl} \tag{11–51}$$

The torsional constant is determined from

$$J = 2\iint\bar{\psi}\,dA + 2C_1A_1 \equiv M_1/Gk_1 \tag{a}$$

Substituting for $\bar{\psi}$ using (11–47), we obtain

$$
\begin{aligned}
J &= J^0 + J^c\\
J^0 &= \tfrac{1}{3}\oint_{S_{cl}} t^3\,dS \qquad J^c = 2C_1A_{cl}
\end{aligned}
\tag{11–52}
$$

Equation (a) was established by substituting for the shearing stresses in terms of $\bar{\psi}$ in the definition equation for M_1 and then transforming the integrand. We could have arrived at (11–52) by first expressing the total torsional moment as

$$M_1 = M^0_1 + M^c_1 \tag{11–53}$$

† See Prob. 11–5.

where M_1^o is the *open section* contribution and M_1^c is due to the *closure*. Next, we write

$$M_1 = Gk_1J \qquad M_1^o = Gk_1J^o \qquad M_1^c = Gk_1J^c \tag{11-54}$$

Then,

$$J = J^o + J^c \tag{11-55}$$

and it follows that

$$M_1^o = \frac{J^o}{J}M_1 \qquad M_1^c = \frac{J^c}{J}M_1 \tag{11-56}$$

Finally, using (11–51), we can express J^c as

$$J^c = M_1^c/(M_1/J) = 2A_{cl}[q/(M_1/J)] \tag{11-57}$$

This result shows that we should work with a modified shear flow,

$$C = q/(M_1/J) \tag{11-58}$$

rather than with the actual shear flow. Note that $C \equiv C_1$ for the single cell.

It remains to determine C_1 by enforcing continuity of the warping function on the centerline curve. Applying (11–32) to S_{cl}, we have

$$\oint_{S_{cl}} \sigma_{1S}^c\, dS = 2\frac{M_1}{J}A_{cl} \tag{11-59}$$

Substituting for σ_{1S}^c,

$$\sigma_{1S}^c = q/t = \frac{M_1}{J}\frac{C_1}{t}$$

leads to†

$$C_1 = \frac{2A_{cl}}{\oint_{S_{cl}} dS/t} \tag{11-60}$$

One should note that C_1 is a property of the cross section. Once C_1 is known, we can evaluate J from (11–52) and the shearing stress from

$$\sigma_{1S} = \frac{M_1}{J}\left(\pm t + \frac{C_1}{t}\right) \tag{11-61}$$

Example 11–1

Consider the rectangular section shown. The thickness is constant and a, b are centerline dimensions. The various cross-sectional properties are

$$A_{cl} = ab \qquad \oint\frac{dS}{t} = \frac{2(a+b)}{t}$$

$$C_1 = \frac{abt}{a+b} \qquad J^o = \tfrac{1}{3}t^3[2(a+b)]$$

$$J^c = \frac{2a^2b^2t}{a+b}$$

† See Prob. 11–6.

We express J as

$$J = J^c\left(1 + \frac{J^o}{J^c}\right)$$

For this section,

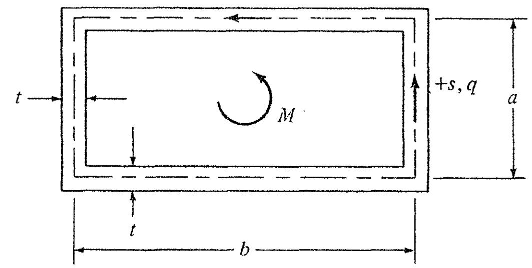
$$\frac{J^o}{J^c} = \frac{1}{3}\left(\frac{t}{b}\right)^2\left(1 + \frac{b}{a}\right)^2$$

We consider $a > b$. Then,

$$\frac{J^o}{J^c} = O\left(\frac{t}{b}\right)^2$$

The section is said to be thin-walled when $t \ll b$. In this case, it is reasonable to neglect J^o vs. J^c.

Fig. E11–1



The stress follows from (11–61),

$$\sigma_{1S} = \frac{M_1}{J}\frac{C_1}{t}\left(1 \pm \frac{t^2}{C_1}\right) = \sigma_{1S}^c\left(1 \pm \frac{\sigma_{1S}^o}{\sigma_{1S}^c}\right)$$

where, for this section,

$$\frac{t^2}{C_1} = \left(1 + \frac{b}{a}\right)\frac{t}{b} = O\left(\frac{t}{b}\right)$$

If the section is thin-walled, we can neglect the contribution of σ_{1S}^o, i.e., we can take

$$\sigma_{1S} \approx \sigma_{1S}^c = q/t = \frac{M_1}{2A_{cl}t}$$

We consider next the section shown in Fig. 11–11. Rather than work with $\bar{\psi}$, it is more convenient to work with the shear flows for the segments. We number the closed cells consecutively and take the $+S$ sense to coincide with the X_2-X_3 sense. The $+S$ sense for the open segments is arbitrary. We define q_j as the shear flow for cell j and write

$$q_j = \frac{M_1}{J}C_j \tag{11-62}$$

Note that C_j is the value of $\bar{\psi}$ on the interior boundary of cell j and the shear flow is constant along a segment. The total shear flow distribution is obtained

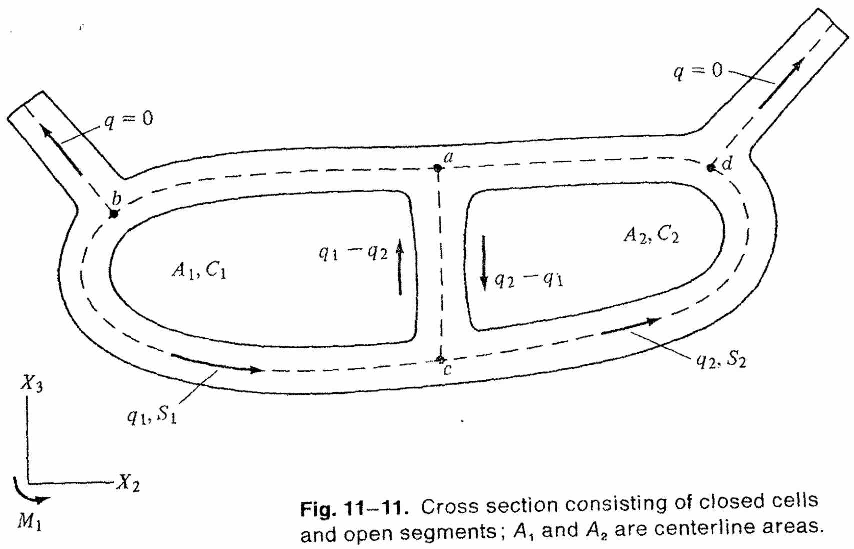


Fig. 11–11. Cross section consisting of closed cells and open segments; A_1 and A_2 are centerline areas.

by superimposing the individual cell flows. Then, the shear flow in the segment common to cells i and j is the difference between q_i and q_j. The sign depends on the sense of S.

$$q = q_1 - q_2 = \frac{M_1}{J}(C_1 - C_2) \qquad \text{for } S_1 \qquad (11\text{–}63)$$
$$q = q_2 - q_1 \qquad \text{for } S_2$$

The shearing stress is assumed to vary linearly over the thickness. For convenience, we drop the subscripts on σ_{1S} and write the limiting values as

$$\sigma = \pm\sigma^o + \sigma^c$$

where

$$\sigma^o = \frac{M_1}{J}t \qquad \sigma^c = q/t = \frac{M_1}{J}\left(\frac{C_{\text{net}}}{t}\right) \qquad (11\text{–}64)$$

It remains to determine C_1, C_2, and J.

We have shown (see (11–55)) that

$$J = J^o + J^c \qquad \text{(a)}$$

and

$$M_1^c = \frac{J^c}{J}M_1 \qquad \text{(b)}$$

We determine J^o from

$$J^o = \sum_{\text{segments}} \tfrac{1}{3}\int t^3\, dS \qquad (11\text{–}65)$$

Substituting for M_1^c,†

$$M_1^c = 2q_1A_1 + 2q_2A_2 = 2\frac{M_1}{J}(C_1A_1 + C_2A_2) \qquad \text{(c)}$$

in (b) leads to

$$J^c = 2(A_1C_1 + A_2C_2) \qquad (11\text{–}66)$$

The constants C_i are obtained by enforcing continuity of ϕ_t on the centerline of each cell. This can also be interpreted as requiring each cell to have the same twist deformation, k_1:‡

$$\oint_{S_i} (q/t)dS = 2\frac{M_1}{J}A_i \qquad i = 1, 2 \qquad (11\text{–}67)$$

Substituting for q in terms of C and letting

$$a_{11} = \oint_{S_1}\frac{dS}{t} \qquad a_{22} = \oint_{S_2}\frac{dS}{t} \qquad a_{12} = a_{21} = -\int_c^a \frac{dS}{t} \qquad (11\text{–}68)$$

where a_{12} involves the segment common to cells 1, 2, the continuity equations take the following form:

$$\begin{aligned} a_{11}C_1 + a_{12}C_2 &= 2A_1 \\ a_{12}C_1 + a_{22}C_2 &= 2A_2 \end{aligned} \qquad (11\text{–}69)$$

We solve this system of equations for C_1, C_2, then determine J^c with (11–66), and finally evaluate the stresses with (11–64).

We can represent the governing equations in compact form by introducing matrix notation. The form of the equations suggests that we define

$$\mathbf{C} = \begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} \qquad \mathbf{A} = \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} \qquad \mathbf{a} = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} \qquad (11\text{–}70)$$

With this notation,

$$\begin{aligned} J^c &= 2\mathbf{A}^T\mathbf{C} \\ \mathbf{aC} &= 2\mathbf{A} \end{aligned} \qquad (11\text{–}71)$$

Substituting for $\mathbf{A}$ in the expression for J^c,

$$J^c = \mathbf{C}^T\mathbf{aC} \qquad \text{(a)}$$

and noting that J^c is positive, we conclude that $\mathbf{a}$ must be positive definite.

The complementary energy per unit length along the centroidal axis is defined by (11–39),

$$\bar{V}^* = \frac{1}{2G}\iint \sigma_{1S}^2\, dA \equiv \frac{M_1^2}{2GJ} \qquad \text{(b)}$$

† We apply (11–51) to each cell.
‡ See (11–32).

Since σ_{1S} varies linearly over the thickness, the open and closed stress distributions are uncoupled, i.e., we can write

$$\bar{V}^* = \bar{V}^*\big|_{\text{open}} + \bar{V}^*\big|_{\text{due to } q}$$

where

$$\begin{aligned}\bar{V}^*\Big|_{\text{open}} &= \frac{1}{2}\,GJ^o k_1^2 = \frac{1}{2G}\left(\frac{M_1^2}{J}\right)\left(\frac{J^o}{J}\right)\\ \bar{V}^*\Big|_q &= \frac{1}{2G}\int q^2\,\frac{dS}{t} = \frac{1}{2G}\,(\mathbf{q}^T\mathbf{a}\mathbf{q})\\ &= \frac{1}{2G}\left(\frac{M_1^2}{J}\right)\left(\frac{J^c}{J}\right)\end{aligned} \tag{11-72}$$

It is reasonable to neglect the *open* contribution when the section is thin-walled.

Example 11-2

The open-section torsional constant for the section shown is

$$J^o = \tfrac{1}{3}[\ell t_3^3 + 2(b + d + h)t_1^3 + ht_2^3] \tag{a}$$

Applying (11-68) to this section, we obtain

$$\begin{aligned}A_1 &= hd\\ A_2 &= hb\\ a_{11} &= \frac{1}{t_1}(h + 2d) + \frac{h}{t_2}\\ a_{12} &= -\frac{h}{t_2}\\ a_{22} &= \frac{1}{t_1}(h + 2b) + \frac{h}{t_2}\end{aligned} \tag{b}$$

and the following equations for C_1, C_2 and J.

$$\begin{aligned}\left(1 + \frac{t_1}{t_2} + 2\frac{d}{h}\right)C_1 - \left(\frac{t_1}{t_2}\right)C_2 &= 2\,dt_1\\ -\left(\frac{t_1}{t_2}\right)C_1 + \left(1 + \frac{t_1}{t_2} + 2\frac{b}{h}\right)C_2 &= 2bt_1\\ J^c &= 2h(C_1 d + C_2 b)\\ J &= J^o + J^c\end{aligned} \tag{c}$$

Finally, the shear stress intensities in the various segments are

$$\begin{aligned}\sigma_1 &= \frac{M_1}{J}\left(\frac{C_1}{t_1} + t_1\right)\\ \sigma_{1,2} &= \frac{M_1}{J}\left(\frac{C_1 - C_2}{t_2} + t_2\right)\\ \sigma_2 &= \frac{M_1}{J}\left(\frac{C_2}{t_1} + t_1\right)\\ \sigma_3 &= \frac{M_1}{J}\,t_3\end{aligned} \tag{d}$$

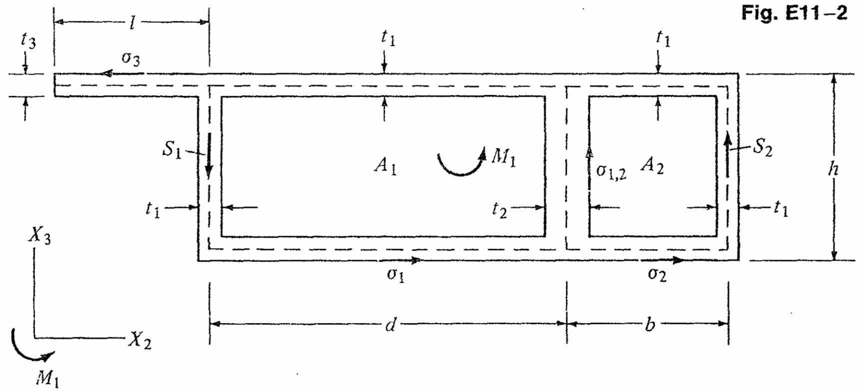


Fig. E11-2

When $d = b$,

$$C_1 = C_2 = \frac{2bt_1}{1 + 2\dfrac{b}{h}} \tag{e}$$

and the section functions as a single cell with respect to shear flow.

11-5. TORSION-FLEXURE WITH UNRESTRAINED WARPING

Consider the prismatic member shown in Fig. 11-12. There are no boundary forces acting on the cylindrical surface. The distribution of boundary forces

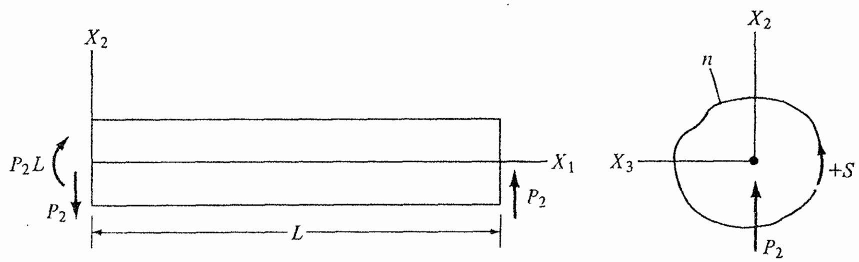


Fig. 11-12. Prismatic member in shear loading.

on the cross section at $x_1 = L$ is statically equivalent to a single force $P_2\bar{i}_2$, acting at the centroid. Also, the end cross sections are *not* restrained against warping, i.e., out-of-plane displacement. In what follows, we describe St. Venant's torsion-flexure formulation for this problem. Later, in Chapter 13, we shall modify the theory to include restraint against warping.

We start by postulating expansions for the stresses. The stress resultants and couples required for equilibrium at x_1 are

$$\begin{aligned} F_1 = F_3 = M_1 = M_2 &= 0 \\ F_2 &= P_2 \\ M_3 &= P_2(L - x_1) \end{aligned} \tag{a}$$

Introducing (a) in the definition equations for the stress resultants and couples leads to the following conditions on the stresses:

$$\begin{aligned} \iint \sigma_{11}\, dA = \iint x_3\sigma_{11}\, dA &= 0 \\ \iint x_2\sigma_{11}\, dA &= P_2(L - x_1) \\ \iint \sigma_{12}\, dA &= P_2 \\ \iint \sigma_{13}\, dA &= 0 \\ \iint (x_2\sigma_{13} - x_3\sigma_{12})dA &= 0 \end{aligned} \tag{b}$$

The expansion,

$$\sigma_{11} = -\frac{M_3}{I_3}x_2 = -\frac{P_2}{I_3}(L - x_1)x_2 \tag{c}$$

satisfies the first three conditions (i.e., F_1, M_2, M_3) identically since

$$\begin{aligned} \iint x_2\, dA = \iint x_2x_3\, dA &= 0 \\ \iint x_2^2\, dA &= I_3 \end{aligned} \tag{d}$$

The last three conditions (i.e., F_2, F_3, M_1) require σ_{12}, σ_{13} to be independent of x_1. This suggests that we consider the following postulated stress behavior:

$$\begin{aligned} \sigma_{11} &= -\frac{M_3}{I_3}x_2 = -\frac{P_2}{I_3}(L - x_1)x_2 \\ \sigma_{12} &= \sigma_{12}(x_2, x_3) \\ \sigma_{13} &= \sigma_{13}(x_2, x_3) \\ \sigma_{22} &= \sigma_{33} = \sigma_{23} = 0 \end{aligned} \tag{11-73}$$

Introducing (11–73) in the stress-equilibrium equations and stress boundary conditions for the cylindrical surface leads to

$$\begin{aligned} \sigma_{21,2} + \sigma_{31,3} + \frac{P_2}{I_3}x_2 &= 0 \qquad \text{(in } A\text{)} \\ \alpha_{n2}\sigma_{21} + \alpha_{n3}\sigma_{31} &= 0 \qquad \text{(on } S\text{)} \end{aligned} \tag{11-74}$$

At this point, we can either introduce a stress function or express (11–74) in terms of a warping function. We will describe the latter approach first.

The displacements can be found by integrating the stress-displacement relations. We suppose the material is linearly elastic, isotropic with respect to the X_2-X_3 plane, and orthotropic with respect to the axial direction. This is a convenient way of keeping track of the coupling between axial and in-plane deformation. Substituting for the stresses in (10–74), we obtain

$$\begin{aligned} \varepsilon_1 &= u_{1,1} = \frac{1}{E_1}\sigma_{11} = -\frac{P_2}{E_1I_3}(L - x_1)x_2 \\ \varepsilon_2 &= u_{2,2} = -\frac{\nu_1}{E}\sigma_{11} = \frac{\nu_1P_2}{EI_3}(L - x_1)x_2 \\ \varepsilon_3 &= u_{3,3} = -\frac{\nu_1}{E}\sigma_{11} = \frac{\nu_1P_2}{EI_3}(L - x_1)x_2 \\ \gamma_{12} &= u_{1,2} + u_{2,1} = \frac{1}{G_1}\sigma_{12} = \text{function of } x_2, x_3 \\ \gamma_{13} &= u_{1,3} + u_{3,1} = \frac{1}{G_1}\sigma_{13} = \text{function of } x_2, x_3 \\ \gamma_{23} &= u_{2,3} + u_{3,2} = 0 \end{aligned} \tag{a}$$

Integrating the first three equations leads to

$$\begin{aligned} u_1 &= -\frac{P_2}{E_1I_3}(Lx_1 - \tfrac{1}{2}x_1^2)x_2 + f_1(x_2, x_3) \\ u_2 &= \frac{\nu_1P_2}{2EI_3}(L - x_1)x_2^2 + f_2(x_1, x_3) \\ u_3 &= \frac{\nu_1P_2}{EI_3}(L - x_1)x_2x_3 + f_3(x_1, x_2) \end{aligned} \tag{b}$$

The functions f_1, f_2, f_3 are determined by substituting (b) in the last three equations. We omit the details and just list the resulting expressions, which involve seven constants:

$$\begin{aligned} f_1 &= C_1 + C_5x_2 + C_6x_3 + \phi(x_2, x_3) \\ f_2 &= C_2 - C_5x_1 + C_4x_3 - k_1x_1x_3 \\ &\quad -\frac{\nu_1P_2}{2EI_3}(L - x_1)x_3^2 + \frac{P_2}{2E_1I_3}(L - \tfrac{1}{3}x_1)x_1^2 \\ f_3 &= C_3 - C_6x_1 - C_4x_2 + k_1x_1x_2 \end{aligned} \tag{c}$$

The constants $C_1, C_2, \ldots, C_6$ are associated with rigid body motion and k_1 is associated with the twist deformation.†

We consider the following displacement boundary conditions:

1. The origin is fixed:

$$u_1 = u_2 = u_3 = 0 \qquad \text{at } (0, 0, 0)$$

2. A line element on the centroidal axis at the origin is fixed:

$$u_{2,1} = u_{3,1} = 0 \qquad \text{at } (0, 0, 0)$$

† See Eq. (11–5).

3. A line element on the X_2 axis at the origin is fixed with respect to rotation in the X_2-X_3 plane:

$$u_{2,3} = 0 \qquad \text{at } (0, 0, 0)$$

These conditions correspond to the "fixed-end" case and are sufficient to eliminate the rigid body terms. The final displacement expressions are

$$\begin{aligned} u_1 &= -\frac{P_2}{E_1 I_3}(Lx_1 - \tfrac{1}{2}x_1^2)x_2 + \phi(x_2, x_3) \\ u_2 &= \frac{\nu_1 P_2}{2EI_3}(L - x_1)(x_2^2 - x_3^2) + \frac{P_2}{2E_1 I_3}\left(Lx_1^2 - \frac{x_1^3}{3}\right) - k_1 x_1 x_3 \qquad (11\text{–}75) \\ u_3 &= \frac{\nu_1 P_2}{EI_3}(L - x_1)x_2 x_3 + k_1 x_1 x_2 \end{aligned}$$

One step remains, namely, to satisfy the equilibrium equation and boundary condition. The transverse shearing stresses are given by

$$\begin{aligned} \frac{1}{G_1}\sigma_{12} &= \phi_{,2} - k_1 x_3 + \frac{\nu_1 P_2}{2EI_3}(x_3^2 - x_2^2) \\ \frac{1}{G_1}\sigma_{13} &= \phi_{,3} + k_1 x_2 - \frac{\nu_1 P_2}{EI_3}x_2 x_3 \end{aligned} \qquad (11\text{–}76)$$

Substituting for the stresses in (11–74), we obtain the following differential equation and boundary condition for ϕ:

$$\begin{aligned} \nabla^2\phi &= \frac{P_2}{I_3}\left(\frac{2\nu_1}{E} - \frac{1}{G_1}\right)x_2 \qquad (\text{in } A) \\ \frac{\partial\phi}{\partial n} &= k_1(\alpha_{n2}x_3 - \alpha_{n3}x_2) + \frac{\nu_1 P_2}{2EI_3}\left[\alpha_{n2}\left(\frac{x_2^2 - x_3^2}{2}\right) + \alpha_{n3}x_2 x_3\right] \end{aligned} \qquad (11\text{–}77)$$

The form of the above equations suggests that we express ϕ as

$$\phi = k_1\phi_t + \frac{P_2}{G_1 I_3}(\phi_{2r} - \tfrac{1}{6}x_2^3) + \frac{\nu_1 P_2}{EI_3}(\phi_{2d} + \tfrac{1}{3}x_2^3) \qquad (11\text{–}78)$$

where ϕ_t is the warping function for *pure* torsion and ϕ_{2r} and ϕ_{2d} are *harmonic* functions which define the warping due to flexure. Substituting for ϕ leads to the following boundary conditions for ϕ_{2r} and ϕ_{2d}:

$$\begin{aligned} \frac{\partial\phi_{2r}}{\partial n} &= \frac{1}{2}\alpha_{n2}x_2^2 \\ \frac{\partial\phi_{2d}}{\partial n} &= -\alpha_{n2}\left(\frac{x_2^2 + x_3^2}{2}\right) + \alpha_{n3}x_2 x_3 \end{aligned} \qquad (11\text{–}79)$$

One can show, by using (11–15), that

$$\oint \frac{\partial\phi_{2r}}{\partial n}\,dS = 0$$

$$\oint \frac{\partial\phi_{2d}}{\partial n}\,dS = 0$$

and therefore the formulation is consistent. Terms involving ν_1/E are due to in-plane deformation, i.e., deformation in the plane of the cross section, and setting $\nu_1/E = 0$ corresponds to assuming the cross section is rigid. Then, ϕ_{2r} defines the flexural warping for a rigid cross section and ϕ_{2d} represents the correction due to in-plane deformation.

The shearing stress is obtained by substituting for ϕ in (11–76). We write the result as

$$\sigma_{1j} = \sigma_{1j,t} + \sigma_{1j,r} + \sigma_{1j,d} \qquad (j = 2, 3) \qquad (11\text{–}80)$$

where $\sigma_{1j,t}$ is the pure-torsion distribution and $\sigma_{1j,r}$, $\sigma_{1j,d}$ are flexural distributions corresponding to ϕ_{2r} and ϕ_{2d}:

$$\begin{aligned} \sigma_{12,r} &= \frac{P_2}{I_3}(\phi_{2r,2} - \tfrac{1}{2}x_2^2) \\ \sigma_{13,r} &= \frac{P_2}{I_3}\phi_{2r,3} \\ \sigma_{12,d} &= \left(\frac{\nu_1 G_1}{E}\right)\frac{P_2}{I_3}[\phi_{2d,2} + \tfrac{1}{2}(x_2^2 + x_3^2)] \\ \sigma_{13,d} &= \left(\frac{\nu_1 G_1}{E}\right)\frac{P_2}{I_3}(\phi_{2d,3} - x_2 x_3) \end{aligned} \qquad (11\text{–}81)$$

The pure torsion distribution is statically equivalent to only a torsional moment, $M_{1,t} = G_1 k_1 J$. One can show that†

$$\begin{aligned} \iint\sigma_{12,r}\,dA &= P_2 \qquad & \iint\sigma_{13,r}\,dA &= 0 \\ \iint\sigma_{12,d}\,dA &= 0 \qquad & \iint\sigma_{13,d}\,dA &= 0 \end{aligned} \qquad (11\text{–}82)$$

Note that the shear stress due to in-plane deformation does not contribute to P_2.

The total torsional moment consists of a pure torsion term and two flexural terms,

$$M_1 = G_1 k_1 J + \frac{P_2}{I_3}\left(S_{2r} + \frac{\nu_1 G_1}{E}S_{2d}\right) \qquad (11\text{–}83)$$

where

$$\begin{aligned} S_{2r} &= \iint(\tfrac{1}{2}x_3 x_2^2 - x_3\phi_{2r,2} + x_2\phi_{2r,3})dA \\ S_{2d} &= \iint(-\tfrac{3}{2}x_2^2 x_3 - \tfrac{1}{2}x_3^3 + x_2\phi_{2d,3} - x_3\phi_{2d,2})dA \end{aligned}$$

Since ϕ_{2r} and ϕ_{2d} depend *only* on the shape of the cross section, it follows that S_{2r} and S_{2d} are properties of the cross section. For convenience, we let

$$\bar{x}_3 = -\frac{1}{I_3}\left(S_{2r} + \frac{\nu_1 G_1}{E}S_{2d}\right) \qquad (11\text{–}84)$$

and (11–83) reduces to

$$M_1 = G_1 k_1 J - P_2\bar{x}_3 \qquad (11\text{–}85)$$

Now, $-P_2\bar{x}_3$ is the statically equivalent torsional moment at the centroid due

† See Prob. 11–10.

to the flexural shear stress distribution. Then, $\bar{x}_3$ defines the location of the resultant of the *flexural shear stress* distribution with respect to the centroid.

The twist deformation is determined from

$$k_1 = \frac{1}{G_1 J}(M_1 + P_2\bar{x}_3) \tag{11-86}$$

where M_1 is the *applied* torsional moment with respect to the centroid. If P_2 is applied at the centroid, $M_1 = 0$, and

$$k_1 = \frac{\bar{x}_3}{G_1 J} P_2 \tag{a}$$

The cross section will twist unless $\bar{x}_3 = 0$. Suppose P_2 has an eccentricity e_3. In this case (see Fig. 11–13), $M_1 = -e_3 P_2$, and

$$k_1 = \frac{P_2}{G_1 J}(\bar{x}_3 - e_3) \tag{b}$$

For flexure alone to occur, e_3 must equal $\bar{x}_3$.

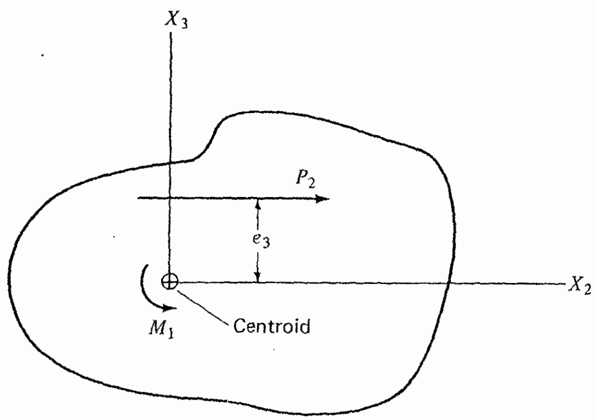


Fig. 11–13. Notation for eccentric load.

Whether twist occurs depends on the relative eccentricity, $e_3 - \bar{x}_3$. Now, to find $\bar{x}_3$, one must determine S_{2r} and S_{2d}. This involves solving two second-order partial differential equations. Exact solutions can be obtained for simple cross sections. In the section following, we present the exact solution for a rectangular cross section. If the section is irregular, one must resort to such numerical procedures as finite differences to solve the equations. In Sec. 11–7, we describe an approximate procedure for determining the flexural shear stress distribution in *thin walled* cross sections.

Suppose the cross section is symmetrical with respect to the X_2 axis. Then, α_{n2} is an even function of x_3 and α_{n3} is an odd function of x_3. The form of the boundary conditions (11–79) requires ϕ_{2r} and ϕ_{2d} to be *even* functions of x_3

for this case. Finally, it follows that† $S_{2r} = 0$ and $S_{2d} = 0$. Generalizing this result, we can state:

> The resultant of the shear stress distribution due to flexure in the X_j direction passes through the centroid when X_j is an axis of symmetry for the cross section.

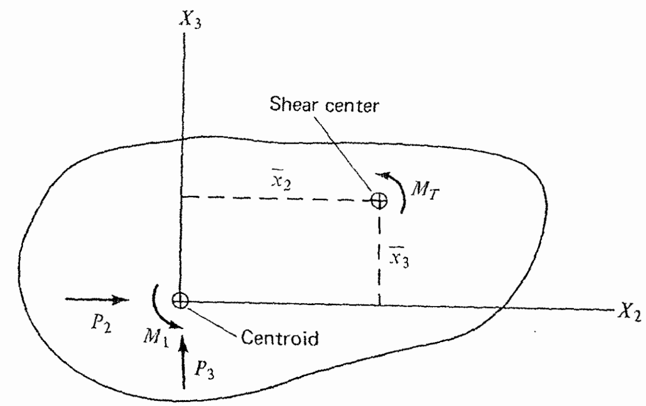


Fig. 11–14. Coordinates of the shear center.

We consider next the case where the member is subjected to P_2, P_3 and M_1 at the right end (see Fig. 11–14). The governing equations for the P_3 loading can be obtained by transforming the equations for the P_2 case according to

$$\begin{aligned} &x_2 \to x_3 && x_3 \to -x_2 \\ &u_2 \to u_3 && u_3 \to -u_2 \\ &\frac{\partial}{\partial x_2} \to \frac{\partial}{\partial x_3} && \frac{\partial}{\partial x_3} \to -\frac{\partial}{\partial x_2} \\ &\sigma_{12} \to \sigma_{13} && \sigma_{13} \to -\sigma_{12} \\ &I_3 \to I_2 && \end{aligned} \tag{a}$$

Two additional flexural warping functions must be determined. The expressions defining the flexural shear stress distributions due to P_3 are

$$\begin{aligned} \sigma_{12,r} &= \frac{P_3}{I_2}\phi_{3r,2} \\ \sigma_{13,r} &= \frac{P_3}{I_2}(\phi_{3r,3} - \tfrac{1}{2}x_3^2) \\ \sigma_{12,d} &= \frac{\nu_1 G_1}{E}\frac{P_3}{I_2}[\phi_{3d,2} - x_2 x_3] \\ \sigma_{13,d} &= \frac{\nu_1 G_1}{E}\frac{P_3}{I_2}[\tfrac{1}{2}(x_2^2 + x_3^2) + \phi_{3d,3}] \end{aligned} \tag{11-87}$$

† $\phi_{,2}$ is even in x_3, $\phi_{,3}$ is odd in x_3, and S_{2r}, S_{2d} involve only integrals of odd functions of x_3.

where ϕ_{3r}, ϕ_{3d} are harmonic functions† satisfying the following boundary conditions:

$$\frac{\partial \phi_{3r}}{\partial n} = \frac{1}{2}\alpha_{n3}x_3^2$$
$$\frac{\partial \phi_{3d}}{\partial n} = \alpha_{n2}x_2x_3 - \alpha_{n3}\left(\frac{x_2^2 + x_3^2}{2}\right) \tag{11-88}$$

Note that the distribution due to ϕ_{3d} leads to *no* shearing stress resultants. Finally, the total normal stress is given by

$$\sigma_1 = +\frac{M_2}{I_2}x_3 - \frac{M_3}{I_3}x_2 = -(L - x_1)\left(\frac{P_3}{I_2}x_3 + \frac{P_2}{I_3}x_2\right) \tag{11-89}$$

Superimposing the shearing stresses and evaluating the torsional moment, we obtain

$$M_1 = G_1k_1J - P_2\bar{x}_3 + P_3\bar{x}_2 \tag{11-90}$$

where $\bar{x}_2$ defines the location of the resultant of the flexural shear stress distribution due to P_3. One can interpret $\bar{x}_2, \bar{x}_3$ as the coordinates of a point, called the *shear center*. The required twist follows from (11–90):

$$k_1 = \frac{1}{G_1J}(M_1 + P_2\bar{x}_3 - P_3\bar{x}_2) \tag{a}$$

Since (see Fig. 11–14)

$$M_1 + P_2\bar{x}_3 - P_3\bar{x}_2 = \text{the applied moment with respect to the shear center} = M_T \tag{11-91}$$

we can write (a) as

$$k_1 = \frac{1}{G_1J}M_T \tag{11-92}$$

To determine the twist deformation (and the resulting torsional stresses), one must work with the torsional moment with respect to the *shear center*, not the centroid. For no twist, the applied force must pass through the shear center. In general, the shear center lies on an axis of symmetry. If the cross section is completely symmetrical, the shear center coincides with the centroid.

It is of interest to determine the complementary energy associated with torsion-flexure. The only finite stress components are σ_{11}, σ_{12}, and σ_{13}. Then $\bar{V}^*$ reduces to

$$\bar{V}^* = \frac{1}{2}\iint_A \left[\frac{1}{E_1}\sigma_{11}^2 + \frac{1}{G_1}\sigma_{12}^2 + \frac{1}{G_1}\sigma_{13}^2\right]dA \tag{a}$$

The contribution from σ_{11} follows directly by substituting (11–89) and using the definition equations for I_2, I_3.

† The total flexural warping function for P_3 is

$$\phi = \frac{P_3}{G_1I_2}\left(\phi_{3r} - \frac{1}{6}x_3^3\right) + \frac{\nu_1P_3}{EI_2}\left(\phi_{3d} + \frac{1}{3}x_3^3\right)$$

$$\bar{V}^*|_{\sigma_{11}} = \frac{1}{2E_1}\left(\frac{M_2^2}{I_2} + \frac{M_3^2}{I_3}\right) \tag{11-93}$$

Now, the total shearing stress is the sum of three terms:

1. σ_t, a pure torsional distribution due to M_T
2. σ_{F_2}, the flexural distribution due to F_2
3. σ_{F_3}, the flexural distribution due to F_3

Each of the flexural distributions can be further subdivided into—

1. σ_{r,F_j}, the distribution corresponding to a rigid cross section (defined by ϕ_{jr})
2. σ_{d,F_j}, the distribution associated with in-plane deformation of the cross section (defined by ϕ_{jd})

We combine the flexural distributions and express the total stress as

$$\sigma_{12} = \sigma_{12,t} + \sigma_{12,r} + \sigma_{12,d}$$
$$\sigma_{13} = \sigma_{13,t} + \sigma_{13,r} + \sigma_{13,d} \tag{a}$$

where the various terms are defined by (11–81) and (11–87). For example,

$$\sigma_{12,r} = \frac{F_2}{I_3}(\phi_{2r,2} - \tfrac{1}{2}x_2^2) + \frac{F_3}{I_2}\phi_{3r,2} \tag{b}$$

The complementary energy due to *pure* torsion follows from (11–38) and (11–92):

$$\iint (\sigma_{12,t}^2 + \sigma_{13,t}^2)dA = \tfrac{1}{2}G_1Jk_1^2 = \frac{M_T^2}{2GJ} \tag{11-94}$$

We express $\sigma_{1j,r}$ as

$$\sigma_{12,r} = \frac{F_2}{I_3}\bar{\phi}_{2r,2} + \frac{F_3}{I_2}\bar{\phi}_{3r,2}$$
$$\sigma_{13,r} = \frac{F_2}{I_3}\bar{\phi}_{2r,3} + \frac{F_3}{I_2}\bar{\phi}_{3r,3} \tag{11-95}$$

where

$$\bar{\phi}_{2r} = \phi_{2r} - \tfrac{1}{6}x_2^3 \qquad \bar{\phi}_{3r} = \phi_{3r} - \tfrac{1}{6}x_3^3$$

Expanding $(\sigma_{12,r}^2 + \sigma_{13,r}^2)$ and integrating over the cross section, we obtain†

$$\iint (\sigma_{12,r}^2 + \sigma_{13,r}^2)dA = \frac{F_2^2}{A_2} + \frac{2F_2F_3}{A_{23}} + \frac{F_3^2}{A_3}$$
$$\frac{1}{A_j} = \frac{1}{I_k^2}\iint (\bar{\phi}_{jr,2}^2 + \bar{\phi}_{jr,3}^2)dA = \frac{1}{I_k^2}\iint x_j\bar{\phi}_{jr}\,dA \qquad \begin{matrix} j \neq k \\ j,k = 2,3 \end{matrix} \tag{11-96}$$
$$\frac{1}{A_{23}} = \frac{1}{I_2I_3}\iint (\bar{\phi}_{2r,2}\bar{\phi}_{3r,2} + \bar{\phi}_{2r,3}\bar{\phi}_{3r,3})dA = \frac{1}{I_2I_3}\iint x_3\bar{\phi}_{2r}\,dA$$

† See Prob. 11–11.

The coupling term, $1/A_{23}$, vanishes when the cross section has an axis of symmetry.

We consider next the coupling between $\sigma_{,t}$ and $\sigma_{,r}$.

$$\iint(\sigma_{12,t}\sigma_{12,r} + \sigma_{13,t}\sigma_{13,r})dA$$
$$= \frac{M_T}{J}\iint\left[(\phi_{t,2} - x_3)\left(\frac{F_2}{I_3}\bar{\phi}_{2r,2} + \frac{F_3}{I_2}\bar{\phi}_{3r,2}\right) + (\phi_{t,3} + x_2)\left(\frac{F_2}{I_3}\bar{\phi}_{2r,3} + \frac{F_3}{I_2}\bar{\phi}_{3r,3}\right)\right]dA$$
$$= \frac{M_T}{J}\oint\left[\frac{F_2}{I_3}\bar{\phi}_{2r} + \frac{F_3}{I_2}\bar{\phi}_{3r}\right]\left[(\phi_{t,2} - x_3)\alpha_{n2} + (\phi_{t,3} + x_2)\alpha_{n3}\right]dS \quad (11\text{–}97)$$
$$- \frac{M_T}{J}\iint\left(\frac{F_2}{I_3}\bar{\phi}_{2r} + \frac{F_3}{I_2}\bar{\phi}_{3r}\right)\nabla^2\phi_t\, dA = 0$$

The remaining terms involve $\sigma_{,d}$, the shearing stress distribution due to in-plane deformation of the cross section. We will not attempt to expand these terms since we are interested primarily in the rigid cross section case.

Summarizing, the complementary energy for flexure-torsion with unrestrained warping is given by

$$\bar{V}^* = \frac{1}{2E_1}\left(\frac{M_2^2}{I_2} + \frac{M_3^2}{I_3}\right) + \frac{M_T^2}{2G_1J} + \frac{1}{2G_1}\left(\frac{F_2^2}{A_2} + 2\frac{F_2F_3}{A_{23}} + \frac{F_3^2}{A_3}\right) \quad (11\text{–}98)$$
$$+ \text{ terms involving } \nu_1/E$$

where $M_T = M_1 + F_2\bar{x}_3 - F_3\bar{x}_2$. We introduce the assumption of negligible in-plane deformation by setting $\nu_1/E = 0$. Similarly, we introduce the assumption of negligible warping due to flexure ($\bar{\phi}_{2r} \approx 0$, $\bar{\phi}_{3r} \approx 0$) by setting $1/A_1 = 1/A_2 = 1/A_{23} = 0$.

In Sec. 11–7, we develop an approximate procedure, called the *engineering theory*, for determining the flexural shear stress distribution, which is based upon integrating the stress-equilibrium equation directly. This approach is similar to the torsional stress analysis procedure described in the previous section. Since the shear stress distribution is statically indeterminate when the cross section is closed, the force redundants have to be determined by requiring the warping function to be continuous. For pure torsion, continuity requires (see (11–32))

$$\oint_S \sigma_{1S,t}\, dS = 2G_1k_1A_S \quad \text{(a)}$$

where the integration is carried out in the X_2-X_3 sense around S, and A_S is the area enclosed by S. To establish the continuity conditions for flexure, we operate on (11–81) and (11–37). There are four requirements:

$$\phi_{jr} \Rightarrow \oint_S (\sigma_{1S,r})_{F_j}\, dS = 0 \qquad j = 2, 3$$
$$\phi_{2d} \Rightarrow \oint_S (\sigma_{1S,d})_{F_2}\, dS = -\frac{2\nu_1G_1P_2}{EI_3}\iint_{A_S} x_3\, dA \quad (11\text{–}99)$$
$$\phi_{3d} \Rightarrow \oint_S (\sigma_{1S,d})_{F_3}\, dS = \frac{2\nu_1G_1P_3}{EI_2}\iint_{A_S} x_2\, dA$$

In the engineering theory of flexural shear stress distribution, the cross section is considered to be rigid, i.e., the distribution due to in-plane deformation is neglected. The *consistent* continuity condition on the flexural shearing stress is

$$\oint_S \sigma_{1S}\, dS = 0 \quad (11\text{–}100)$$

One can take the $+S$ direction as either clockwise or counterclockwise. By definition, the positive sense for σ_{1S} coincides with the $+S$ direction.

11–6. EXACT FLEXURAL SHEAR STRESS DISTRIBUTION FOR A RECTANGULAR CROSS SECTION

We consider the problem of determining the exact shear stress distribution due to F_2 for the rectangular cross section shown in Fig. 11–15. For convenience, we first list the governing equations:

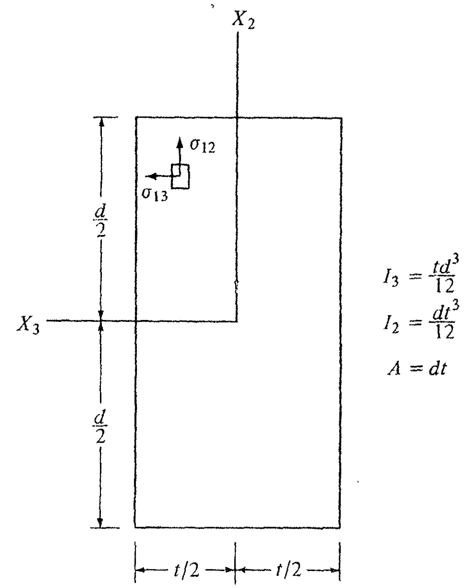


Fig. 11–15. Notation for rectangular cross section.

1. Warping functions

$$\phi = \frac{F_2}{G_1I_3}(\phi_{2r} - \tfrac{1}{6}x_2^3) + \frac{\nu_1F_2}{EI_3}(\phi_{2d} + \tfrac{1}{3}x_2^3)$$
$$\nabla^2\phi_{2r} = 0 \qquad \nabla^2\phi_{2d} = 0$$
$$\frac{\partial\phi_{2r}}{\partial n} = \tfrac{1}{2}\alpha_{n2}x_2^2$$
$$\frac{\partial\phi_{2d}}{\partial n} = -\alpha_{n2}\left(\frac{x_2^2 + x_3^2}{2}\right) + \alpha_{n3}x_2x_3$$

2. Shearing stresses

$$\sigma_{12} = \frac{F_2}{I_3}(\phi_{2r,\,2} - \tfrac{1}{2}x_2^2) + \frac{\nu_1 G_1 F_2}{E I_3}[\phi_{2d,\,2} + \tfrac{1}{2}(x_2^2 + x_3^2)]$$
$$\sigma_{13} = \frac{F_2}{I_3}(\phi_{2r,\,3}) + \frac{\nu_1 G_1 E_2}{E I_3}(\phi_{2d,\,3} - x_2 x_3)$$

Determination of ϕ_{2r}

The boundary conditions for ϕ_{2r} are

$$\begin{aligned} \phi_{2r,\,2} &= \frac{1}{2}\left(\frac{d}{2}\right)^2 \qquad && \text{at } x_2 = \pm\frac{d}{2} \\ \phi_{2r,\,3} &= 0 && \text{at } x_3 = \pm\frac{t}{2} \end{aligned} \tag{a}$$

We can take the solution as

$$\phi_{2r} = \tfrac{1}{8}d^2 x_2 \tag{b}$$

The corresponding stresses and warping function are

$$\begin{aligned} \phi_{2r} &= \phi_{2r} - \tfrac{1}{6}x_2^3 = \tfrac{1}{8}d^2 x_2 - \tfrac{1}{6}x_2^3 \\ \sigma_{12,\,r} &= \frac{F_2}{2I_3}\left(\frac{d^2}{4} - x_2^2\right) \\ \sigma_{13,\,r} &= 0 \end{aligned} \tag{11-101}$$

One can readily show that

$$\iint \sigma_{12,\,r}\, dA \equiv F_2$$

Finally, we evaluate $1/A_2$ using (11–96):

$$\frac{1}{A_2} = \frac{1}{I_3^2}\iint x_2 \phi_{2r}\, dA = \frac{6}{5}\frac{1}{A} \tag{11-102}$$

Determination of ϕ_{2d}

The boundary conditions for ϕ_{2d} are

$$\begin{aligned} \phi_{2d,\,2} &= -\frac{1}{2}\left(\frac{d^2}{4} + x_3^2\right) \qquad && \text{at } x_2 = \pm\frac{d}{2} \\ \phi_{2d,\,3} &= \pm\frac{t}{2}x_2 && \text{at } x_3 = \pm\frac{t}{2} \end{aligned} \tag{a}$$

Now, the form of (a) suggests that we express ϕ_{2d} as

$$\phi_{2d} = \tfrac{1}{2}x_2 x_3^2 - \tfrac{1}{6}x_2^3 - f(x_2, x_3) \tag{b}$$

where f is an harmonic function. The shearing stresses and boundary conditions expressed in terms of f are

$$\begin{aligned} \sigma_{12,\,d} &= \frac{\nu_1 G_1}{E}\frac{F_2}{I_3}(x_3^2 - f_{,2}) \\ \sigma_{13,\,d} &= \frac{\nu_1 G_1}{E}\frac{F_2}{I_3}(-f_{,3}) \end{aligned} \tag{c}$$

and

$$\begin{aligned} f_{,2} &= x_3^2 \qquad && \text{at } x_2 = \pm\frac{d}{2} \\ f_{,3} &= 0 && \text{at } x_3 = \pm\frac{t}{2} \end{aligned} \tag{d}$$

It remains to solve $\nabla^2 f = 0$ subject to (d).

Since the cross section is symmetrical, f must be an even function of x_3 and an odd function of x_2. We express f as

$$f = B_0 x_2 + \sum_{n=1,2,\ldots} B_n \cos\left(\frac{2n\pi x_3}{t}\right)\sinh\left(\frac{2n\pi x_2}{t}\right) \tag{e}$$

This expansion satisfies $\nabla^2 f = 0$ and the boundary condition at $x_3 = \pm t/2$. The remaining boundary condition requires

$$B_0 + \sum_{n=1,2,\ldots} B_n\left(\frac{2n\pi}{t}\cosh\frac{n\pi d}{t}\right)\cos\frac{2n\pi x_3}{t} = x_3^2 \qquad \left(-\frac{t}{2} < x_3 < +\frac{t}{2}\right) \tag{f}$$

Expanding x_3^2 in a Fourier cosine series and equating coefficients leads to

$$\begin{aligned} B_0 &= \frac{t^2}{12} \\ B_n &= \frac{1}{2}\left(\frac{t}{n\pi}\right)^3 \frac{(-1)^n}{\cosh\dfrac{n\pi d}{t}} \end{aligned} \tag{g}$$

The final expressions for the shearing stresses are

$$\begin{aligned} \sigma_{12,\,d} &= \frac{\nu_1 G_1}{E}\frac{F_2}{I_3}\left[\left(\frac{t}{\pi}\right)^2 \sum_{n=1,2,\ldots}\frac{(-1)^n}{n^2}\cos\frac{2n\pi x_3}{t}\left(1 - \frac{\cosh\dfrac{2n\pi x_2}{t}}{\cosh\dfrac{n\pi d}{t}}\right)\right] \\ \sigma_{13,\,d} &= \frac{\nu_1 G_1}{E}\frac{F_2}{I_3}\left[\left(\frac{t}{\pi}\right)^2 \sum_{n=1,2,\ldots}\frac{(-1)^n}{n^2}\frac{\sinh\dfrac{2n\pi x_2}{t}}{\cosh\dfrac{n\pi d}{t}}\sin\frac{2n\pi x_3}{t}\right] \end{aligned} \tag{11-103}$$

This system is statically equivalent to zero.

To investigate the error involved in assuming the cross section is rigid, we note that the maximum value of $\sigma_{12,r}$ occurs at $x_2 = 0$:

$$\sigma_{12,r}\big|_{\max} = \frac{F_2}{8I_3} d^2 \tag{a}$$

Specializing $\sigma_{12,d}$ for $x_2 = 0$,

$$(\sigma_{12,d})_{x_2=0} = \frac{\nu_1 G_1}{E} \frac{F_2}{I_3} \frac{d^2}{4} \sum_{n=1,2,\ldots} C_n \cos \frac{2n\pi x_3}{t} \tag{b}$$

where

$$C_n = \frac{4(-1)^n}{\lambda_n^2}\left(1 - \frac{1}{\cosh \lambda_n}\right) \qquad \lambda_n = \frac{n\pi d}{t} \tag{c}$$

Now, C_n decreases rapidly with n. Retaining only the first term in (b) leads to the following error estimate,

$$\frac{|\sigma_{12,d}|}{\sigma_{12,r}} \approx \left(\frac{2\nu_1 G_1}{E}\right)\left[\frac{4}{\left(\frac{\pi d}{t}\right)^2}\left(1 - \frac{1}{\cosh \frac{\pi d}{t}}\right)\right] \tag{d}$$

Results for a representative range of d/t and isotropic material are listed below. They show that it is reasonable to neglect the corrective stress system for a rectangular cross section. The error decreases as the section becomes thinner, i.e., as d/t becomes large with respect to unity:

d/t	$\lvert\sigma_{12,d}\rvert/\sigma_{12,r}$
2	0.024
1	0.092
$\frac{1}{2}$	0.122

11–7. ENGINEERING THEORY OF FLEXURAL SHEAR STRESS DISTRIBUTION IN THIN-WALLED CROSS SECTIONS

The "exact" solution of the flexure problem involves solving four second-order partial differential equations. If one assumes the cross section is rigid with respect to in-plane deformation, only two equations have to be solved. Even in this case, solutions can be found for only simple cross sections. When the cross section is irregular, one must resort to a numerical procedure such as finite differences or, alternatively, introduce simplifying assumptions as to the stress distribution. In what follows, we describe the latter approach for a thin-walled cross section. The resulting theory is generally called the *engineering* theory of shear stress. We apply the engineering theory to typical cross sections

and also illustrate the determination of the shear center and the energy coefficients, $1/A_j$ $(j = 2, 3)$.

Figure 11–16 shows a segment defined by cutting planes at x_1 and $x_1 + dx_1$. Since the cross section is thin-walled, it is reasonable to assume that the normal stress, σ_{11}, is constant through the thickness and to neglect σ_{1n}. Also, we work

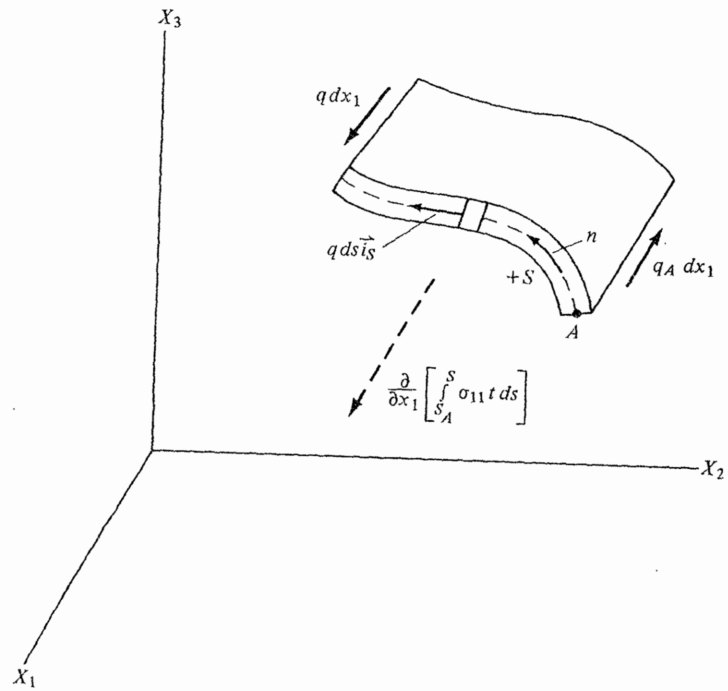


Fig. 11–16. Differential thin-walled segment.

with the shear flow, q, rather than with σ_{1S}. Integrating the axial force-equilibrium equation,

$$\frac{\partial q}{\partial S} + \frac{\partial}{\partial x_1}(\sigma_{11} t) = 0 \tag{a}$$

with respect to S, we obtain the following expression for q,

$$q = q_A - \frac{\partial}{\partial x_1}\int_{S_A}^{S} \sigma_{11} t\, dS \tag{11–104}$$

Equation (11–104) is the starting point for the engineering theory of shear stress distribution. Once the variation of σ_{11} over the cross section is known, we can evaluate q. Now, we have shown that the normal stress varies *linearly* over the cross section when the member is subjected to a constant shear (F_2, F_3

constant) and the end sections can warp freely. Noting that the member is prismatic, the derivative of σ_{11} for this case is

$$\begin{aligned}\frac{\partial \sigma_{11}}{\partial x_1} &= \frac{x_3}{I_2}\frac{dM_2}{dx_1} - \frac{x_2}{I_3}\frac{dM_3}{dx_1} \\ &= \frac{F_3}{I_2}x_3 + \frac{F_2}{I_3}x_2\end{aligned} \tag{a}$$

and (11–104) expands to

$$q = q_A - \frac{F_2}{I_3}\int_{S_A}^{S} x_2 t\, dS - \frac{F_3}{I_2}\int_{S_A}^{S} x_3 t\, dS \tag{b}$$

The integrals represent the moment of the segmental area with respect to X_2, X_3 and are generally denoted by Q_2, Q_3:

$$\begin{aligned}Q_2 &= \int_{S_A}^{S} x_3 t\, dS = Q_2(S, S_A) \\ Q_3 &= \int_{S_A}^{S} x_2 t\, dS = Q_3(S, S_A)\end{aligned} \tag{11–105}$$

With this notation, (b) simplifies to

$$q = q_A - \frac{F_2}{I_3}Q_3 - \frac{F_3}{I_2}Q_2 \tag{11–106}$$

Equation (11–106) defines the shear flow distribution for the case of *negligible* restraint against warping, i.e., for a linear variation in normal stress. Note that q is positive when pointing in the $+S$ direction.

We consider first the open section shown in Fig. 11–17. The end faces are unstressed, i.e.,

$$q_A = q_B = 0 \tag{a}$$

Taking the origin for S at A, (11–106) reduces to

$$\begin{aligned}q &= -\frac{F_2}{I_3}Q_3 - \frac{F_3}{I_2}Q_2 \\ Q_3 &= \int_0^S x_2 t\, dS \qquad Q_2 = \int_0^S x_3 t\, dS\end{aligned} \tag{11–107}$$

We determine Q_2, Q_3 and then combine according to (11–107).

The shearing stress distribution corresponding to F_2,

$$q = -\frac{F_2}{I_3}Q_3 \tag{a}$$

satisfies

$$\begin{aligned}\iint \sigma_{12}\, dA &= F_2 \\ \iint \sigma_{13}\, dA &= 0\end{aligned} \tag{b}$$

identically. To show this, we expand $\vec{q}$,

$$\vec{q} = q\vec{i}_S = (q\alpha_{S2})\vec{i}_2 + (q\alpha_{S3})\vec{i}_3 \tag{c}$$

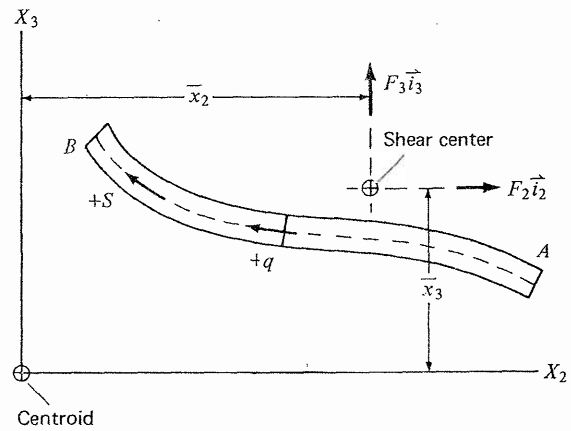


Fig. 11–17. Flexural shear flow—open segment.

and evaluate the shear stress resultants:

$$\begin{aligned}\iint \sigma_{12}\, dA &= \int q\alpha_{S2}\, dS = -\frac{F_2}{I_3}\int \alpha_{S2}Q_3\, dS \\ \iint \sigma_{13}\, dA &= \int q\alpha_{S3}\, dS = -\frac{F_2}{I_3}\int \alpha_{S3}Q_3\, dS\end{aligned} \tag{d}$$

Equation (b) requires

$$\begin{aligned}\int_0^{S_B} \alpha_{S2}Q_3 &= -I_3 \\ \int_0^{S_B} \alpha_{S3}Q_3\, dS &= 0\end{aligned} \tag{e}$$

Now,

$$\alpha_{Sj} = \frac{dx_j}{dS} \tag{f}$$

Integrating (e) by parts and noting that X_2, X_3 are principal centroidal axes,† we obtain

$$\begin{aligned}\int_0^{S_B} \alpha_{S2}Q_3 dS &= [x_2Q_3]_0^{S_B} - \int_0^{S_B} x_2^2 t\, dS = -I_3 \\ \int_0^{S_B} \alpha_{S3}Q_3 dS &= [x_3Q_3]_0^{S_B} - \int_0^{S_B} x_2x_3 t\, dS = 0\end{aligned} \tag{g}$$

The shear stress distribution predicted by (a) is statically equivalent to a force $F_2\vec{i}_2$. To determine the location of its line of action,‡ we evaluate the moment with respect to a convenient moment center. By applying the same argument, one can show that the shear flow corresponding to F_3 is statically

† See Eq. (11–2).
‡ See Prob. 11–12.

equivalent to a force, $F_3\bar{\imath}_3$. The intersection of the lines of action of the two resultants is the *shear center* for the cross section (see Fig. 11–17).

Example 11–3

Consider the thin rectangular section shown. We take $+S$ in the $+X_2$ direction. Then, $+q$ points in the $+X_2$ direction and $q/t = \sigma_{12}$. The various terms are

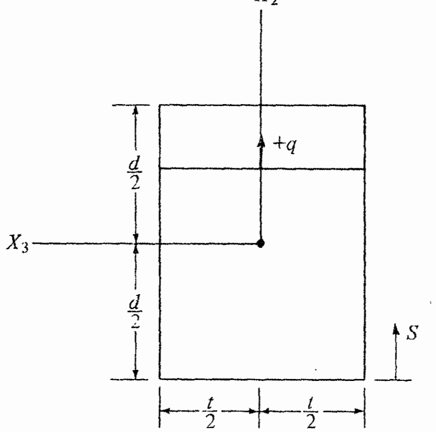
$$Q_3 = \int_{-d/2}^{x_2} t x_2\, dx_2 = -\frac{t}{2}\left(\frac{d^2}{4} - x_2^2\right)$$

$$q = -\frac{F_2}{I_3} Q_3 = \frac{tF_2}{2I_3}\left(\frac{d^2}{4} - x_2^2\right)$$

$$\sigma_{12} = \frac{q}{t} = \frac{F_2}{2I_3}\left(\frac{d^2}{4} - x_2^2\right)$$

This result coincides with the solution for $\sigma_{12,r}$ obtained in Sec. 11–6. Actually, the engineering theory is *exact* for a rigid cross section, i.e., for $\nu_1/E = 0$.

Fig. E11–3



Example 11–4

We determine the distribution of q corresponding to F_2 for the symmetrical section of Fig. E11–4A. Only two segments, AB and BC, have to be considered since $|Q_3|$ is symmetrical.

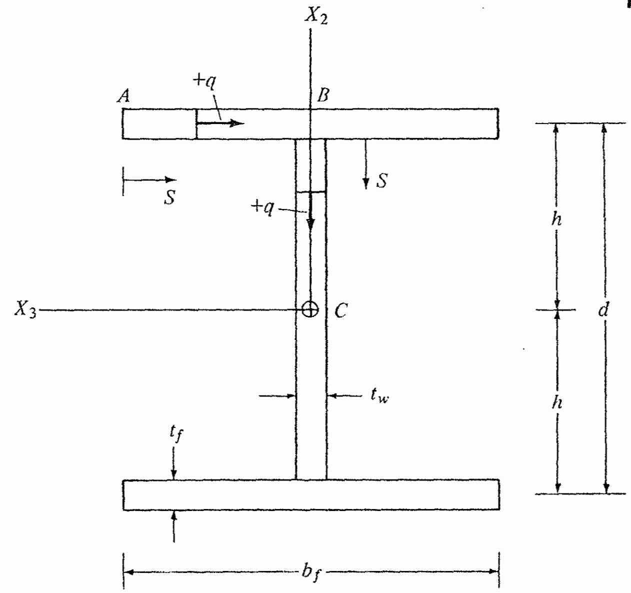
Segment AB

$$Q_3 = ht_f S$$

$$q = -\frac{F_2}{I_3} Q_3 = -\frac{F_2}{I_3}(ht_f S)$$

According to our definition, $+q$ points in the $+S$ direction (from A to B). Since q is negative for this segment, it actually acts in the negative S direction (from B to A).

Fig. E11–4A



Segment BC

We measure S from B to C. Then,

$$Q_3 = hb_f t_f + \tfrac{1}{2}t_w(h^2 - x_2^2)$$

$$q = -\frac{F_2}{I_3}\left[hb_f t_f + \tfrac{1}{2}t_w(h^2 - x_2^2)\right]$$

Note that the actual sense of q is from C to B. The distribution and sense of q are shown in Fig. E11–4B.

It is of interest to evaluate A_2. Specializing (11–96) for a thin-walled section,

$$\iint (\sigma_{1S,r}^2)_{F_2}\, dA \Rightarrow \int (q^2)_{F_2} \frac{dS}{t} = \frac{F_2^2}{A_2} \tag{a}$$

and substituting for q yields

$$\frac{1}{A_2} = \frac{1}{I_3^2}\int Q_3^2 \frac{dS}{t} \tag{b}$$

We let

$$A_w = \text{area of the web} = dt_w$$
$$A_f = \textit{total}\text{ flange area} = 2b_f t_f \tag{c}$$
$$A_2 = kA_w$$

The resulting expression for k is

$$k = \frac{1 + \dfrac{2}{3}\dfrac{A_w}{A_f}\left(1 + \dfrac{1}{6}\dfrac{A_w}{A_f}\right)}{1 + \dfrac{2}{3}\dfrac{A_w}{A_f}\left[1 + \dfrac{1}{5}\dfrac{A_w}{A_f} + \dfrac{1}{2}\left(\dfrac{b_f}{d}\right)^2\right]} \tag{d}$$

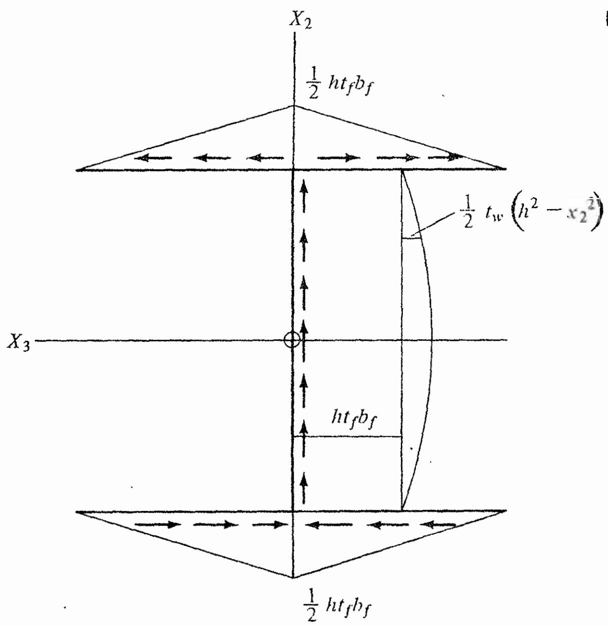


Fig. E11-4B

This factor is quite close to unity. For example, taking as typical, for a wide-flange section,

$$t_f = 2t_w$$
$$b_f = \tfrac{3}{4}d$$

we find

$$A_f = 3A_w$$
$$k = 0.95$$

The shearing stress corresponding to F_3 varies parabolically in the flanges and is zero in the web. Each flange carries half the shear and

$$\frac{1}{A_3} = \frac{6}{5}\frac{1}{A_f} = \frac{3}{5}\frac{1}{bt_f} \tag{e}$$

Example 11-5

Cross-Sectional Properties

This section (Fig. E11-5A) is symmetrical with respect to X_2. The shift in the centroid from the center of the web due to the difference in flange areas is

$$\Delta = \frac{b_2 t_2 - b_1 t_1}{b_1 t_1 + b_2 t_2 + d t_w} \tag{a}$$

We neglect the contribution of the web in I_2 since it involves t_w^3:

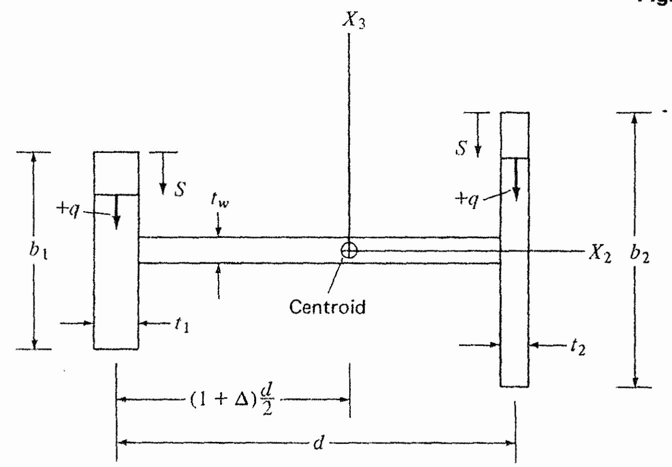
$$I_2 \approx (I_2)_1 + (I_2)_2 = \tfrac{1}{12}(t_1 b_1^3 + t_2 b_2^3) \tag{b}$$

Determination of Q_2

Taking S as shown in the sketch, we have

$$\begin{aligned} Q_2|_1 &= \frac{t_1}{2}\left[\left(\frac{b_1}{2}\right)^2 - x_3^2\right] \\ Q_2|_2 &= \frac{t_2}{2}\left[\left(\frac{b_2}{2}\right)^2 - x_3^2\right] \\ Q_2|_w &= 0 \qquad \text{since } X_2 \text{ is an axis of symmetry.} \end{aligned} \tag{c}$$

Fig. E11-5A



Distribution of q Corresponding to F_3

The shear flow corresponding to F_3 is obtained by applying

$$q = -\frac{F_3}{I_2} Q_2 \tag{d}$$

and is shown in Fig. E11-5B. The shear stress vanishes in the web and varies parabolically in each flange.

Integrating the shear flow over each flange, we obtain

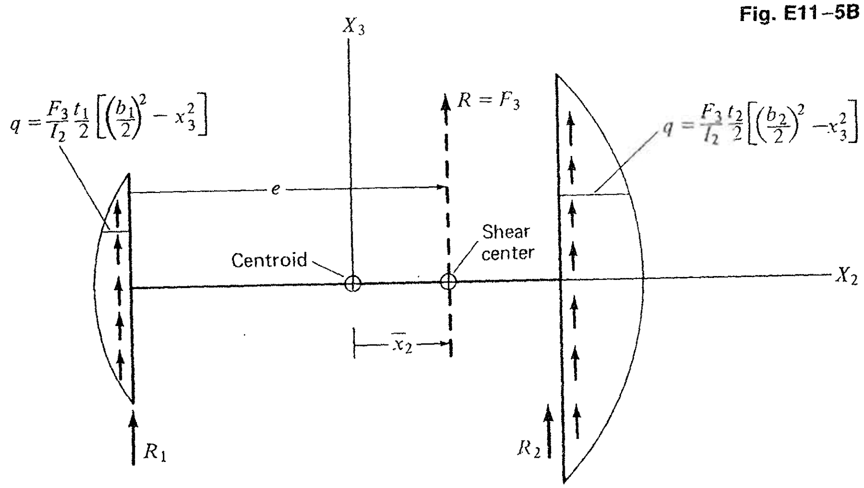
$$R_j = F_3 \frac{(I_2)_j}{I_2} \tag{e}$$

Then, the distribution is statically equivalent to $F_3\bar{\imath}_3$ acting at a distance e from the left flange, where

$$e = \frac{R_2 d}{R} = d\frac{(I_2)_2}{I_2} \tag{f}$$

Since X_2 is an axis of symmetry, the shear center is located at the intersection of R and X_2,

Fig. E11-5B



The coordinates of the shear center with respect to the *centroid* are

$$
\begin{aligned}
\bar{x}_3 &= 0 \\
\bar{x}_2 &= e - (1 + \Delta)\frac{d}{2} \\
&= d\left[\frac{(I_2)_2}{I_2} - \frac{1 + \Delta}{2}\right]
\end{aligned}
\tag{g}
$$

Torsional Shear Stress

The flexural shear stress distribution is statically equivalent to a torsional moment equal to $F_3\bar{x}_2$ with respect to the centroid. We have defined M_1 as the required torsional moment with respect to the centroid. Then, the moment which must be balanced by torsion is $M_1 - F_3\bar{x}_2 = M_T$, the required torsional moment with respect to the *shear center*. Using the approximate theory developed in Sec. 11-3, the maximum torsional shear stress in a segment is

$$\sigma_{\max}|_j = \frac{M_T}{J} t_j \tag{h}$$

where

$$J = \tfrac{1}{3}(b_1 t_1^3 + b_2 t_2^3 + d t_w^3) \tag{i}$$

We consider next the closed cross section shown in Fig. 11-18. We take the origin for S at some *arbitrary* point and apply (11-106) to the segment S_P-S:

$$
q = q_1 - \frac{F_2}{I_3} Q_3 - \frac{F_3}{I_2} Q_2 \tag{11-108}
$$

$$
Q_2 = \int_{S_P}^{S} x_3 t\, dS \qquad Q_3 = \int_{S_P}^{S} x_2 t\, dS
$$

where q_1 is the shear flow at P. The shear flow distribution is statically indeterminate since q_1 in unknown. We have previously shown that $q = q_1 =$ con-

stant is statically equivalent to only a torsional moment equal to $2q_1 A_{cl}\bar{i}_3$. The second and third terms are statically equivalent to $F_2\bar{i}_2$ and $F_3\bar{i}_3$.

The constant q_1 is determined by applying the continuity requirement to the centerline curve. Since the engineering theory corresponds to assuming the cross section is rigid with respect to in-plane deformation, we use (11-100).

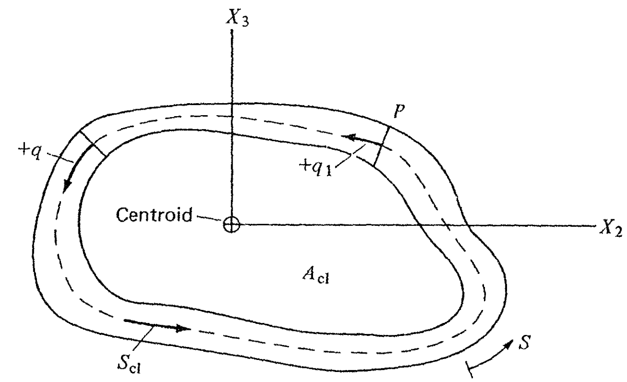


Fig. 11-18. Notation for closed cell.

The flexural shear stress distribution must satisfy

$$\oint_S \sigma_{1S}\, dS = \oint_S q \frac{dS}{t} = 0 \tag{11-109}$$

for an arbitrary closed curve.† Substituting for q,

$$q_1 \oint_{S_{cl}} \frac{dS}{t} = \frac{F_2}{I_3}\oint_{S_{cl}} Q_3 \frac{dS}{t} + \frac{F_3}{I_2}\oint_{S_{cl}} Q_2 \frac{dS}{t} \tag{a}$$

and considering separately the distributions corresponding to F_2 and F_3, we obtain

$$
\begin{gathered}
q = q_{F_2} + q_{F_3} \\
q_{F_2} = \frac{F_2}{I_3}(B_2 - Q_3) \qquad q_{F_3} = \frac{F_3}{I_2}(B_3 - Q_2) \\
B_2 = \frac{\oint_{S_{cl}} Q_3 \frac{dS}{t}}{\oint_{S_{cl}} \frac{dS}{t}} \qquad B_3 = \frac{\oint_{S_{cl}} Q_2 \frac{dS}{t}}{\oint \frac{dS}{t}}
\end{gathered}
\tag{11-110}
$$

Each distribution satisfies (11-109) identically. Also, the distribution $(q)_{F_j}$ is statically equivalent to a force $F_j\bar{i}_j$, located $\bar{x}_k$ units from the centroid. Note that $q = B_j$ leads only to a torsional moment equal to $2B_j A_{cl}$.

† One can interpret (11-109) as requiring the flexural shear stress distribution to lead to no twist deformation. See Prob. 11-14 for the more general expression, which allows for a variable shear modulus.

The general expression for $1/A_j$ follows from (11-96):

$$\sum \int_S (q^2)_{F_j} \frac{dS}{t} = \frac{F_j^2}{A_j} \qquad j = 2, 3 \tag{11-111}$$

Substituting for $(q)_{F_j}$,

$$\frac{1}{A_j} = \frac{1}{I_k^2} \oint_{S_{cl}} (B_j^2 - 2B_j Q_k + Q_k^2) \frac{dS}{t} \qquad (j \neq k; j, k = 2, 3) \tag{b}$$

and noting that

$$B_j \oint_{S_{cl}} \frac{dS}{t} = \oint_{S_{cl}} Q_k \frac{dS}{t} \tag{c}$$

we obtain

$$\frac{1}{A_j} = \frac{1}{I_k^2} \oint_{S_{cl}} Q_k^2 \frac{dS}{t} - B_j \oint_{S_{cl}} Q_k \frac{dS}{t} \qquad (j \neq k; j, k = 2, 3) \tag{11-112}$$

which applies for an arbitrary single cell.

Example 11-6

We illustrate the determination of $(q)_{F_3}$ for the square section of Fig. E11-6A. It is convenient to take P at the midpoint since the centerline is symmetrical.

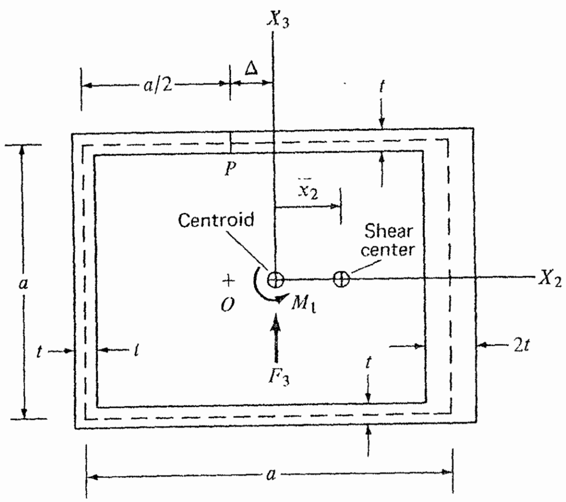


Fig. E11-6A

Cross-Sectional Properties

$$A_{cl} = a^2$$

$$I_2 = 3t\left(\frac{a^3}{12}\right) + 2(at)\frac{a^2}{4} = \frac{3}{4}a^3 t$$

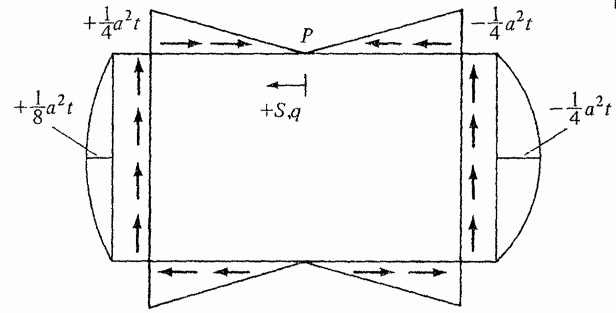
$$\Delta = \frac{(at)(a/2)}{5at} = \frac{a}{10}$$

$$\oint \frac{dS}{t} = 3.5\frac{a}{t}$$

Determination of Q_2

We start at P and work counterclockwise around the centerline. The resulting distribution and actual sense of q due to Q_2 are shown in Fig. E11-6B. Note that $+Q_2$ corresponds to a negative i.e., clockwise, q.

Fig. E11-6B



Evaluation of B_3

By definition,

$$B_3 = \frac{\oint Q_2 \frac{dS}{t}}{\oint \frac{dS}{t}} = \frac{1}{3.5\frac{a}{t}} \oint Q_2 \frac{dS}{t}$$

Using the above results, and noting that the area of a parabola is equal to (2/3) (base) × (height), we obtain

$$\oint Q_2 \frac{dS}{t} = +\frac{a^3}{8}$$

$$B_3 = \frac{a^2 t}{28}$$

Distribution of Flexural Shear Flow for F_3

The shear flow is given by

$$q = \frac{F_3}{I_2}(B_3 - Q_2) = \frac{F_3}{a}\left(\frac{1}{21} - \frac{4}{3}\frac{Q_2}{a^2 t}\right)$$

(+ sense clockwise). The two distributions are plotted in Fig. E11-6C.

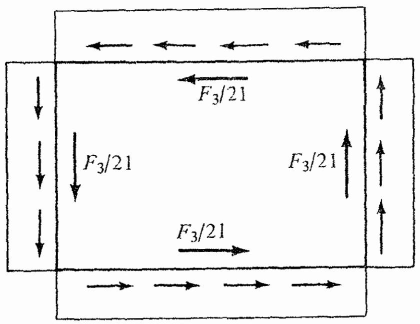
To locate the line of action of the resultant, we sum moments about the midpoint (O in the sketch:

$$(M)_O = \frac{1}{9}F_3\left(\frac{a}{2}\right) + \frac{4F_3}{21}\left(\frac{a}{2}\right) = \frac{19}{126}aF$$

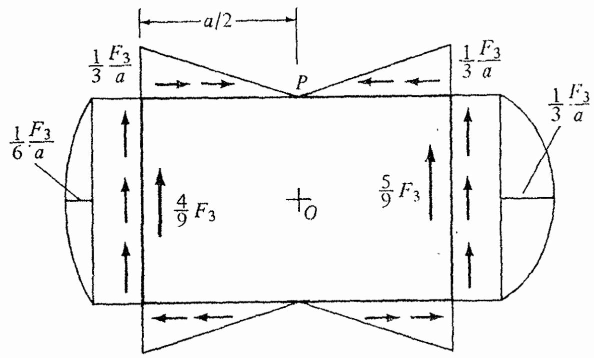
The resultant acts e units to the *right* of O, where

$$e = \frac{19}{126}a$$

Fig. E11–6C



$$q = \frac{F_3}{a}\left(\frac{1}{21}\right)$$



$$q = \frac{F_3}{a}\left(-\frac{4}{3}\frac{Q_2}{a^2 t}\right)$$

Finally, the coordinates of the shear center with respect to the centroid (which is Δ units to the right of O) are

$$\bar{x}_2 = e - \Delta = +\frac{16}{315}a$$

$$\bar{x}_3 = 0$$

Torsional Shear Flow

The shear flow for pure torsion is due to M_T, the torsional moment with respect to the shear center. For this section,

$$M_T = M_1 - \bar{x}_2 F_3 + \bar{x}_3 F_2 = M_1 - \frac{16}{315}aF_3$$

We apply the theory developed in Sec. 11–4. One just has to replace M_1 with M_T in

Equation (11–61):

$$\sigma_{1S}\big|_{\max} = \frac{M_T}{J}\left(t + \frac{C_1}{t}\right)$$

$$C_1 = \tfrac{4}{7}at$$

$$J = \tfrac{8}{7}a^3 t + \tfrac{11}{3}at^3$$

Determination of $1/A_3$

Applying (11–112), we find

$$\frac{1}{A_3} = \frac{1}{I_2^2}\left(\oint Q_2^2 \frac{dS}{t} - B_3 \oint Q_2 \frac{dS}{t}\right) = \frac{1.276}{3at}$$

Note that $3at$ is the total web area.

We consider next the analysis of a two-cell section and include open segments for generality. There is one redundant shear flow for each cell. We select a convenient point in each cell and take the shear flow at the point as the redundant for the cell. This is illustrated in Fig. 11–19: q_j represents the shear flow redundant for cell j and the $+S$ sense coincides with the X_2-X_3 sense to be consistent with the pure-torsion analysis. The $+S$ sense for the open segments is inward from the free edge. For convenience, we drop the CL (centerline) subscript on S and A.

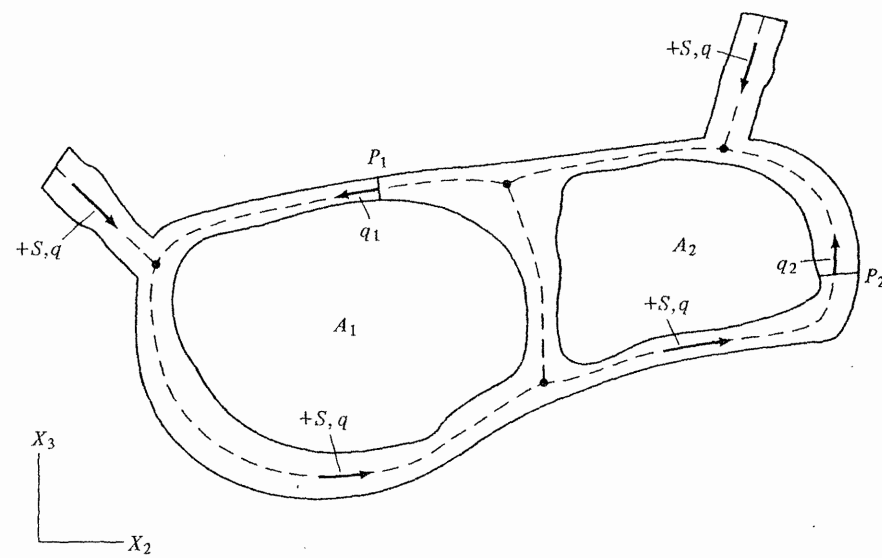


Fig. 11–19. Notation for multi-cell section.

The total shear flow is the sum of q_0, the *open cross-section* distribution ($q_1 = q_2 = 0$), and q_R, the distribution due to the redundants:

$$q = q_0 + q_R \tag{11-113}$$

We determine q_0 by applying (11–107) to the various segments. The redundant shear-flow distribution is the same as for pure torsion (see Fig. 11–11). Finally, we obtain a system of equations relating q_1, q_2 to F_2, F_3 by applying the continuity requirement to each centerline,†

$$\oint_{S_j} q\frac{dS}{t} = 0 \qquad j = 1, 2 \tag{11-114}$$

where q is positive if it points in the $+S$ direction. Using the a_{jk} notation defined by (11–68), the equations take the following form:

$$\begin{aligned} a_{11}q_1 + a_{12}q_2 &= D_1 \\ a_{12}q_1 + a_{22}q_2 &= D_2 \\ D_j = -\oint_{S_j} q_0\frac{dS}{t} &= D_j(F_2, F_3) \end{aligned} \tag{11-115}$$

The shear flows $(q_{1,t}, q_{2,t})$ for *pure* torsion are related by (we multiply (11–71) by M_T/J and note (11–62))

$$\begin{aligned} a_{11}q_{1,t} + a_{12}q_{2,t} &= 2A_1\frac{M_T}{J} \\ a_{12}q_{1,t} + a_{22}q_{2,t} &= 2A_2\frac{M_T}{J} \end{aligned} \tag{11-116}$$

Thus, the complete shear stress analysis involves solving $\mathbf{aq} = \mathbf{b}$ for three different right-hand sides. The equations developed above can be readily generalized.

Example 11–7

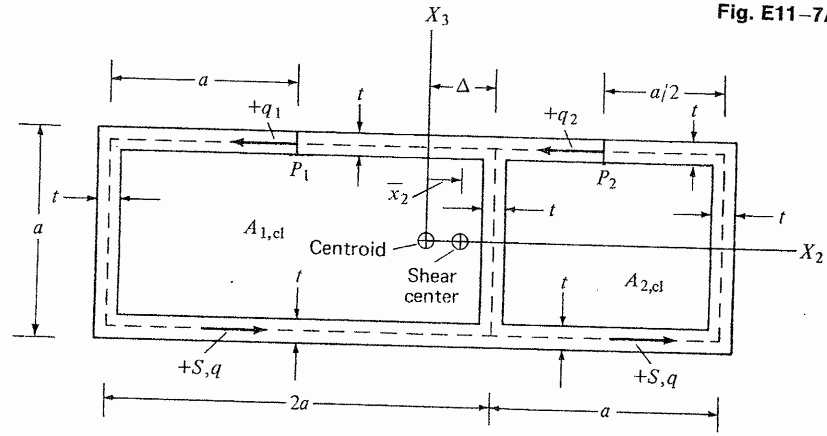
We determine the flexural shear stress distribution corresponding to F_3 for the section shown in Fig. E11–7A. We locate P_1 and P_2 at the midpoints to take advantage of symmetry.

Cross-Sectional Properties

$$A_1 = 2a^2$$
$$A_2 = a^2$$
$$\Delta = \tfrac{4}{9}a$$
$$I_2 = 3\left(\frac{a^3t}{12}\right) + 2\left[(3at)\frac{a^2}{4}\right] = \frac{7}{4}a^3t$$
$$a_{11} = \frac{6a}{t} \qquad a_{22} = \frac{4a}{t} \qquad a_{12} = -\frac{a}{t}$$

† See Prob. 11–14 for the more general expression, which allows for a variable shear modulus.

Fig. E11–7A



Distribution of q_R

This system (Fig. E11–7B) is statically equivalent to a moment

$$2a^2(2q_1 + q_2)\bar{\imath}_3$$

Distribution of q_0 Due to F_3

We apply

$$q = -\frac{F_3}{I_2}Q_2$$

to the various segments starting at points P_1, P_2. The resulting distribution is shown in Fig. E11–7C.

Determination of q_1, q_2

$$D_1 = -\oint_{S_1} q_0\frac{dS}{t} = -\frac{1}{7}\frac{F_3}{t}$$
$$D_2 = -\oint_{S_2} q_0\frac{dS}{t} = +\frac{2}{7}\frac{F_3}{t}$$

The equations for q_1 and q_2 are

$$\begin{aligned} 6q_1 - q_2 &= -\frac{1}{7}\frac{F_3}{a} \\ -q_1 + 4q_2 &= \frac{2}{7}\frac{F_3}{a} \end{aligned} \tag{a}$$

Solving (a), we find

$$\begin{aligned} q_1 &= -\frac{2}{161}\frac{F_3}{a} \\ q_2 &= +\frac{11}{161}\frac{F_3}{a} \end{aligned} \tag{b}$$

The total distribution is obtained by adding q_R and q_0 algebraically.

Fig. E11-7B

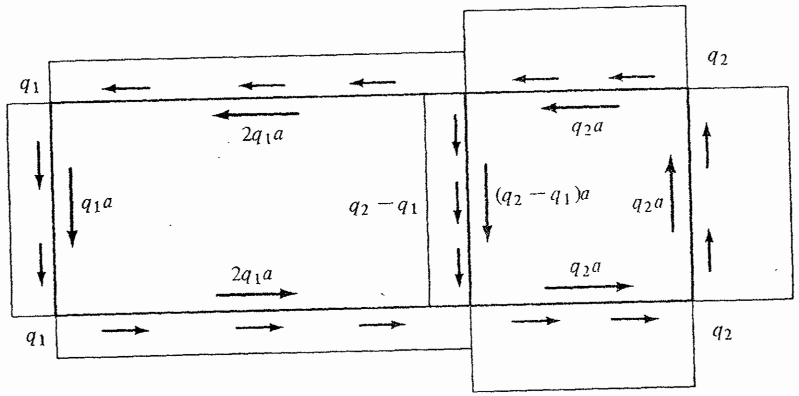
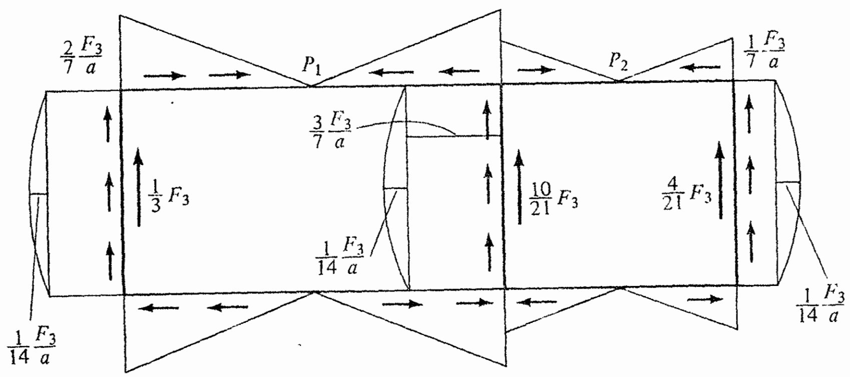


Fig. E11-7C



Location of the Shear Center

Taking moments about the midpoint of the left web, and letting e be the distance to the line of action of the resultant, we obtain

$$M(+\circlearrowleft) = 2a^2(2q_1 + q_2) + (2a)\left(\frac{10}{21}F_3\right) + (3a)\left(\frac{4}{21}F_3\right) = eF_3$$

$$e = \left(\frac{2}{23} + \frac{32}{21}\right)a = 1.61a$$

The shear center is located on the X_2 axis and

$$\bar{x}_2 = e - (2a - \Delta) = +0.055a$$

REFERENCES

1. Wang, C. T.: *Applied Elasticity*, McGraw-Hill, New York, 1953.
2. Timoshenko, S. J., and J. N. Goodier: *Theory of Elasticity*, 3d ed., McGraw-Hill, New York, 1970.

3. Den Hartog, J. P.: Advanced Strength of Materials, McGraw-Hill, New York, 1952.
4. Herrmann, L. R.: "Elastic Torsional Analysis of Irregular Shapes," *J. Eng. Mech. Div.*, A.S.C.E., December 1965.
5. Sokolnikoff, I.S.: *Mathematical Theory of Elasticity*, McGraw-Hill, New York, 1956.
6. Timoshenko, S. J.: *Strength of Materials*, Part 2, Van Nostrand, New York, 1941.
7. Cernica, J. N.: Strength of Materials, Holt, Rinehart, New York, 1966.
8. Dabrowski, R.: *Gekrümmte dünnwandige Träger* ("Curved Thin-walled Girders"), Springer-Verlag, Berlin, 1968.
9. Kollbrunner, C. F., and K. Basler: *Torsion in Structures*, Springer-Verlag, Berlin, 1969.
10. Vlasov, V. Z.: *Thin-Walled Elastic Beams*, Israel Program for Scientific Translations, Jerusalem, 1961.
11. Oden, J. T.: *Mechanics of Elastic Structures*, McGraw-Hill, New York, 1967.

PROBLEMS

11–1. The pure-torsion formulation presented in Sec. 11–2 considers the cross section to rotate about the centroid, i.e., it takes

$$\begin{aligned} u_2 &= -\omega_1 x_3 \qquad & \omega_1 &= k_1 x_1 \\ u_3 &= +\omega_1 x_2 \qquad & u_1 &= k_1 \phi_t \end{aligned} \tag{a}$$

Suppose we consider the cross-section to rotate about an arbitrary point (x_2^*, x_3^*). The general form of (a) is

$$\begin{aligned} u_2 &= -\omega_1(x_3 - x_3^*) \qquad & \omega_1 &= k_1 x_1 + c_1 \\ u_3 &= +\omega_1(x_2 - x_2^*) \qquad & u_1 &= k_1 \phi_t^* \end{aligned} \tag{b}$$

(a) Starting with Equation (b), derive the expressions for σ_{12}, σ_{13} and the governing equations for ϕ_t^*.
(b) What form do the equations take if we write

$$\phi_t^* = \phi_t + c_2 - x_2 x_3^* + x_3 x_2^*$$

Do the torsional shearing stress distribution and torsional constant J depend on the center of twist?

11–2. Show that J can be expressed as

$$\begin{aligned} J &= \iint [x_2^2 + x_3^2 - (\phi_{t,2})^2 - (\phi_{t,3})^2]\, dA \\ &= I_p - \iint [(\phi_{t,2})^2 + (\phi_{t,3})^2]\, dA \end{aligned}$$

Hint:

$$\iint \phi_t \nabla^2 \phi_t \, dA = 0$$

Compare this result with the solution for a circular cross section and comment on the relative efficiency of circular vs. noncircular cross section for torsion.

11–3. Derive the governing differential equation and boundary condition for ϕ_t for the case where the material is orthotropic and the material symmetry axes coincide with the X_1, X_2, X_3 directions.

11–4. The variation in the warping function ϕ_t along an arbitrary curve S is obtained by integrating (11–29),

$$\gamma_{1S} = k_1\left(\frac{\partial \phi_t}{\partial S} + \rho_{ct}\right) \tag{a}$$

where ρ_{ct} is the perpendicular distance from the center of twist to the tangent. One selects a positive sense for S. The sign of ρ is positive when a rotation about the center of twist results in translation in the $+S$ direction. We express γ_{1S} as

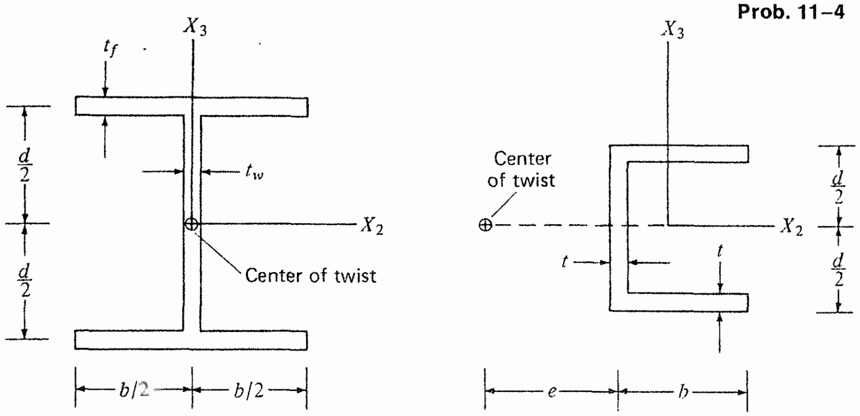
$$\gamma_{1S} = \frac{1}{G}\sigma_{1S} = \frac{1}{G}\frac{M_1}{J}\bar{\sigma}_{1S} \tag{b}$$

and (a) reduces to

$$\frac{\partial \phi_t}{\partial S} = -\rho_{ct} + \bar{\sigma}_{1S} \tag{c}$$

Determine the variation of ϕ_t along the centerline for the two thin-walled open sections shown.

Prob. 11–4



(a) (b)

11–5. Verify that the distribution, q = const, satisfies

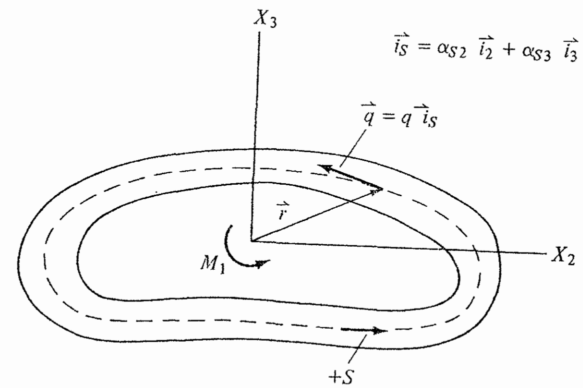
$$F_2 = \iint \sigma_{12}\, dA = \oint q\alpha_{S2}\, dS = 0$$
$$F_3 = \iint \sigma_{13}\, dA = \oint q\alpha_{S3}\, dS = 0$$
$$M_1 = \oint (\vec{r} \times \vec{q} \cdot \vec{i}_1) dS = 2qA_{cl}$$

for the closed cross section sketched.

11–6. Refer to Prob. 11–4. To apply Equation (c) to the centerline of a closed cell, we note that (see (11–50))

$$\bar{\sigma}_{1S} = \frac{J}{M_1}\sigma_{1S}\Big|_{cl} = \frac{J}{M_1}\frac{q}{t} = \frac{C}{t} \tag{a}$$

Prob. 11–5

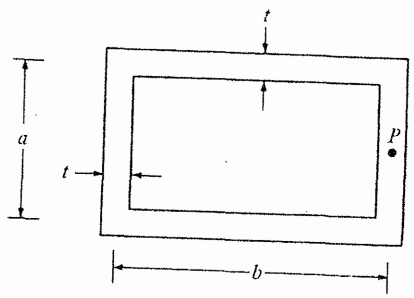


Then,

$$\frac{\partial \phi_t}{\partial S} = -\rho_{ct} + \frac{C}{t} \tag{b}$$

Integrating (b) leads to the distribution of ϕ_t. Apply (b) to the section shown. Take $\phi_t = 0$ at point P. Discuss the case where $a = b$.

Prob. 11–6



11–7. Determine the torsional shear stress distribution and torsional constant J for the section shown. Specialize for $t \ll a$.

11–8. Determine the equations for C_j $(j = 1, 2, 3)$ and J for the section shown. Generalize for a section consisting of "n" cells.

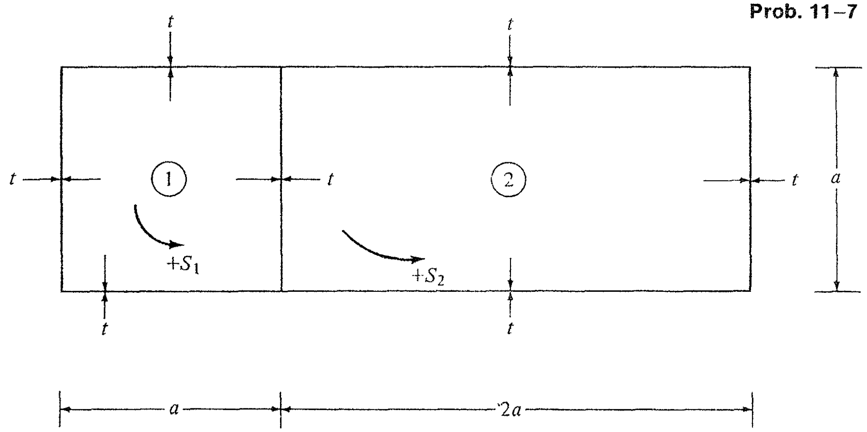
11–9. Determine the distribution of torsional shear stress, the torsional constant J, and the distribution of the warping function for the section shown. Take $\phi_t = 0$ on the symmetry axis and use the results presented in Prob. 11–6.

11–10. Verify Equation (11–82). Utilize (11–15).

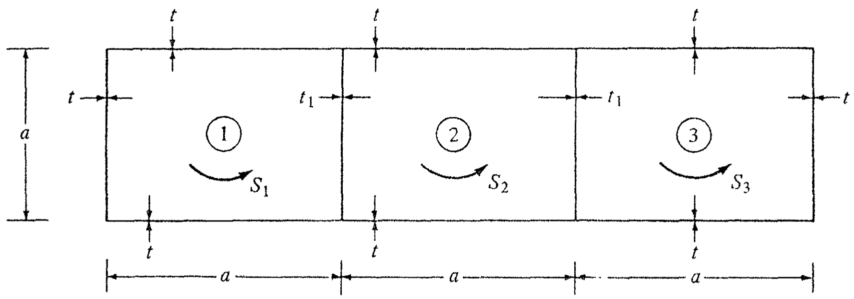
11–11. The flexural warping function $\bar{\phi}_{jr}$ satisfy

$$\nabla^2 \bar{\phi}_{jr} = -x_j \quad \text{in } A$$
$$\frac{\partial \bar{\phi}_{jr}}{\partial n} = 0 \quad \text{on } S$$

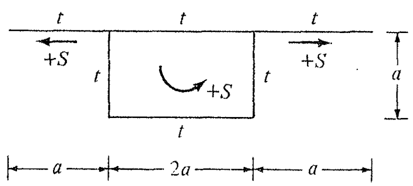
Prob. 11–7



Prob. 11–8



Prob. 11–9



Utilizing the following integration formula,

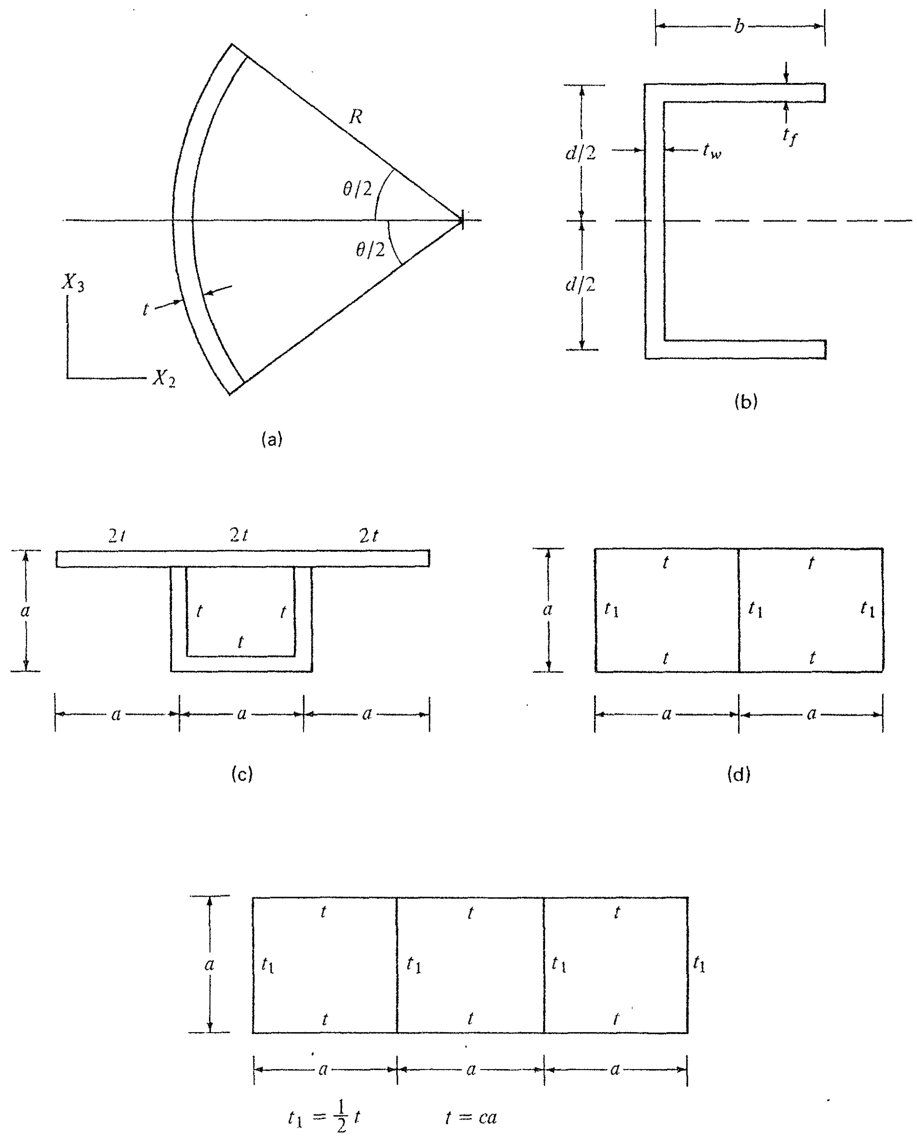
$$\iint (f_{1,2}f_{2,2} + f_{1,3}f_{2,3})dx_2\,dx_3 = \oint f_1 \frac{\partial f_2}{\partial n}\,dS - \iint f_1 \nabla^2 f_2\,dx_2\,dx_3$$

where f_1, f_2 are arbitrary functions, verify Equation (11–96).

11–12. Refer to Fig. 11–17. Starting with (11–107), derive the expressions for the coordinates of the shear center in terms of the cross-sectional parameters.

11–13. Determine the flexural shear flow distributions due to F_2, F_3 and locate the shear center for the five thin-walled sections shown.

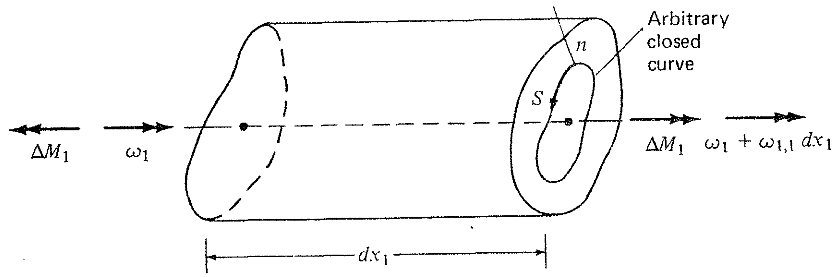
Prob. 11–13



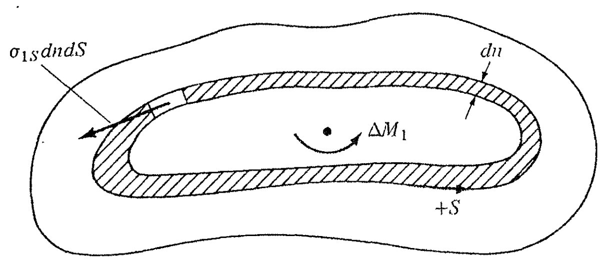
(a) (b) (c) (d) (e)

11–14. We established the expression for the twist deformation (Equation (11–31) by requiring the torsional warping function to be continuous. One can also obtain this result by applying the principle of virtual forces to the segment shown as part of the accompanying figure.

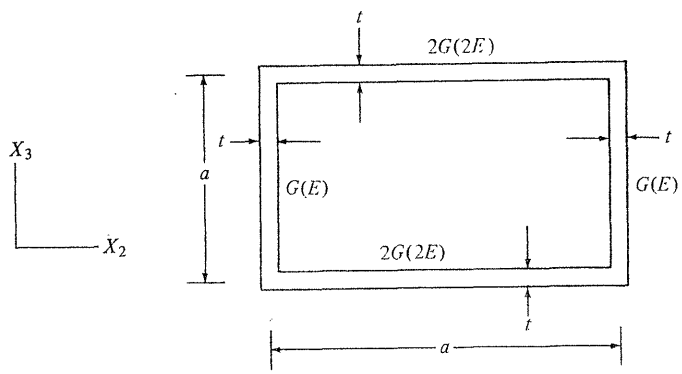
Prob. 11–14



(a)



(b)



(c)

The general principle states that

$$\left(\iint \boldsymbol{\varepsilon}^T \,\Delta\boldsymbol{\sigma}\, dA\right)dx_1 = \left(\iint \Delta\mathbf{b}^T\mathbf{u}\, dA\right)dx_1 + \left|\iint \Delta\mathbf{p}^T\,\Delta\mathbf{u}\, dA\right|_{x_1,\, x+dx} \tag{a}$$

for a statically permissible force system. Now, we select a force system acting on the end faces which is statically equivalent to only a torsional moment M_1. If we consider the cross section to be *rigid*, the right-hand side of (a) reduces to $\Delta M_1 \omega_{1,1}\, dx_1$, and we can write

$$\omega_{1,1} = k_1 = \frac{1}{\Delta M_1}\iint \boldsymbol{\varepsilon}^T\,\Delta\boldsymbol{\sigma}\, dA \tag{b}$$

Next, we select an arbitrary closed curve, S (part b of figure), and consider the region defined by S and the differential thickness dn. We specialize the virtual-stress system such that $\Delta\boldsymbol{\sigma} = \mathbf{0}$ outside this domain and only $\Delta\sigma_{1S}$ is finite inside the domain. Finally, using (11–51), we can write

$$dn(\Delta\sigma_{1S}) = \frac{\Delta M_1}{2A_S} \tag{c}$$

and Equation (b) reduces to

$$k_1 = \frac{1}{2A_S}\oint_S \gamma_{1S}\, dS \tag{d}$$

The derivations presented in the text are based on a constant shear modulus G throughout the section, so we replace (d) with

$$Gk_1 = \frac{1}{2A_S}\oint \sigma_{1S}\, dS \tag{e}$$

If G is a variable, say $G = fG^*$ (where $f = f(x_2, x_3)$), we have to work with

$$G^*k_1 = \frac{1}{2A_S}\oint\left(\frac{1}{f}\right)\sigma_{1S}\, dS \tag{f}$$

Also, we define the torsional constant J according to

$$G^*k_1 J \equiv M_1 \tag{g}$$

Consider a thin-walled section comprising discrete elements having different material properties. Develop the expressions for the torsional and flexural shear flow distributions accounting for variable G and E. Determine the normal stress distribution from the stress-strain relation. Assume a *linear* variation in extensional strain and evaluate the coefficients of the strain expansion from the definition equations for F_1, M_2, and M_3. Apply your formulation to the section shown in part c of the figure.