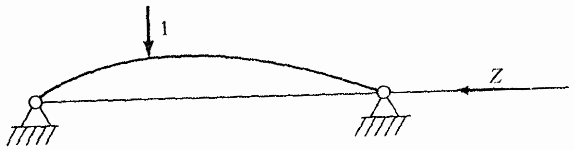
14–17. Consider the arbitrary two-hinged arch shown. Discuss how you

Prob. 14–17



would generate the influence line for the horizontal reaction. Utilize the results contained in Examples 14–10 and 14–11.

15

Engineering Theory of an Arbitrary Member

15–1. INTRODUCTION; GEOMETRICAL RELATIONS

In the first part of this chapter, we establish the governing equations for a member whose centroidal axis is an arbitrary space curve. The formulation is restricted to linear geometry and negligible warping and is referred to as the *engineering theory*. Examples illustrating the application of the displacement and force methods are presented. Next, we outline a restrained warping formulation and apply it to a planar circular member. Lastly, we cast the force method for the engineering theory in matrix form and develop the member force-displacement relations which are required for the analysis of a system of member elements.

The geometrical relations for a member are derived in Chapter 4. For convenience, we summarize the differentiation formulas here. Figure 15–1 shows the *natural* and *local* frames. They are related by

$$\begin{aligned} \bar{t}_1 &= \bar{t} \\ \bar{t}_2 &= \cos\phi\bar{n} + \sin\phi\bar{b} \\ \bar{t}_3 &= -\sin\phi\bar{n} + \cos\phi\bar{b} \end{aligned} \qquad \text{(a)}$$

where $\phi = \phi(s)$. Differentiating (a) and using the Frenet equations (4–20), we obtain

$$\frac{d\mathbf{t}}{dS} = \mathbf{at}$$

$$\begin{Bmatrix} \dfrac{d\bar{t}_1}{dS} \\ \dfrac{d\bar{t}_2}{dS} \\ \dfrac{d\bar{t}_3}{dS} \end{Bmatrix} = \begin{bmatrix} 0 & K\cos\phi & -K\sin\phi \\ -K\cos\phi & 0 & \tau + \dfrac{d\phi}{dS} \\ K\sin\phi & -\left(\tau + \dfrac{d\phi}{dS}\right) & 0 \end{bmatrix} \begin{Bmatrix} \bar{t}_1 \\ \bar{t}_2 \\ \bar{t}_3 \end{Bmatrix} \qquad (15\text{–}1)$$

Note that **a** is skew-symmetric.

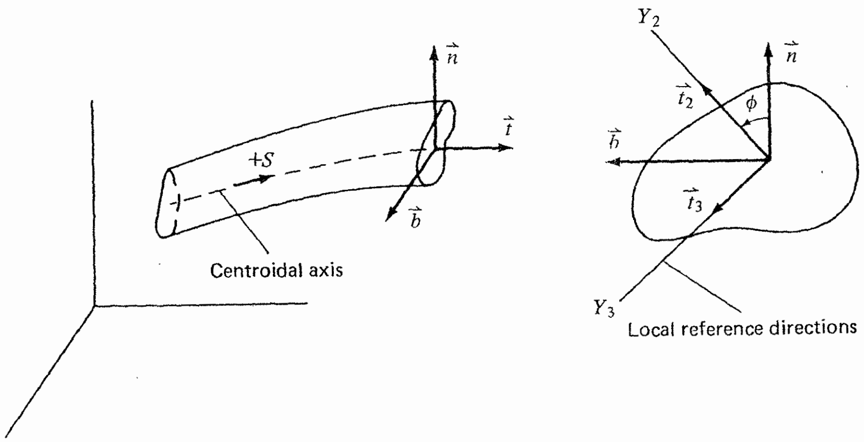


Fig. 15–1. Natural and local reference frames for a member element.

The curvilinear coordinates† of a point, say Q, are taken as S and y_2, y_3. Letting $\vec{R}$ be the position vector to Q (see Fig. 15–2),

$$\vec{R} = \vec{r}(S) + y_2\vec{t}_2(S) + y_3\vec{t}_3(S) \tag{15–2}$$

and differentiating, we find

$$\begin{aligned} \frac{\partial \vec{R}}{\partial S} &= (1 - y_2 a_{12} - y_3 a_{13})\vec{t}_1 - y_3 a_{23}\vec{t}_2 + y_2 a_{23}\vec{t}_3 \\ \frac{\partial \vec{R}}{\partial y_2} &= \vec{t}_2 \\ \frac{\partial \vec{R}}{\partial y_3} &= \vec{t}_3 \end{aligned} \tag{15–3}$$

The differential volume at Q is

$$\begin{aligned} d(\text{vol.}) &= (1 - y_2 a_{12} - y_3 a_{13})dS\, dy_2\, dy_3 \\ &= \left(1 - \frac{y_n}{R_c}\right) dS\, dy_2\, dy_3 \end{aligned} \tag{15–4}$$

where y_n is the coordinate of Q in the normal ($\vec{n}$) direction and $R_c = 1/K$ is the radius of curvature. Also,

$$\begin{aligned} \frac{\partial \vec{R}}{\partial S} \cdot \vec{t}_2 &= -y_3 a_{23} = -y_3\left(\frac{1}{R_t} + \frac{d\phi}{dS}\right) \\ \frac{\partial \vec{R}}{\partial S} \cdot \vec{t}_3 &= y_2 a_{23} = y_2\left(\frac{1}{R_t} + \frac{d\phi}{dS}\right) \end{aligned} \tag{15–5}$$

† See Sec. 4–8.

and the local vectors at Q are orthogonal when $a_{23} = 0$, which requires

$$\begin{gathered} a_{23} = 0 \\ \Downarrow \\ \frac{d\phi}{dS} = -\tau = -\frac{1}{R_t} \end{gathered} \tag{15–6}$$

It is reasonable to neglect y/R terms with respect to unity when the member is *thin*, i.e., when the cross-sectional dimensions are small in comparison to

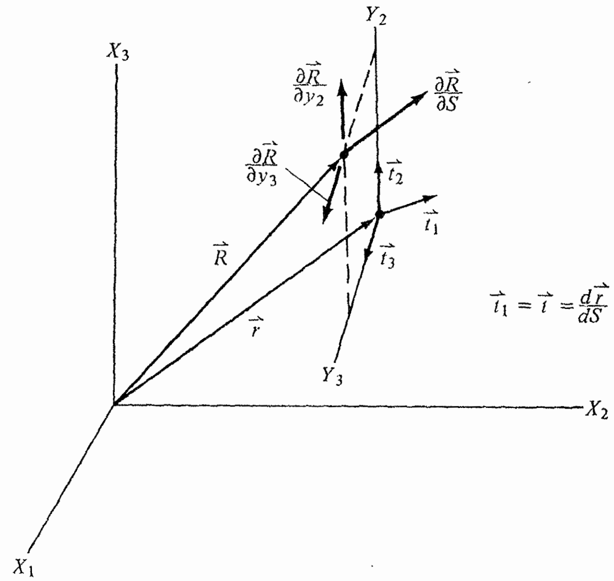


Fig. 15–2. Curvilinear directions.

R_c and R_t. We express $d\phi/dS$ as

$$\frac{d\phi}{dS} = 0\left(\frac{\Delta\phi}{L}\right) \tag{15–7}$$

where L is the total arc length and $\Delta\phi$ is the total increment in ϕ. The non-orthogonality due to ϕ can be neglected when the member is only slightly twisted, i.e., when

$$\frac{b\,\Delta\phi}{L} \ll 1 \tag{15–8}$$

where b is a typical cross-sectional dimension. In what follows, we will assume the member is *thin*, (15–8) is satisfied, and ϕ defines the orientation of the *principal inertia directions.*

Example 15–1

The curvature and torsion for a circular helix are derived in Example 4–5:

$$K = \frac{1}{R_c} = \frac{1}{R} \frac{1}{1 + \left(\frac{H}{2\pi R}\right)^2}$$

$$\tau = \frac{1}{R_t} = \left(\frac{H}{2\pi R}\right) \frac{1}{R_c}$$

where R is the radius of the base circle and H is the rise in one full revolution. The helix is *thin* when $b/R \ll 1$, where b is a typical cross-sectional dimension.

Example 15–2

By definition, a member is planar if $\tau = 0$ and the normal direction ($\vec{n}$) is an axis of symmetry for the cross section. We take the centroidal axis to be in the X_1-X_2 plane and define the sense of $\vec{t}_2$ according to $\vec{t}_2 \times \vec{t}_3 = \vec{t}_3$. The angle ϕ is constant and equal to either 0° ($\vec{t}_2 = \vec{n}$) or 180° ($\vec{t}_2 = -\vec{n}$). Only a_{12} is finite for a planar member:

$$a_{13} = a_{23} = 0 \qquad a_{12} \equiv \frac{1}{R} = \pm K$$

Example 15–3

Consider the case where the centroidal axis is straight and ϕ varies *linearly* with S. The member is said to be *naturally twisted.* Only a_{23} is finite for this case:

$$a_{12} = a_{13} = 0$$

$$a_{23} = \frac{d\phi}{dS} = \text{const} = k$$

If $bk \ll 1$, we can assume $\partial\vec{R}/\partial S$ is orthogonal to $\vec{t}_2, \vec{t}_3$.

15–2. FORCE-EQUILIBRIUM EQUATIONS

To establish the force-equilibrium equations, we consider the differential element shown in Fig. 15–3. We use the same notation as for the planar case. The vector equilibrium equations follow from the requirement that the resultant force and moment vectors must vanish:

$$\frac{d\vec{F}_+}{dS} + \vec{b} = \vec{0}$$

$$\frac{d\vec{M}_+}{dS} + \vec{m} + \vec{t}_1 \times \vec{F}_+ = \vec{0} \qquad (15\text{–}9)$$

We express the force and moment vectors in terms of components referred to the local frame,

$$\vec{F}_+ = \sum F_j \vec{t}_j = \mathbf{F}^T\mathbf{t}$$
$$\vec{M}_+ = \mathbf{M}^T\mathbf{t}$$
$$\vec{b} = \mathbf{b}^T\mathbf{t} \qquad (15\text{–}10)$$
$$\vec{m} = \mathbf{m}^T\mathbf{t}$$

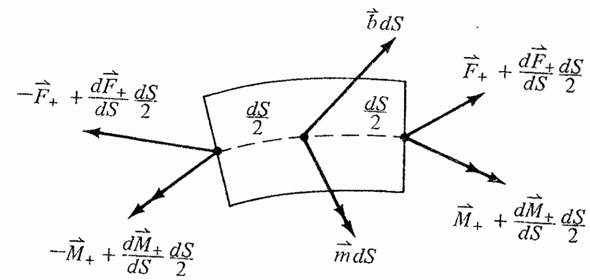


Fig. 15–3. Differential element for equilibrium analysis.

where $\mathbf{F} = \{F_1, F_2, F_3\}$ etc. The vector derivatives are

$$\frac{d\vec{F}_+}{dS} = \frac{d\mathbf{F}^T}{dS}\mathbf{t} + \mathbf{F}^T\mathbf{a}\mathbf{t}$$

$$\frac{d\vec{M}_+}{dS} = \frac{d\mathbf{M}^T}{dS}\mathbf{t} + \mathbf{M}^T\mathbf{a}\mathbf{t} \qquad (a)$$

Also,

$$\vec{t}_1 \times \vec{F}_+ = F_2\vec{t}_3 - F_3\vec{t}_2 = \{0, -F_3, F_2\}^T\mathbf{t} \qquad (b)$$

Substituting in (15–9), and noting that $\mathbf{a}^T = -\mathbf{a}$, lead to the following equilibrium equations:

$$\frac{d\mathbf{F}}{dS} - \mathbf{aF} + \mathbf{b} = \mathbf{0}$$

$$\frac{d\mathbf{M}}{dS} - \mathbf{aM} + \mathbf{m} + \begin{Bmatrix} 0 \\ -F_3 \\ +F_2 \end{Bmatrix} = \mathbf{0}$$

$$\Downarrow$$

$$\frac{dF_1}{dS} - a_{12}F_2 - a_{13}F_3 + b_1 = 0$$
$$\frac{dF_2}{dS} + a_{12}F_1 - a_{23}F_3 + b_2 = 0$$
$$\frac{dF_3}{dS} + a_{13}F_1 + a_{23}F_2 + b_3 = 0 \qquad (15\text{–}11)$$
$$\frac{dM_1}{dS} - a_{12}M_2 - a_{13}M_3 + m_1 = 0$$
$$\frac{dM_2}{dS} + a_{12}M_1 - a_{23}M_3 + m_2 - F_3 = 0$$
$$\frac{dM_3}{dS} + a_{13}M_1 + a_{23}M_2 + m_3 + F_2 = 0$$

When the member is planar, $a_{13} = a_{23} = 0$ and the equations uncouple naturally into two systems, one associated with *in-plane* loading (b_1, b_2, m_3,

F_1, F_2, M_3) and the other with *out-of-plane* loading ($b_3, m_1, m_2, F_3, M_1, M_2$). The in-plane equations coincide with (14–14) when we set $a_{12} = 1/R$ and the out-of-plane equations take the form

$$
\begin{aligned}
\frac{dF_3}{dS} + b_3 &= 0 \\
\frac{dM_1}{dS} - \frac{1}{R}M_2 + m_1 &= 0 \\
\frac{dM_2}{dS} + \frac{1}{R}M_1 + m_2 - F_3 &= 0
\end{aligned}
\tag{15–12}
$$

15–3. FORCE-DISPLACEMENT RELATIONS—NEGLIGIBLE WARPING RESTRAINT; PRINCIPLE OF VIRTUAL FORCES

We consider the material to be elastic and define $\bar{V}^*$ as the complementary energy per unit arc length. Since we are neglecting warping restraint, $\bar{V}^*$ is a function only of **F** and **M**. We let

$$
\begin{aligned}
\mathbf{e} &= \{e_i\} = \left\{\frac{\partial \bar{V}^*}{\partial F_i}\right\} \\
\mathbf{k} &= \{k_i\} = \left\{\frac{\partial \bar{V}^*}{\partial M_i}\right\}
\end{aligned}
\qquad i = 1, 2, 3
\tag{15–13}
$$

and write the one-dimensional principle of virtual forces as

$$
\int_S d\bar{V}^*\, dS = \int_S (\mathbf{e}^T \Delta\mathbf{F} + \mathbf{k}^T \Delta\mathbf{M})dS = \sum d_i\, \Delta P_i
\tag{15–14}
$$

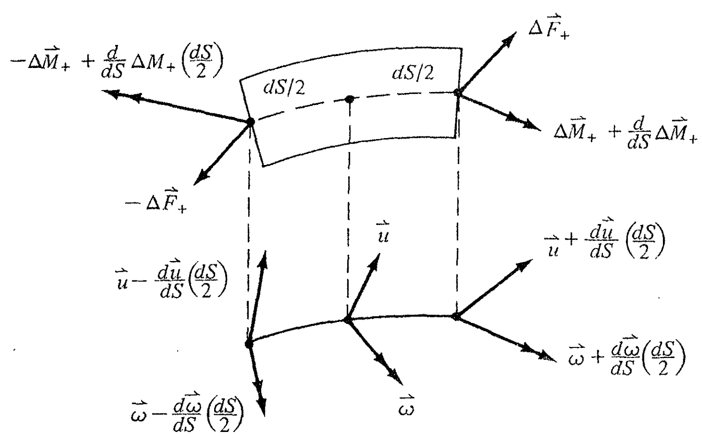


Fig. 15–4. Virtual force system.

Now, we apply the principle of virtual forces to the element shown in Fig. 15–4. We define $\bar{u}$ and $\bar{\omega}$ as

$$
\begin{aligned}
\bar{u} &= \sum u_j \bar{t}_j = \mathbf{u}^T\mathbf{t} = \text{equivalent rigid body translation vector at the centroid} \\
\bar{\omega} &= \sum \omega_j \bar{t}_j = \boldsymbol{\omega}^T\mathbf{t} = \text{equivalent rigid body rotation vector}
\end{aligned}
\tag{15–15}
$$

The virtual system satisfies the equilibrium equations (15–5) identically and therefore is statically permissible. Evaluating $\sum d_i\, \Delta P_i$,

$$
\begin{aligned}
\sum d_i\, \Delta P_i &= \left[\Delta\bar{F}_+ \cdot \left(\frac{d\bar{u}}{dS} + \bar{t}_1 \times \bar{\omega}\right) + \Delta\bar{M}_+ \cdot \frac{d\bar{\omega}}{dS}\right] dS \\
&= \Delta\mathbf{F}^T\left(\frac{d\mathbf{u}}{dS} - \mathbf{a}\mathbf{u} + \begin{Bmatrix} 0 \\ -\omega_3 \\ +\omega_2 \end{Bmatrix}\right) dS + \Delta\mathbf{M}^T\left(\frac{d\boldsymbol{\omega}}{dS} - \mathbf{a}\boldsymbol{\omega}\right) dS
\end{aligned}
\tag{a}
$$

and substituting in (15–14) lead to the following force-displacement relations:

$$
\begin{aligned}
\mathbf{e} &= \frac{d\mathbf{u}}{dS} - \mathbf{a}\mathbf{u} + \begin{Bmatrix} 0 \\ -\omega_3 \\ +\omega_2 \end{Bmatrix} \\
\mathbf{k} &= \frac{d\boldsymbol{\omega}}{dS} - \mathbf{a}\boldsymbol{\omega}
\end{aligned}
\tag{15–16}
$$

$$\Downarrow$$

$$
\begin{aligned}
e_1 &= \frac{\partial \bar{V}^*}{\partial F_1} = \frac{du_1}{dS} - a_{12}u_2 - a_{13}u_3 \\
e_2 &= \frac{\partial \bar{V}^*}{\partial F_2} = \frac{du_2}{dS} + a_{12}u_1 - a_{23}u_3 - \omega_3 \\
e_3 &= \frac{\partial \bar{V}^*}{\partial F_3} = \frac{du_3}{dS} + a_{13}u_1 + a_{23}u_2 + \omega_2 \\
k_1 &= \frac{\partial \bar{V}^*}{\partial M_1} = \frac{d\omega_1}{dS} - a_{12}\omega_2 - a_{13}\omega_3 \\
k_2 &= \frac{\partial \bar{V}^*}{\partial M_2} = \frac{d\omega_2}{dS} + a_{12}\omega_1 - a_{23}\omega_3 \\
k_3 &= \frac{\partial \bar{V}^*}{\partial M_3} = \frac{d\omega_3}{dS} + a_{13}\omega_1 + a_{23}\omega_2
\end{aligned}
$$

Once $\bar{V}^*$ is specified, the left-hand terms can be expanded. The form of $\bar{V}^*$ depends on the material properties, the particular stress expansions selected, and the member geometry. In what follows, we consider the material to be linearly elastic and approximation $\bar{V}^*$ with the complementary energy function for the prismatic case, which is developed in Sec. 12–3:

$$\bar{V}^* = F_1 e_1^0 + \frac{1}{2AE} F_1^2 + \frac{1}{2GA_2} F_2^2 + \frac{1}{2GA_3} F_3^2 + \frac{1}{2GJ} M_T^2 + k_2^0 M_2 + \frac{1}{2EI_2} M_2^2 + k_3^0 M_3 + \frac{1}{2EI_3} M_3^2 \qquad (15\text{–}17)$$

where

$M_T = M_1 + F_2\bar{y}_3 - F_3\bar{y}_2$ = torsional moment with respect to the shear center

$\bar{y}_2, \bar{y}_3$ = coordinates of the shear center with respect to the centroid

$$e_1^0 = \frac{1}{A}\iint \varepsilon_1^0\, dA$$

$$k_2^0 = \frac{1}{I_2}\iint y_3 \varepsilon_1^0\, dA$$

$$k_3^0 = \frac{-1}{I_3}\iint y_2 \varepsilon_1^0\, dA$$

Note that (15–17) is based on taking a linear expansion for the normal stress,

$$\sigma_{11} = \frac{F_1}{A} + \frac{M_2}{I_2} y_3 - \frac{M_3}{I_3} y_2 \qquad (a)$$

and using the shear stress distribution predicted by the engineering theory,

$$\sigma_{1j} = \sigma_{1j}^t + \sigma_{1j}^f \qquad (b)$$

where σ^t is the unrestrained torsional distribution due to M_T and σ^f is the flexural distribution due to F_2, F_3. In addition to these approximations, we are also neglecting the effect of curvature, i.e., we are considering the member to be *thin*. The approximate force-displacement relations for a linearly elastic thin curve member are

$$\begin{aligned}
e_1 &= e_1^0 + \frac{F_1}{AE} = \frac{du_1}{dS} - a_{12}u_2 - a_{13}u_3 \\
e_2 &= \frac{F_2}{GA_2} + \frac{M_T}{GJ}\bar{y}_3 = \frac{du_2}{dS} + a_{12}u_1 - a_{23}u_3 - \omega_3 \\
e_3 &= \frac{F_3}{GA_3} - \frac{M_T}{GJ}\bar{y}_2 = \frac{du_3}{dS} + a_{13}u_1 + a_{23}u_2 + \omega_2 \\
k_1 &= \frac{M_T}{GJ} = \frac{d\omega_1}{dS} - a_{12}\omega_2 - a_{13}\omega_3 \\
k_2 &= k_2^0 + \frac{M_2}{EI_2} = \frac{d\omega_2}{dS} + a_{12}\omega_1 - a_{23}\omega_3 \\
k_3 &= k_3^0 + \frac{M_3}{EI_3} = \frac{d\omega_3}{dS} + a_{13}\omega_1 + a_{23}\omega_2
\end{aligned} \qquad (15\text{–}18)$$

When the member is planar, the shear center is on the Y_2 axis† and there is no coupling between in-plane (u_1, u_2, ω_3) and out-of-plane (u_3, ω_1, ω_2) displacements. That is, an out-of-plane loading will produce only out-of-plane displacements. The approximate force-displacement relations for out-of-plane deformation for a thin planar member are

$$\begin{aligned}
e_3 &= \frac{F_3}{GA_3} - \frac{M_T}{GJ}\bar{y}_2 = \frac{du_3}{dS} + \omega_2 \\
k_1 &= \frac{M_T}{GJ} = \frac{d\omega_1}{dS} - \frac{1}{R}\omega_2 \\
k_2 &= k_2^0 + \frac{M_2}{EI_2} = \frac{d\omega_2}{dS} + \frac{1}{R}\omega_1
\end{aligned} \qquad (15\text{–}19)$$

where $M_T = M_1 - \bar{y}_2 F_3$. Note that flexure and twist are *coupled*, due to the curvature, even when the shear center coincides with the centroid.

15–4. DISPLACEMENT METHOD—CIRCULAR PLANAR MEMBER

Since the displacement method involves integrating the governing differential equations, its application is restricted to simple geometries. In what follows, we apply the displacement method to a circular planar member subjected to *out-of-plane* loading. We suppose the cross section is constant and the shear center coincides with the centroid. It is convenient to take the polar angle θ as the independent variable. The governing equations are summarized below and the notation is defined in Fig. 15–5.

Equilibrium Equations (see (15–12))

$$\begin{aligned}
\frac{dF_3}{d\theta} + Rb_3 &= 0 \\
\frac{dM_1}{d\theta} - M_2 + Rm_1 &= 0 \\
\frac{dM_2}{d\theta} + M_1 + Rm_2 - RF_3 &= 0
\end{aligned} \qquad (a)$$

Force-Displacement Relations (see (15–19))

$$\begin{aligned}
e_3 &= \frac{F_3}{GA_3} = \frac{1}{R}\frac{du_3}{d\theta} + \omega_2 \\
k_1 &= \frac{M_1}{GJ} = \frac{1}{R}\left(\frac{d\omega_1}{d\theta} - \omega_2\right) \\
k_2 &= k_2^0 + \frac{M_2}{EI_2} = \frac{1}{R}\left(\frac{d\omega_2}{d\theta} + \omega_1\right)
\end{aligned} \qquad (b)$$

† The shear center axis lies in the plane containing the centroidal axis, which, by definition, is a plane of symmetry for the cross section.

Boundary Conditions

$$\left.\begin{array}{lll} F_3 & \text{or} & u_3 \\ M_1 & \text{or} & \omega_1 \\ M_2 & \text{or} & \omega_2 \end{array}\right\} \text{prescribed at each end (pts. } A, B) \tag{c}$$

The solution of the equilibrium equations is quite straightforward. We integrate the first equation directly:

$$F_3 = C_1 - R \int_\theta b_3 \, d\theta \tag{15–20}$$

The remaining two equations can be transformed to

$$\frac{d^2 M_1}{d\theta^2} + M_1 = R\left(F_3 - m_2 - \frac{dm_1}{d\theta}\right) \tag{d}$$

$$M_2 = \frac{dM_1}{d\theta} + Rm_1 \tag{e}$$

We solve (d) for M_1 and determine M_2 from (e). The resulting expressions are

$$\begin{aligned} M_1 &= C_2 \cos\theta + C_3 \sin\theta + M_{1,p} \\ M_2 &= -C_2 \sin\theta + C_3 \cos\theta + \frac{d}{d\theta} M_{1,p} + Rm_1 \end{aligned} \tag{15–21}$$

where $M_{1,p}$ is the particular solution of (d).

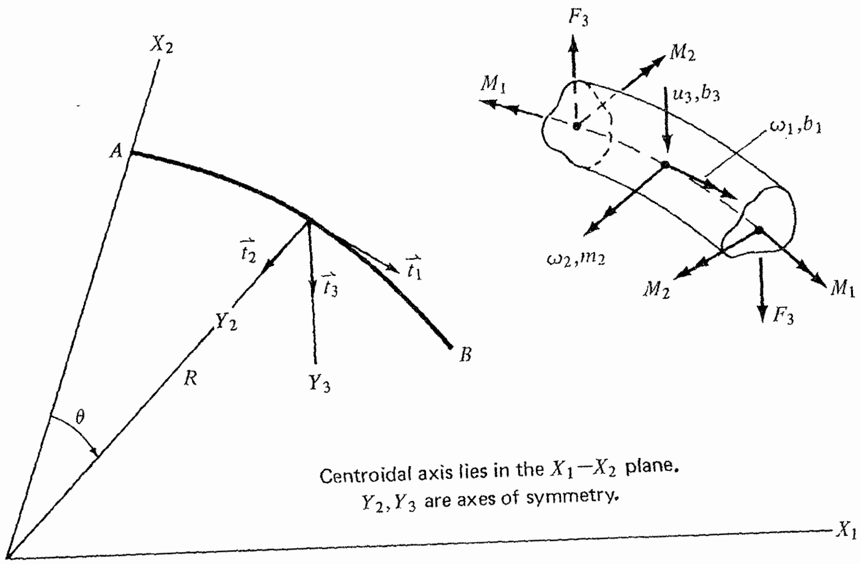


Fig. 15–5. Notation for circular member.

The solution of the force-displacement relations is also straightforward. First, we transform (b) to

$$\begin{aligned} \frac{d^2\omega_1}{d\theta^2} + \omega_1 &= Rk_2^0 - \frac{R^2}{GJ} m_1 + \frac{R}{EI_2}(1 + c_t)M_2 \\ \omega_2 &= \frac{d\omega_1}{d\theta} - \frac{RM_1}{GJ} \\ \frac{du_3}{d\theta} &= \frac{RF_3}{GA_3} - R\omega_2 \end{aligned} \tag{f}$$

where c_t is a dimensionless parameter,

$$c_t = \frac{EI_2}{GJ} \tag{15–22}$$

which is an indicator for torsional deformation. Solving the first equation for ω_1 and then determining ω_2 and u_3 from the second and third equations lead to

$$\begin{aligned} \omega_1 &= C_4 \cos\theta + C_5 \sin\theta + \omega_{1,p} \\ \omega_2 &= -C_4 \sin\theta + C_5 \cos\theta + \frac{d}{d\theta}\omega_{1,p} - \frac{RM_1}{GJ} \\ u_3 &= C_6 - R\omega_1 + \int_\theta \left[\frac{RF_3}{GA_3} + \frac{R^2 M_1}{GJ}\right] d\theta \end{aligned} \tag{15–23}$$

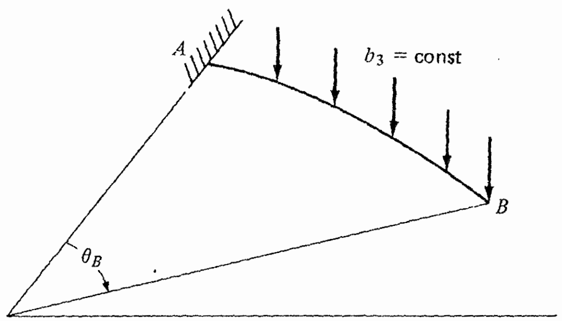
where $\omega_{1,p}$ is the particular solution for ω_1.

The complete solution involves six integration constants which are determined by enforcing the boundary conditions. The following examples illustrate the application of the above equations.

Example 15–4

The member shown is fixed at A and subjected to a uniform distributed loading. Taking

Fig. E15–4



b_3 = const in (15–20), we obtain

$$F_3 = C_1 - Rb_3\theta \tag{a}$$

The equation for M_1 reduces to

$$\frac{d^2M_1}{d\theta^2} + M_1 = RF_3 = RC_1 - R^2b_3\theta \tag{b}$$

Then,

$$M_{1,p} = RF_3 = RC_1 - R^2b_3\theta \tag{c}$$

and the solution for M_1 and M_2 follows from (15–21),

$$\begin{aligned} M_1 &= C_2\cos\theta + C_3\sin\theta + RF_3 \\ M_2 &= -C_2\sin\theta + C_3\cos\theta - R^2b_3 \end{aligned} \tag{d}$$

The boundary conditions at B require

$$F_3 = M_1 = M_2 = 0 \quad \text{at} \quad \theta = \theta_B$$

$$\Downarrow$$

$$\begin{aligned} C_1 &= Rb_3\theta_B \\ C_2 &= -R^2b_3\sin\theta_B \\ C_3 &= R^2b_3\cos\theta_B \end{aligned} \tag{e}$$

Replacing $\theta_B - \theta$ by η, the final solution is

$$\begin{aligned} F_3 &= Rb_3\eta \\ M_1 &= R^2b_3[\eta - \sin\eta] \\ M_2 &= -R^2b_3[1 - \cos\eta] \end{aligned} \tag{f}$$

Example 15–5

The force system due to the end action, $\bar{F}_{B3}$, can be determined by applying the equilibrium conditions directly to the segment shown in Fig. E15–5A. This leads to

$$\begin{aligned} F_3 &= \bar{F}_{B3} \\ M_1 &= \bar{F}_{B3}R(1 - \cos\eta) = \bar{F}_{B3}R[1 - \cos(\theta_B - \theta)] \\ M_2 &= -\bar{F}_{B3}R\sin\eta = -\bar{F}_{B3}R\sin(\theta_B - \theta) \end{aligned} \tag{a}$$

We suppose there is no initial deformation. Using (a), the equation for ω_1 becomes

$$\frac{d^2\omega_1}{d\theta^2} + \omega_1 = \frac{-R^2\bar{F}_{B3}}{EI_2}(1 + c_t)\sin(\theta_B - \theta) \tag{b}$$

The particular solution of (b) is

$$\omega_{1,p} = -\frac{R^2\bar{F}_{B3}}{2EI_2}(1 + c_t)\left[\theta\cos(\theta_B - \theta)\right] \tag{c}$$

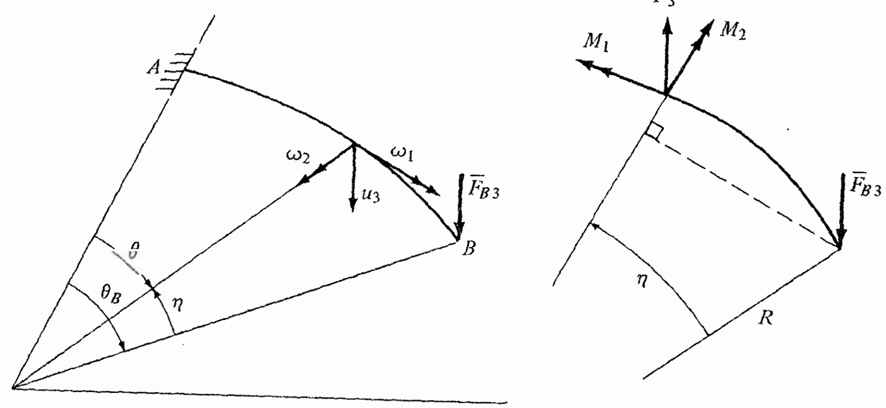
Using the above results and specializing (15–23) for this support condition lead to the following expressions for the displacements:

$$\omega_1 = \bar{\omega}_{A1}\cos\theta + \bar{\omega}_{A2}\sin\theta + \frac{R^2\bar{F}_{B3}}{EI_2}\left\{\left[\frac{1-c_t}{2}\cos\theta_B + c_t\right]\sin\theta - \frac{1+c_t}{2}\theta\cos(\theta_B - \theta)\right\}$$

$$\begin{aligned} \omega_2 &= -\bar{\omega}_{A1}\sin\theta + \bar{\omega}_{A2}\cos\theta + \frac{R^2\bar{F}_{B3}}{EI_2}\left\{c_t[-1 + \cos\theta] - \left(\frac{1-c_t}{2}\right)\sin\theta_B\sin\theta - \left(\frac{1+c_t}{2}\right)\theta\sin(\theta_B - \theta)\right\} \\ u_3 &= \bar{u}_{A3} + R\bar{\omega}_{A1}(1 - \cos\theta) - R\bar{\omega}_{A2}\sin\theta \\ &\quad + \frac{R^3\bar{F}_{B3}}{EI_2}\left\{\left[-\frac{1-c_t}{2}\cos\theta_B - c_t\right]\sin\theta - c_t\sin\theta_B \right. \\ &\quad \left. + \theta\left[\frac{1+c_t}{2}\cos(\theta_B - \theta) + c_t + \frac{EI_2}{GA_3R^2}\right] + c_t\sin(\theta_B - \theta)\right\} \end{aligned} \tag{d}$$

Terms involving $\bar{\omega}_{A1}$, $\bar{\omega}_{A2}$ and $\bar{u}_{A3}$ define the rigid body displacements due to support movement. Also, terms involving c_t are due to twist deformation. The rotations and translation at B are listed below:

Fig. E15–5A



$$\begin{aligned} \omega_{B1} &= \bar{\omega}_{A1}\cos\theta_B + \bar{\omega}_{A2}\sin\theta_B + \frac{R^2\bar{F}_{B3}}{EI_2}\left\{\left[\frac{1-c_t}{2}\cos\theta_B + c_t\right]\sin\theta_B - \frac{1+c_t}{2}\theta_B\right\} \\ \omega_{B2} &= -\bar{\omega}_{A1}\sin\theta_B + \bar{\omega}_{A2}\cos\theta_B + \frac{R^2\bar{F}_{B3}}{EI_2}\left\{c_t[\cos\theta_B - 1] - \frac{1-c_t}{2}\sin^2\theta_B\right\} \\ u_{B3} &= \bar{u}_{A3} + R\bar{\omega}_{A1}(1 - \cos\theta_B) - R\bar{\omega}_{A2}\sin\theta_B \\ &\quad + \frac{R^3\bar{F}_{B3}}{EI_2}\left\{-\frac{1-c_t}{2}\cos\theta_B\sin\theta_B - 2c_t\sin\theta_B + \theta_B\left[\frac{1}{2} + \frac{3}{2}c_t + \frac{EI_2}{GA_3R^2}\right]\right\} \end{aligned} \tag{e}$$

To investigate the relative importance of the various deformation terms, we consider the rectangular cross section shown in Fig. E15-5B. The cross-sectional properties are†

$$\frac{1}{A_3} = \frac{6}{5}\frac{1}{A} = \frac{6}{5}\frac{1}{d_2 d_3}$$
$$J = \frac{k}{3} d_3 d_2^3 \text{ (for } d_2 \leqslant d_3) \qquad \text{(f)}$$
$$I_2 = \frac{d_2 d_3^3}{12}$$

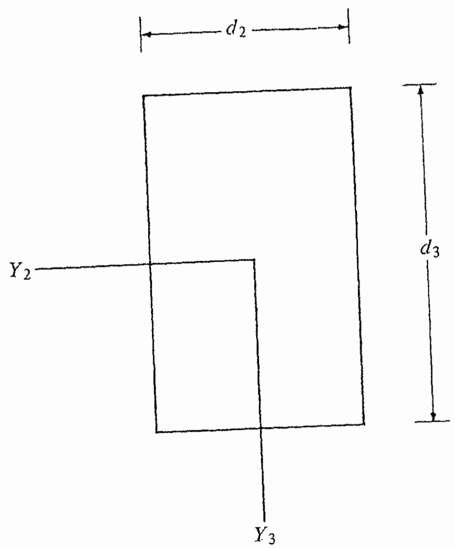
Then,

$$c_t = \frac{E}{G}\frac{I_2}{J} = \frac{E}{G}\left[\frac{1}{4k}\left(\frac{d_3}{d_2}\right)^2\right]$$
$$\frac{EI_2}{GA_3R^2} = \frac{E}{G}\left[\frac{1}{10}\left(\frac{d_3}{d_2}\right)^2\right]\left(\frac{d_2}{R}\right)^2 \qquad \text{(g)}$$

The values of $4k$ and c_t for $d_3/d_2 = 1, 2, 3$ and $\nu = 0.3$ are tabulated below:

d_3/d_2	$4k$	$c_t = EI_2/GJ$ (for $\nu = 0.3$)
1	1.69	1.54
2	2.75	3.8
3	3.16	7.4

Fig. E15-5B



† The torsional constant for a rectangular cross section is developed in Sec. 11-3.

Since $(d_2/R)^2 \ll 1$, we see that it is reasonable to neglect transverse shear deformation. In general, we *cannot* neglect twist deformation when the member is *not* shallow. For the shallow case, we can neglect c_t in the expressions for ω_{B2}, u_{B3}.

Example 15-6

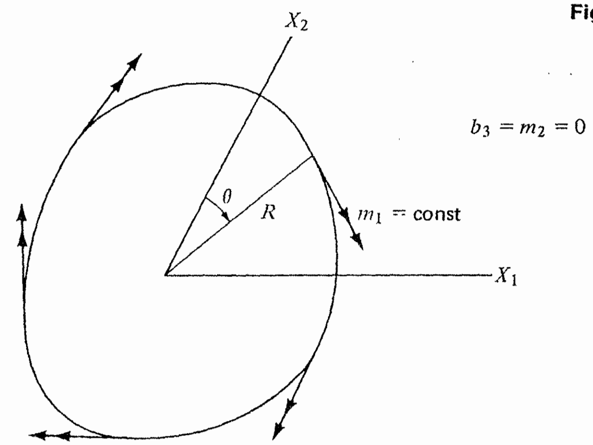
Consider a closed circular ring (Fig. E15-6) subjected to a uniformly distributed twisting moment. From symmetry, $F_3 = 0$ and M_1, M_2 are constant. Then, using (15-16), we find

$$M_1 = 0$$
$$M_2 = Rm_1 \qquad \text{(a)}$$

The displacements follow from (15-18)

$$u_3 = \omega_2 = 0$$
$$\omega_1 = \frac{RM_2}{EI_2} = \frac{R^2 m_1}{EI_2} \qquad \text{(b)}$$

Fig. E15-6



15-5. FORCE METHOD—EXAMPLES

In this section, we illustrate the application of the principle of virtual forces to curved members. The steps involved are the same as for the prismatic or planar case and therefore we will not reiterate them here. We restrict this discussion to the case where the material is linearly elastic, the member is thin and slightly twisted, and warping is neglected. The general form of the expression for the displacement at an arbitrary point and the compatibility equations corresponding to these restrictions (see (15-14), (15-17)) follow.

Displacement at Point Q

$$\mathbf{d}_Q = -\sum_i R_{i,Q}\,\bar{d}_i + \int_S \left[\left(e_1^0 + \frac{F_1}{AE}\right) F_{1,Q} + \left(\frac{F_2}{GA_2}\right) F_{2,Q} + \left(\frac{F_3}{GA_3}\right) F_{3,Q} + \left(\frac{M_T}{GJ}\right) M_{T,Q} + \left(k_2^0 + \frac{M_2}{EI_2}\right) M_{2,Q} + \left(k_3^0 + \frac{M_3}{EI_3}\right) M_{3,Q} \right] dS \qquad (15\text{–}24)$$

Compatibility Equations

$$Z_1, Z_2, \ldots, Z_r = \text{force redundants} \qquad \text{(a)}$$

$$\left.\begin{aligned} F_j &= F_{j,0} + \sum_{k=1}^{r} F_{j,k} Z_k \\ M_j &= M_{j,0} + \sum_{k=1}^{r} M_{j,k} Z_k \\ R_i &= R_{i,0} + \sum_{k=1}^{r} R_{i,k} Z_k \end{aligned}\right\} \qquad \text{(b)}$$

$$\sum_{j=1}^{r} f_{kj} Z_j = \Delta_k \qquad (k = 1, 2, \ldots, r) \qquad (15\text{–}25)$$

where

$$f_{kj} = f_{jk} = \int_S \left[\frac{1}{AE} F_{1,j} F_{1,k} + \frac{1}{GA_2} F_{2,j} F_{2,k} + \frac{1}{GA_3} F_{3,j} F_{3,k} + \frac{1}{GJ} M_{T,j} M_{T,k} + \frac{1}{EI_2} M_{2,j} M_{2,k} + \frac{1}{EI_3} M_{3,j} M_{3,k} \right] dS$$

$$\Delta_k = \sum_i R_{i,k}\,\bar{d}_i - \int_S \left[\left(e_1^0 + \frac{F_{1,0}}{AE}\right) F_{1,k} + \left(\frac{F_{2,0}}{GA_2}\right) F_{2,k} + \left(\frac{F_{3,0}}{GA_3}\right) F_{3,k} + \left(\frac{M_{T,0}}{GJ}\right) M_{T,k} + \left(k_2^0 + \frac{M_{2,0}}{EI_2}\right) M_{2,k} + \left(k_3^0 + \frac{M_{3,0}}{EI_3}\right) M_{3,k} \right] dS$$

$$M_T = M_1 + y_3 F_2 - y_2 F_3$$

The reduced form for out-of-plane deformation is obtained by setting $F_1 = F_2 = M_3 = e_1^0 + k_3^0 = 0$.

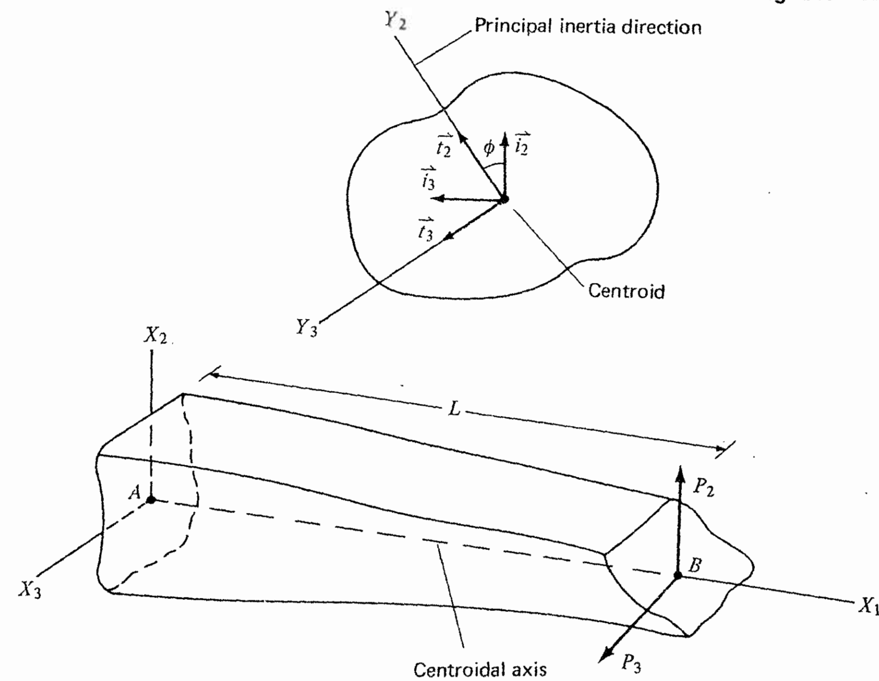
Example 15–7

Consider the *nonprismatic* member shown below. The centroidal axis is *straight* but the orientations of the principal inertia axes *vary*. We take X_1 to coincide with the centroidal axis and X_2, X_3 to coincide with the principal inertia directions at the left end (point A). The principal inertia directions are defined by the unit vectors $\vec{t}_2$, $\vec{t}_3$.

$$\begin{aligned} \vec{t}_2 &= \cos\phi\,\vec{\imath}_2 + \sin\phi\,\vec{\imath}_3 \\ \vec{t}_3 &= -\sin\phi\,\vec{\imath}_2 + \cos\phi\,\vec{\imath}_3 \\ \phi &= 0 \quad \text{at} \quad x_1 = 0 \end{aligned} \qquad \text{(a)}$$

Now, we consider the problem of determining the translations of the centroid at B due to the loading shown in Fig. E15–7A. It is convenient to work with translation components (v_{B2}, v_{B3}) referred to the basic frame, i.e., the X_2, X_3 directions. We suppose that the shear

Fig. E15–7A



center coincides with the centroid and transverse shear deformation is negligible. Specializing (15–24), and noting that $M_1 = 0$ for a transverse load applied at the centroid, the displacement expression reduces to

$$d_Q = \frac{1}{E}\int_0^L \left(\frac{1}{I_2} M_2 M_{2,Q} + \frac{1}{I_3} M_3 M_{3,Q} \right) dx_1 \qquad \text{(b)}$$

Force Systems

The moment vectors acting on a positive cross section due to P_2, P_3 applied at B (Fig. E15–7B) are

$$\begin{aligned} (\vec{M})_{P_2} &= P_2(L - x_1)\vec{\imath}_3 \\ (\vec{M})_{P_3} &= -P_3(L - x_1)\vec{\imath}_2 \end{aligned} \qquad \text{(c)}$$

To find M_2, M_3, we must determine the components of $\vec{M}$ with respect to the local frame. These follow from Fig. E15–7C:

For P_2,

$$\begin{aligned} M_2 &= P_2(L - x_1)\sin\phi \\ M_3 &= P_2(L - x_1)\cos\phi \end{aligned} \qquad \text{(d)}$$

Fig. E15-7B

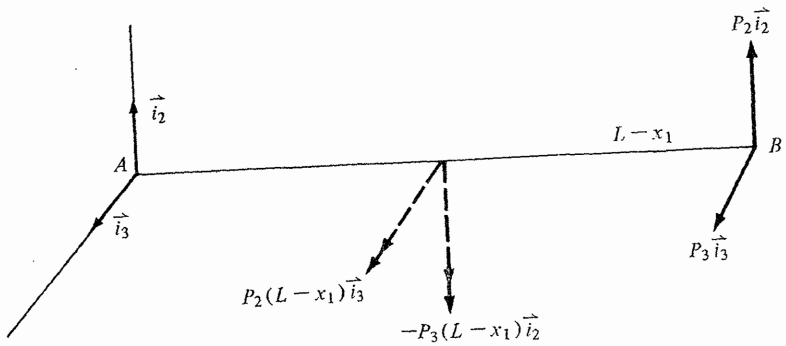
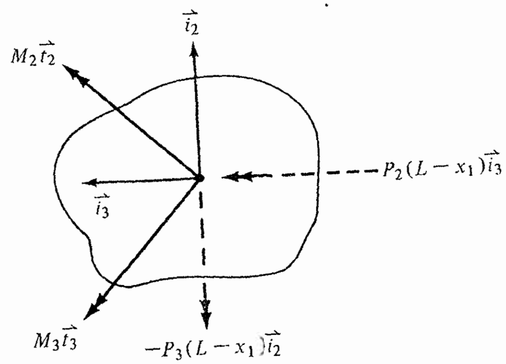


Fig. E15-7C



For P_3,

$$
\begin{aligned}
M_2 &= -P_3(L - x_1)\cos\phi \\
M_3 &= +P_3(L - x_1)\sin\phi
\end{aligned}
\tag{e}
$$

Determination of v_{B2} Due to P_2

The virtual-force system for v_{B2} corresponds to $P_2 = +1$. Introducing (d) in (b), we obtain

$$
v_{B2} = \frac{P_2}{E}\int_0^L \left[\frac{\sin^2\phi}{I_2} + \frac{\cos^2\phi}{I_3}\right](L - x_1)^2\,dx_1 \tag{f}
$$

Determination of v_{B3} Due to P_2

The virtual-force system for v_{B3} corresponds to $P_3 = +1$. Using (e) leads to

$$
v_{B3} = \frac{P_2}{E}\int_0^L \left(-\frac{1}{I_2} + \frac{1}{I_3}\right)(L - x_1)^2 \sin\phi\cos\phi\,dx_1 \tag{g}
$$

Example 15-8

We rework Example 15-6 with the force method. Using symmetry, we see that

$$
\begin{aligned}
M_1 &= 0 \\
M_2 &= m_1 R
\end{aligned}
\tag{a}
$$

Suppose the rotation ω_1 in the direction of m_1 is desired. The virtual loading for this displacement is $m_1 = +1$. Starting with

$$
\oint \omega_1\,\Delta m_1\,dS = \oint \frac{M_2}{EI_2}\,\Delta M_2\,dS \tag{b}
$$

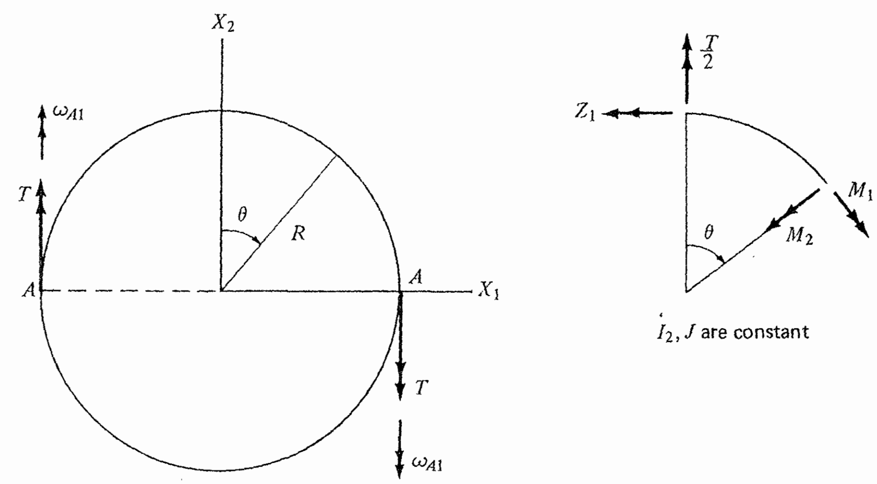
and substituting for M_2, we obtain

$$
\omega_1 = \frac{m_1 R^2}{EI_2} \tag{c}
$$

Example 15-9

Consider the closed ring shown. Only M_1 and M_2 are finite for this loading. Also, the behavior is symmetrical with respect to X_1 and we have to analyze only one half the ring.

Fig. E15-9



We take the torsional moment at $\theta = 0$ as the force redundant. The moment distributions are

$$
\begin{aligned}
M_1 &= \frac{T}{2}\sin\theta + Z_1\cos\theta = M_{1,0} + Z_1 M_{1,1} \\
M_2 &= \frac{T}{2}\cos\theta - Z_1\sin\theta = M_{2,0} + Z_1 M_{2,1} \qquad -\frac{\pi}{2} \leqslant \theta \leqslant \pi/2
\end{aligned}
\tag{a}
$$

Specializing (15–25) for this problem,

$$f_{11}Z_1 = \Delta_1$$

$$\Delta_1 = -2R\int_{-\pi/2}^{\pi/2}\left[\frac{M_{1,0}M_{1,1}}{GJ} + \frac{M_{2,0}M_{2,1}}{EI_2}\right]d\theta \qquad \text{(b)}$$

$$f_{11} = 2R\int_{-\pi/2}^{\pi/2}\left[\frac{M_{1,1}^2}{GJ} + \frac{M_{2,1}^2}{EI_2}\right]d\theta$$

and then substituting for M_1, M_2,

$$\Delta_1 = -RT\int_{-\pi/2}^{\pi/2}\left(\frac{1}{GJ} - \frac{1}{EI_2}\right)\sin\theta\cos\theta\, d\theta$$

$$= -\frac{RT}{2}\left(\frac{1}{GJ} - \frac{1}{EI_2}\right)[\sin^2\theta]_{-\pi/2}^{\pi/2} = 0 \qquad \text{(c)}$$

and it follows that $Z_1 = 0$. We could have arrived at this result by noting that the behavior is also symmetrical with respect to X_2. This requires M_2 to be an *even* function of θ.

The virtual-force system for ω_{A1} is $T = +1$. Using (15–24) and (a) leads to

$$2\omega_{A1} = 2R\int_{-\pi/2}^{\pi/2}\left[\left(\frac{T\sin\theta}{2GJ}\right)\frac{\sin\theta}{2} + \left(\frac{T\cos\theta}{2EI_2}\right)\frac{\cos\theta}{2}\right]d\theta$$

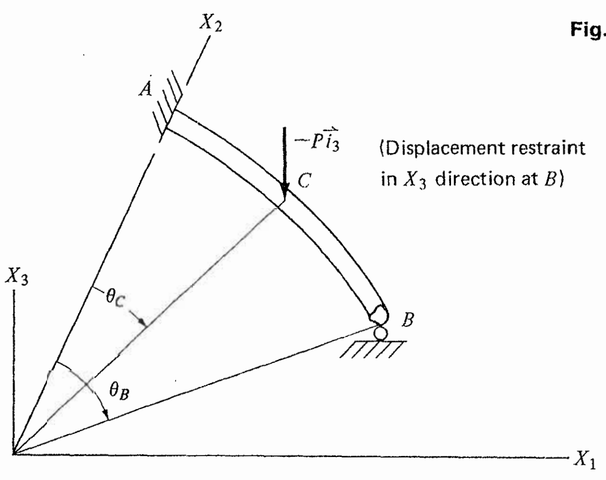
$$\Downarrow \qquad \text{(d)}$$

$$\omega_{A1} = \frac{RT\pi}{8}\left[\frac{1}{GJ} + \frac{1}{EI_2}\right]$$

Example 15–10

We analyze the planar circular member shown in Fig. E15–10A. The loading is out-of-plane, and only F_3, M_1, and M_2 are finite. To simplify the algebra, we consider the shear center to coincide with the centroid and neglect transverse shear deformation. It is convenient to take the reaction at B as the force redundant.

Fig. E15–10A

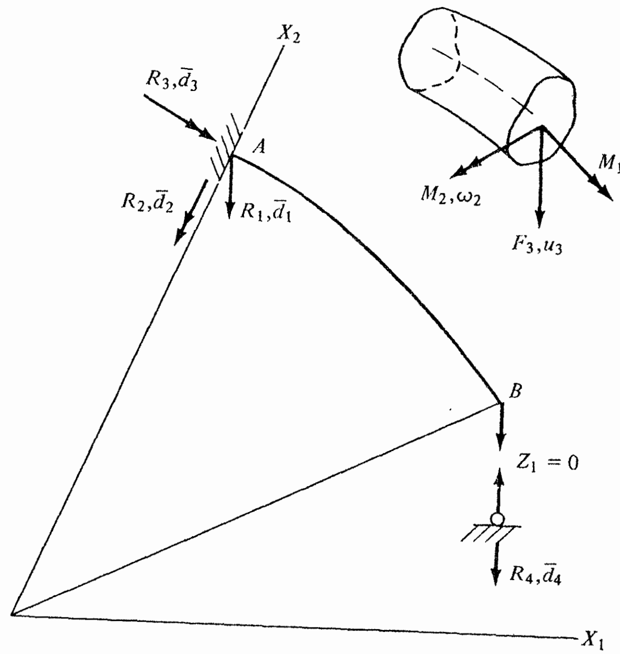


Primary Structure

The primary structure is defined in Fig. E15–10B:

$$\begin{aligned} R_1 &= -F_{A3} & R_2 &= -M_{A2} & R_3 &= -M_{A1} \\ d_1 &= \bar{u}_{A3} & d_2 &= \bar{\omega}_{A2} & d_3 &= \bar{\omega}_{A1} \\ R_4 &= Z_1 = F_{B3} & d_4 &= \bar{u}_{B3} \end{aligned} \qquad \text{(a)}$$

Fig. E15–10B



Force Analyses

The force solutions for the loadings shown in Fig. E15–10C are:

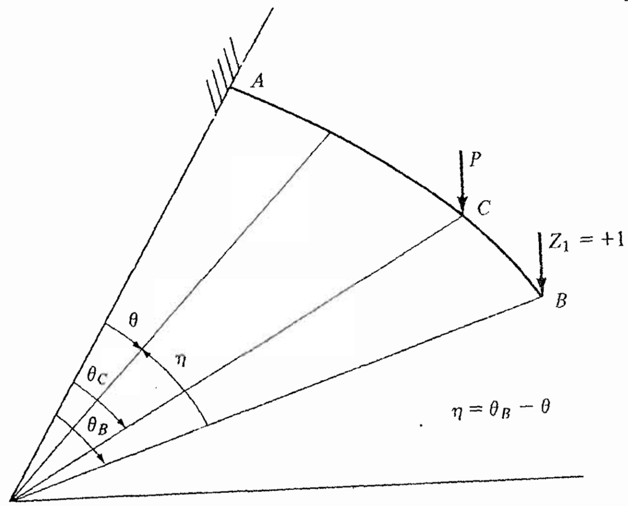
For P:

$$\begin{aligned} F_{3,0} &= +P \\ M_{1,0} &= PR[1 - \cos(\eta - \eta_C)] \\ M_{2,0} &= -PR\sin(\eta - \eta_C) \\ \eta_C &\leq \eta \leq \theta_B \end{aligned} \qquad \text{(b)}$$

For $Z_1 = +1$:

$$\left.\begin{aligned} F_{3,1} &= +1 \\ M_{1,1} &= R(1 - \cos\eta) \\ M_{2,1} &= -R\sin\eta \end{aligned}\right\} \qquad \text{(c)}$$

Fig. E15–10C



Compatibility Equation (15–25)

$$Z_1 = \frac{\Delta_1}{f_{11}}$$

$$f_{11} = R\int_0^{\theta_B}\left[\frac{M_{1,1}^2}{GJ} + \frac{M_{2,1}^2}{EI_2}\right]d\eta \qquad \text{(d)}$$

$$\Delta_1 = \sum_{i=1}^{4} R_{i,1}\bar{d}_i - R\int_{\theta_B-\theta_C}^{\theta_B}\left[\frac{M_{1,0}M_{1,1}}{GJ} + \left(k_2^0 + \frac{M_{2,0}}{EI_2}\right)M_{2,1}\right]d\eta$$

Substituting for the internal force and reactions, we obtain the following expressions for f_{11} and Δ_1:

$$f_{11} = \frac{R^3}{EI_2}\left[\left(\frac{1+3c_t}{2}\right)\theta_B - 2c_t\sin\theta_B - \left(\frac{1-c_t}{2}\right)\sin\theta_B\cos\theta_B\right]$$

$$\begin{aligned}\Delta_1 = {} & \bar{u}_{B3} - \bar{u}_{A3} + R\bar{\omega}_{A2}\sin\theta_B - R\bar{\omega}_{A1}(1-\cos\theta_B) \\ & + R^2\int_{\theta_B-\theta_c}^{\theta_B} k_2^0\sin(\theta_B-\theta)d\theta \\ & - \frac{PR^3}{EI_2}\left[c_t\left\{\theta_c\left[1+\frac{1}{2}\cos(\theta_B-\theta_c)\right] - \sin\theta_B - \sin\theta_c\right.\right. \\ & \left. + \sin\theta_B\cos\theta_c - \frac{1}{2}\cos\theta_B\sin\theta_c\right\} \\ & \left. + \frac{1}{2}\left\{\theta_c\cos(\theta_B-\theta_c) - \cos\theta_B\sin\theta_c\right\}\right]\end{aligned} \qquad \text{(e)}$$

$$c_t = \frac{EI_2}{GJ}$$

Note that we could have determined Δ_1 and f_{11} using the results of Example 15–5.

15–6. RESTRAINED WARPING FORMULATION

In what follows, we consider the member to be thin and slightly twisted. Referring to Fig. 15–2, these restrictions lead to

$$\frac{d\vec{R}}{dS} \approx \vec{t}_1 \qquad d(\text{vol.}) \approx dS\,dy_2\,dy_3 \qquad (15\text{–}26)$$

Therefore, in analyzing the strain at Q (S, y_2, y_3), we can treat the differential line elements as if they were orthogonal. The approach followed for the prismatic case is also applicable here. One has only to work with stress and strain measures referred to the local frame $(\vec{t}_1, \vec{t}_2, \vec{t}_3)$ rather than the global frame.

Our formulation is based on Reissner's principle (13–33):

$$\begin{aligned}\delta\left[\iiint(\boldsymbol{\sigma}^T\boldsymbol{\varepsilon} - \mathbf{b}^T\hat{\mathbf{u}} - V^*)d(\text{vol.}) - \iint\mathbf{p}^T\hat{\mathbf{u}}\,d(\text{surface area})\right] &= 0 \\ \boldsymbol{\sigma}, \hat{\mathbf{u}} &= \text{independent quantities} \\ \boldsymbol{\varepsilon} &= \boldsymbol{\varepsilon}(\hat{\mathbf{u}}) \\ \mathbf{p}, \mathbf{b} &= \text{prescribed forces} \\ V^* = V^*(\boldsymbol{\sigma}) &= \text{complementary energy density}\end{aligned} \qquad \text{(a)}$$

We introduce expansions for $\hat{\mathbf{u}}$, $\boldsymbol{\sigma}$ in terms of one-dimensional displacement and force measures (functions of S) and integrate over the cross section. The force-equilibrium equations follow from the stationary requirement with respect to displacement measures.

We start with the strain measures, $\boldsymbol{\varepsilon} = \{\varepsilon_1, \gamma_{12}, \gamma_{13}\}$. One can show that†

$$\begin{aligned}\varepsilon_1 &\approx \vec{t}_1\cdot\frac{\partial\vec{\hat{u}}}{\partial S} \\ \gamma_{12} &\approx \vec{t}_1\cdot\frac{\partial\vec{\hat{u}}}{\partial y_2} + \vec{t}_2\cdot\frac{\partial\vec{\hat{u}}}{\partial S} \\ \gamma_{13} &\approx \vec{t}_1\cdot\frac{\partial\vec{\hat{u}}}{\partial y_3} + \vec{t}_3\cdot\frac{\partial\vec{\hat{u}}}{\partial S}\end{aligned} \qquad (15\text{–}27)$$

where $\vec{\hat{u}}$ is the displacement vector for Q (S, y_1, y_2). We use the same displacement expansion as for the prismatic case:

$$\begin{aligned}\vec{\hat{u}} &= \hat{u}_1\vec{t}_1 + \hat{u}_2\vec{t}_2 + \hat{u}_3\vec{t}_3 \\ \hat{u}_1 &= u_1 + \omega_2 y_3 - \omega_3 y_2 + f\phi \\ \hat{u}_2 &= u_{s2} - \omega_1(y_3 - \bar{y}_3) \\ \hat{u}_3 &= u_{s3} + \omega_1(y_2 - \bar{y}_2) \\ \phi &= \phi(y_2, y_3)\end{aligned} \qquad (15\text{–}28)$$

Expanding

$$\iint\boldsymbol{\sigma}^T\boldsymbol{\varepsilon}\,dy_2\,dy_3 = \iint(\sigma_{11}\varepsilon_1 + \sigma_{12}\gamma_{12} + \sigma_{13}\gamma_{13})dy_2\,dy_3 \qquad \text{(a)}$$

† See Prob. 15–5.

leads to

$$\iint \sigma^T \varepsilon \, dy_2 \, dy_3 = F_1 e_1 + F_2 e_2 + F_3 e_3 + M_T k_1 + M_2 k_2 + M_3 k_3 + M_R f + M_\phi f_{,s}$$
$$e_1, e_2, \ldots, k_3 \quad \text{(defined by (15–16))} \tag{15–29}$$
$$M_\phi = \iint \sigma_{11} \phi \, dy_2 \, dy_3$$
$$M_R = \iint [\sigma_{12}(\phi_{,2} + a_{12}\phi) + \sigma_{13}(\phi_{,3} + a_{13}\phi)] dy_2 \, dy_3$$

The equilibrium equations consist of (15–11) and the equation due to warping restraint,

$$M_R = M_{\phi,s} \tag{15–30}$$

which can be interpreted as the stress equilibrium equation for the $\vec{t}_1$ direction weighted with respect to ϕ.

Now, we use the stress expansion developed for the prismatic case. The derivation is discussed in Sec. 13–5, so we only list the essential results here. The normal stress is expressed as

$$\sigma_{11} = \frac{F_1}{A} + \frac{M_2}{I_2} y_3 - \frac{M_3}{I_3} y_2 + \frac{M_\phi}{I_\phi} \phi \tag{15–31}$$

where $\phi = -\phi_t^{sc}$, the St. Venant warping function referred to the shear center. We write the transverse shear stress distribution as

$$\sigma = \psi_2 F_2 + \psi_3 F_3 + \psi_u M_T^u + \psi_r M_T^r$$
$$M_T = M_T^u + M_T^r \tag{15–32}$$

(ψ's are functions of y_2, y_3.) The corresponding complementary energy function is

$$V^* = \iint V^* \, dy_2 \, dy_3 = \frac{1}{2E}\left(\frac{F_1^2}{A} + \frac{M_2^2}{I_2} + \frac{M_3^2}{I_3}\right) + \frac{1}{2E_r}\left(\frac{M_\phi^2}{I_\phi}\right)$$
$$+ \frac{1}{2G}\left(\frac{F_2^2}{A_2} + \frac{2F_2F_3}{A_{23}} + \frac{F_3^2}{A_3}\right) + \frac{1}{2GJ}\left((M_T^u)^2 + C_r(M_T^r)^2\right) \tag{15–33}$$
$$+ \frac{1}{GJ}(F_2 y_{3r} + F_3 y_{2r}) M_T^r$$

Also, (15–32) satisfies (see(13–50))

$$\iint (\sigma_{12}\phi_{,2} + \sigma_{13}\phi_{,3}) dy_2 \, dy_3 \equiv M_T^r \tag{b}$$

Finally, noting (b), we express M_R as

$$M_R = (1 + b_r) M_T^r + b_2 F_2 + b_3 F_3 \tag{15–34}$$

where the b's involve the curvature (a_{12}, a_{13}). If the cross section is symmetrical,

$$A_{23} = y_{3r} = y_{2r} = b_r = 0 \tag{c}$$

and b_2, b_3 are due to self-equilibrating stress distributions.† It is reasonable, in this case, to take $b_2 \approx b_3 \approx 0$ and compute the shear coefficients (A_2, A_3) based on the primary flexural shear stress distributions.

Expanding the stationary requirement with respect to force measures yields the force-displacement relations,

$$e_1 = \frac{\partial \bar{V}^*}{\partial F_1} \qquad k_2 = \frac{\partial \bar{V}^*}{\partial M_2} \qquad k_3 = \frac{\partial \bar{V}^*}{\partial M_3} \qquad f_{,s} = \frac{\partial \bar{V}}{\partial M_\phi}$$
$$e_2 + b_2 f = \frac{\partial \bar{V}^*}{\partial F_2} \qquad e_3 + b_3 f = \frac{\partial \bar{V}^*}{\partial F_3} \tag{15–35}$$
$$k_1 = \frac{\partial \bar{V}^*}{\partial M_T^u} \qquad k_1 + (1 + b_r) f = \frac{\partial \bar{V}^*}{\partial M_T^r}$$

where $e_1, e_2, \ldots, k_3$ are defined by (15–16). The corresponding *unrestrained* warping relations are (15–18).

Example 15–11

To investigate the influence of warping restraint, we consider a planar circular member having a *doubly symmetrical* cross section (Fig. E15–11), clamped at one end and subjected

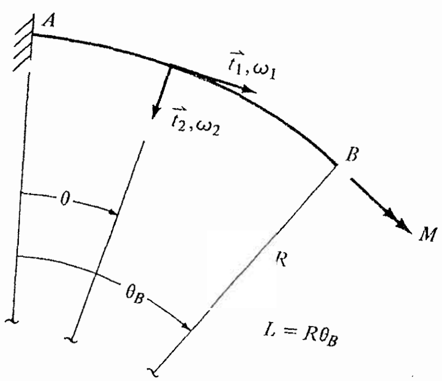


Fig. E15–11

to a torsional moment at the other end. We neglect transverse shear deformation due to restrained torsion. The governing equations for this loading (see Sec. 15–4) follow.

Equilibrium Equations

$$\frac{dM_1}{d\theta} = M_2$$
$$\frac{dM_2}{d\theta} = -M_1$$
$$M_1^r = \frac{1}{R}\frac{dM_\phi}{d\theta} \tag{a}$$
$$M_1 = M_1^u + M_1^r$$

† See Prob. 15–6.

Force-Displacement Relations

$$M_2 = EI_2 k_2 = \frac{EI_2}{R}\left(\frac{d\omega_2}{d\theta} + \omega_1\right)$$

$$M_\phi = \frac{E_r I_\phi}{R}\frac{df}{d\theta} \qquad \text{(b)}$$

$$f = -k_1 = -\frac{1}{R}\left(\frac{d\omega_1}{d\theta} - \omega_2\right)$$

$$M_1^u = GJk_1$$

Boundary Conditions

$$\begin{aligned} &\theta = 0 && \omega_1 = \omega_2 = f = 0 \\ &\theta = \theta_B && M_1 = M \\ & && M_\phi = 0 \\ & && M_2 = 0 \end{aligned} \qquad \text{(c)}$$

One can write the equilibrium solution directly from the sketch:

$$M_1 = M\cos(\theta_B - \theta) \qquad M_2 = M\sin(\theta_B - \theta) \qquad \text{(d)}$$

We substitute for the moments in the force-displacement relations.

$$M_1 = GJk_1 - \frac{E_r I_\phi}{R^2}\frac{d^2 k_1}{d\theta^2} = M\cos(\theta_B - \theta)$$

$$k_1 = \frac{1}{R}\left(\frac{d\omega_1}{d\theta} - \omega_2\right) \qquad \text{(e)}$$

$$k_2 = \frac{1}{R}\left(\frac{d\omega_2}{d\theta} + \omega_1\right) = \frac{M}{EI_2}\sin(\theta_B - \theta)$$

and solve for k_1, and then ω_1. The resulting expressions are

$$\lambda^2 = \frac{GJ}{E_r I_\phi} \qquad \bar{\lambda} = R\lambda \qquad c_t = \frac{EI_2}{GJ}$$

$$k_1 = \frac{M}{GJ + \frac{E_r I_\phi}{R^2}}\{\cos(\theta_B - \theta) - \cos\theta_B[\cosh\bar{\lambda}\theta - \sinh\bar{\lambda}\theta\tanh\bar{\lambda}\theta_B]\}$$

$$\frac{\omega_1}{\left(\frac{RM}{EI_2}\right)} = \frac{1}{2}\left\{\theta\cos(\theta_B - \theta) - \sin\theta\cos\theta_B\right\} \qquad \text{(f)}$$

$$+ \frac{c_t}{\left(1 + \frac{1}{\bar{\lambda}^2}\right)}\left\{\frac{1}{2}\theta\cos(\theta_B - \theta) + \left[\frac{1}{1 + \frac{1}{\bar{\lambda}^2}} - \frac{1}{2}\right]\sin\theta\cos\theta_B\right.$$

$$\left. - \frac{\bar{\lambda}\cos\theta_B}{1 + \bar{\lambda}^2}[\sinh\bar{\lambda}\theta - \tanh\bar{\lambda}\theta_B\cosh\bar{\lambda}\theta + \cos\theta\tanh\bar{\lambda}\theta_B]\right\}$$

Warping restraint is neglected by setting $E_r = 0$ and $\bar{\lambda} = \infty$.

The rotation at B is

$$\left.\frac{\omega_1}{\left(\frac{RM}{EI_2}\right)}\right|_{\theta_B} = \frac{1}{2}(\theta_B - \sin\theta_B\cos\theta_B) + c_t K$$

$$K = \frac{1}{1 + \frac{1}{\bar{\lambda}^2}}\left\{\frac{1}{2}\theta_B + \sin\theta_B\cos\theta_B\left[\frac{1}{1 + \frac{1}{\bar{\lambda}^2}} - \frac{1}{2}\right] - \frac{\bar{\lambda}}{1 + \bar{\lambda}^2}\cos^2\theta_B\tanh\bar{\lambda}\theta_B\right\} \qquad \text{(g)}$$

If we set

$$R = \frac{L}{\theta_B} \qquad \bar{\lambda} = \frac{L\lambda}{\theta_B} \qquad \text{(h)}$$

and let $\theta_B \rightarrow 0$, (g) reduces to (13–57), the prismatic solution. The influence of warping restraint depends on $\bar{\lambda}$ and θ_B. Values of K vs. $\bar{\lambda}$ for $\theta_B = \pi/4, \pi/2$ are tabulated below:

$$\frac{K}{K_\infty} = \frac{1}{1 + \frac{1}{\bar{\lambda}^2}} \qquad \text{for } \theta_B = \frac{\pi}{2}$$

$$\frac{K}{K_\infty} = \frac{1}{1 + \frac{1}{\bar{\lambda}^2}}\left\{\frac{\frac{\pi}{2} - 1}{\frac{\pi}{2} + 1} + \frac{2}{1 + \frac{\pi}{2}}\left(\frac{1}{1 + \frac{1}{\bar{\lambda}^2}}\right)\left(1 - \frac{\tanh\frac{\pi\bar{\lambda}}{4}}{\bar{\lambda}}\right)\right\} \qquad \text{for } \theta_B = \frac{\pi}{4} \qquad \text{(i)}$$

$$K_\infty = \frac{1}{2}(\theta_B + \sin\theta_B\cos\theta_B)$$

$\bar{\lambda}$	K/K_∞ for $\theta_B = \pi/4$	K/K_∞, $\theta_B = \pi/2$
1	0.179	0.500
5	0.786	0.96
10	0.907	0.99

We showed in Chapter 13 that

$$\begin{aligned} \lambda &= 0\left(\frac{t}{h^2}\right) && \text{(open section)} \\ \lambda &= 0\left(\frac{1}{h}\right) && \text{(closed section)} \end{aligned} \qquad \text{(j)}$$

where t is the wall thickness and h is a depth measure. Since $\bar{\lambda} = R\lambda$ and $R/h \gg 1$ for a thin curved member, the influence of warping restraint is not as significant as for the prismatic case.

15–7. MEMBER FORCE-DISPLACEMENT RELATIONS—COMPLETE END RESTRAINT

In the analysis of a member system, one needs the relations between the forces and displacements at the ends of the member. For a truss, these equations reduce to a single relation between the bar force and the elongation. Matrix notation is particularly convenient for this derivation so we start by expressing the principle of virtual forces and the complementary energy density in terms of generalized force and deformation matrices.

Referring back to Sec. 15–3, we define

$$\mathscr{E} = \begin{Bmatrix} \mathbf{e} \\ \mathbf{k} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial \bar{V}^*}{\partial F_i} \\ \dfrac{\partial \bar{V}^*}{\partial M_j} \end{Bmatrix} \qquad \mathscr{F} = \begin{Bmatrix} \mathbf{F} \\ \mathbf{M} \end{Bmatrix} = \begin{Bmatrix} F_i \\ M_j \end{Bmatrix} \tag{15–36}$$

and write the principle of virtual forces as

$$\int_S d\bar{V}^*\, dS \equiv \int_S \mathscr{E}^T \Delta\mathscr{F}\, dS = \mathbf{d}^T \Delta\mathbf{P} \tag{15–37}$$

Note that we are working with M_1, not M_T. We use the complementary energy function for a thin slightly twisted member with *negligible warping* restraint (i.e., (15–17)). With the above notation,

$$\text{Eq. (15–17)} \Rightarrow \bar{V}^* = (\mathscr{E}^0)^T \mathscr{F} + \tfrac{1}{2}\mathscr{F}^T \mathbf{g} \mathscr{F}$$

where

$$\mathbf{g} = \begin{bmatrix} \mathbf{g}_f & \mathbf{g}_{fm} \\ \mathbf{g}_{fm}^T & \mathbf{g}_m \end{bmatrix} \tag{15–38}$$

$$\mathbf{g}_f = \begin{bmatrix} \dfrac{1}{AE} & 0 & 0 \\ & \dfrac{1}{A_2 G} + \dfrac{\bar{y}_3^2}{GJ} & -\dfrac{\bar{y}_2 \bar{y}_3}{GJ} \\ \text{Sym} & & \dfrac{1}{A_3 G} + \dfrac{\bar{y}_2^2}{GJ} \end{bmatrix}$$

$$\mathbf{g}_{fm} = \begin{bmatrix} 0 & 0 & 0 \\ \dfrac{\bar{y}_3}{GJ} & 0 & 0 \\ -\dfrac{\bar{y}_2}{GJ} & 0 & 0 \end{bmatrix} \qquad \mathbf{g}_m = \begin{bmatrix} \dfrac{1}{GJ} & 0 & 0 \\ & \dfrac{1}{EI_2} & 0 \\ \text{Sym} & & \dfrac{1}{EI_3} \end{bmatrix}$$

The force-deformation relation implied by (15–38) is

$$\mathscr{E} = \mathscr{E}^0 + \mathbf{g}\mathscr{F} \tag{15–39}$$

We will use these general expressions for planar and out-of-plane deformation as well as for the arbitrary case. One has only to delete the rows and columns

of $\mathbf{g}$ corresponding to the zero force measures. For example,

$$\mathscr{F} \Rightarrow \{F_1 F_2 M_3\}$$
$$\bar{y}_3 = 0$$

$$\mathbf{g} \Rightarrow \begin{bmatrix} \dfrac{1}{AE} & 0 & 0 \\ 0 & \dfrac{1}{A_2 G} & 0 \\ 0 & 0 & \dfrac{1}{EI_3} \end{bmatrix} \tag{15–40}$$

for planar loading applied to a *planar* member.

Finally, we substitute for $\mathscr{E}$ in (15–37) and distinguish between prescribed and unknown displacements. The principle of virtual forces expands to

$$\int_S (\mathscr{E}^0 + \mathbf{g}\mathscr{F})^T \Delta\mathscr{F}\, dS - \bar{\mathbf{d}}^T \Delta\mathbf{R} = \mathbf{d}^T \Delta\mathbf{P} \tag{15–41}$$

where $\bar{\mathbf{d}}$ contains prescribed displacements and $\mathbf{R}$ are the corresponding reactions; $\mathbf{d}$ contains unknown displacements and $\Delta\mathbf{P}$ are forces corresponding to $\mathbf{d}$. The virtual-force system ($\Delta\mathbf{P}$, $\Delta\mathbf{R}$, $\Delta\mathscr{F}$) must satisfy the force-equilibrium equations, (15–11). It is more convenient to generate $\mathscr{F}$ and $\mathbf{R}$ with the equilibrium equations for a finite segment rather than attempt to solve (15–11).

Consider the arbitrary member shown in Fig. 15-6. Each end is completely restrained against displacement. The positive sense of S is from A toward B.

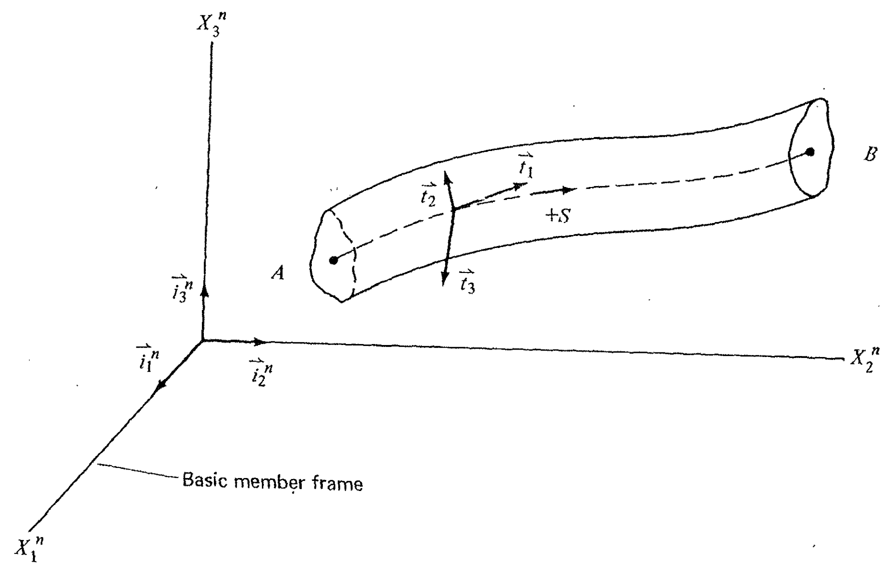


Fig. 15–6. Arbitrary curved member.

We suppose the geometry of the member is defined with respect to a basic frame which we refer to as frame n, and take the end forces at B as the force redundants. Then, the primary structure consists of the member cantilevered from A.

Throughout the remaining portion of the chapter, we will employ the notation for force and displacement transformations that is developed in Chapter 5. A superscript n is used to denote a quantity referred to the *basic* frame. When *no* frame superscript is used, it is understood the quantity is referred to the *local* frame. For example, $\mathcal{F}_Q$ represents the internal force matrix at point Q referred to the local frame at Q. Note that $\mathcal{F}_Q$ acts on the positive face. The force matrix for the negative face is $-\mathcal{F}_Q$. The end forces at A, B are denoted by $\bar{\mathcal{F}}^n_A$, $\bar{\mathcal{F}}^n_B$ and are related to the internal force matrices by

$$\begin{aligned} \bar{\mathcal{F}}^n_B &= +\mathcal{F}^n_B = \mathcal{R}^{bn}\mathcal{F}_B \\ \bar{\mathcal{F}}^n_A &= -\mathcal{F}^n_A = -\mathcal{R}^{an}\mathcal{F}_A \end{aligned} \tag{15-42}$$

Also, the displacement matrix at point Q is written as $\mathcal{U}_Q$.

$$\mathcal{U}_Q = \{u_1, u_2, u_3 \mid \omega_1, \omega_2, \omega_3\}_Q = \begin{Bmatrix} \mathbf{u} \\ \boldsymbol{\omega} \end{Bmatrix}_Q \tag{15-43}$$

For this system, $\mathcal{U}^n_A$ and $\mathcal{U}^n_B$ are prescribed.

We determine $\mathcal{U}^n_B$ for the primary structure, i.e. the member cantilevered from A, due to displacement of A, temperature, loads applied along the member, and the end forces at B and then equate it to the actual $\mathcal{U}^n_B$. The virtual-force system is

$$\begin{aligned} \Delta\bar{\mathbf{P}} &= \Delta\bar{\mathcal{F}}^n_B \\ \Delta\mathbf{R} &= \Delta\bar{\mathcal{F}}^n_A = -\mathcal{X}^n_{BA}\,\Delta\bar{\mathcal{F}}^n_B \\ \Delta\mathcal{F}_Q &= \mathcal{T}^{nq}_{BQ}\,\Delta\bar{\mathcal{F}}^n_B = \mathcal{R}^{nq}\mathcal{X}^n_{BQ}\Delta\bar{\mathcal{F}}^n_B \end{aligned} \tag{a}$$

Also,

$$\begin{aligned} \bar{\mathbf{d}} &= \mathcal{U}^n_A \\ \mathbf{d} &= \mathcal{U}^n_B \end{aligned} \tag{b}$$

Introducing (a), (b) in (15–41), we obtain

$$\begin{gathered} (\Delta\bar{\mathcal{F}}^n_B)^T\mathcal{U}^n_B = (\Delta\bar{\mathcal{F}}^n_B)^T\left[(\mathcal{X}^n_{BA})^T\mathcal{U}^n_A + \int_{S_A}^{S_B}(\mathcal{T}^{nq}_{BQ})^T(\mathcal{E}^0_Q + \mathbf{g}_Q\mathcal{F}_Q)dS\right] \\ \Downarrow \\ \mathcal{U}^n_B = \mathcal{X}^{n,T}_{BA}\mathcal{U}^n_A + \int_{S_A}^{S_B}\mathcal{T}^{nq,T}_{BQ}(\mathcal{E}^0_Q + \mathbf{g}_Q\mathcal{F}_Q)dS \end{gathered} \tag{c}$$

Next, we express $\mathcal{F}_Q$ as

$$\mathcal{F}_Q = \mathcal{F}_{Q,0} + \mathcal{T}^{nq}_{BQ}\bar{\mathcal{F}}^n_B \tag{15-44}$$

where $\mathcal{F}_{Q,0}$ is the internal force matrix at Q due to the prescribed external loading applied to the member cantilevered from A. Finally, substituting for

$\mathcal{F}_Q$ leads to

$$\begin{aligned} \mathcal{U}^n_B = {} & \mathcal{X}^{n,T}_{BA}\mathcal{U}^n_A + \int_{S_A}^{S_B}\mathcal{T}^{nq,T}_{BQ}(\mathcal{E}^0_Q + \mathbf{g}_Q\mathcal{F}_{Q,0})dS \\ & + \left[\int_{S_A}^{S_B}(\mathcal{T}^{nq,T}_{BQ}\mathbf{g}_Q\mathcal{T}^{nq}_{BQ})dS\right]\bar{\mathcal{F}}^n_B \end{aligned} \tag{15-45}$$

The first term is due to rigid body motion of the member about A whereas the second and third terms are due to deformation of the member. We define $\mathcal{V}^n$ as the *member deformation* matrix:

$$\mathcal{V}^n = \mathcal{U}^n_B\big|_{\text{actual}} - \mathcal{U}^n_B\big|_{\substack{\text{due to rigid body} \\ \text{motion about } A}} \tag{15-46}$$

By definition, $\mathcal{V}^n$ is equal to the sum of the second and third terms in (15–45). We also define

$$\begin{aligned} \mathcal{V}^n_0 &= \int_{S_A}^{S_B}\mathcal{T}^{nq,T}_{BQ}(\mathcal{E}^0_Q + \mathbf{g}_Q\mathcal{F}_{Q,0})dS = \text{initial deformation matrix} \\ \mathbf{f}^n &= \int_{S_A}^{S_B}(\mathcal{T}^{nq,T}_{BQ}\mathbf{g}_Q\mathcal{T}^{nq}_{BQ})dS = \text{member flexibility matrix} \end{aligned} \tag{15-47}$$

and (15–45) reduces to

$$\mathcal{V}^n = \mathcal{U}^n_B - \mathcal{X}^{n,T}_{BA}\mathcal{U}^n_A = \mathcal{V}^n_0 + \mathbf{f}^n\bar{\mathcal{F}}^n_B \tag{15-48}$$

Equation (15–48) is the force-displacement relation for an arbitrary member with complete end restraint. It is analogous to the force-elongation relation for the ideal truss element that we developed in Chapter 6.

The member flexibility matrix, $\mathbf{f}^n$, is a *natural* property of the member since it depends only on the geometry and material properties. For simple members such as a prismatic member or a planar circular member with constant cross section, one can obtain the explicit form of $\mathbf{f}$. When the geometry is complex, one must generally resort to numerical integration such as described in Sec. 14–8 in order to determine $\mathbf{f}$ and $\mathcal{V}_0$. This problem is discussed in the next section. Finally, we point out that the general definitions of $\mathbf{f}$, $\mathcal{V}_0$ are also valid for in-plane or out-of-plane deformation of a planar member. One simply has to use the appropriate forms for the various matrices.

Up to this point, we have considered only a simple member. Now suppose the actual member consists of a set of members rigidly connected to each other and the flexibility matrix for each member is known. We can obtain the total flexibility matrix by *compounding* the flexibility matrices for the individual elements. To illustrate the procedure, we consider two members, AA_1 and A_1B, shown in Fig. 15–7.

The matrix, $\mathbf{f}^n$, contains the displacements at B due to the end forces at B with A fixed:

$$\mathcal{U}^n_B = \mathbf{f}^n\bar{\mathcal{F}}^n_B \tag{a}$$

Now, suppose point A_1 is fixed. Then, the displacement at B due to the deformation of member A_1B is

$$\mathcal{U}^n_B\big|_{\substack{\text{member} \\ A_1B}} = \mathbf{f}^n_{A_1B}\bar{\mathcal{F}}^n_B \tag{b}$$

where $\mathbf{f}^n_{A_1B}$ is the flexibility matrix for member A_1B referred to frame n. The additional displacement at B due to movement of A_1 is

$$(\mathscr{U}^n_B)_{\text{displacement at } A_1} = \mathscr{X}^{n,T}_{BA_1}\mathscr{U}^n_{A_1} \tag{c}$$

It remains to determine $\mathscr{U}^n_{A_1}$.

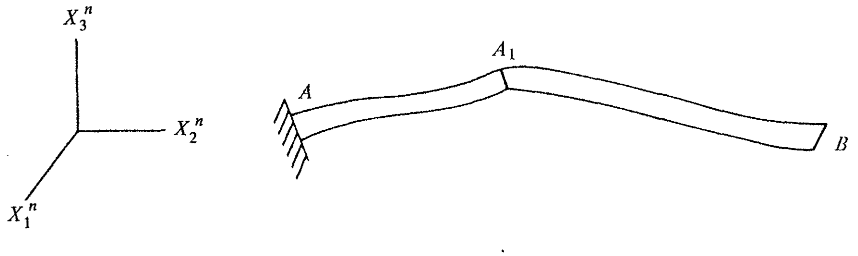


Fig. 15–7. Segmented member.

The force system at A_1 due to the end forces at B is given by

$$\mathscr{F}^n_{A_1} = \mathscr{X}^n_{BA_1}\bar{\mathscr{F}}^n_B \tag{d}$$

and the resulting deformation of member AA_1 is

$$\mathscr{U}^n_{A_1}\big|_{\text{member } AA_1} = \mathbf{f}^n_{AA_1}\mathscr{F}^n_{A_1} = \mathbf{f}^n_{AA_1}\mathscr{X}^n_{BA_1}\bar{\mathscr{F}}^n_B \tag{e}$$

Finally, we have

$$\mathscr{U}^n_B = (\mathbf{f}^n_{A_1B} + \mathscr{X}^{n,T}_{BA_1}\mathbf{f}^n_{AA_1}\mathscr{X}^n_{BA_1})\bar{\mathscr{F}}^n_B = \mathbf{f}^n\bar{\mathscr{F}}^n_B \tag{15–49}$$

The end forces at B are found by inverting (15–48):

$$\mathbf{k}^n = (\mathbf{f}^n)^{-1} = \text{member stiffness matrix}$$

$$\begin{aligned}\bar{\mathscr{F}}^n_B &= \mathbf{k}^n(\mathscr{V}^n - \mathscr{V}^n_0)\\ &= -\mathbf{k}^n\mathscr{V}^n_0 + \mathbf{k}^n\mathscr{U}^n_B - \mathbf{k}^n\mathscr{X}^{n,T}_{BA}\mathscr{U}^n_A\end{aligned} \tag{15–50}$$

The first term is due to external load applied along the member and represents the initial (or fixed-end) forces at B. For convenience, let

$$\bar{\mathscr{F}}^n_{B,i} = -\mathbf{k}^n\mathscr{V}^n_0 \tag{15–51}$$

The second and third terms are the end forces at B due to end displacement at B, A. Once $\bar{\mathscr{F}}^n_B$ is known, we can evaluate the interior force matrix at a point using (15–44),

$$\mathscr{F}_Q = \mathscr{F}_{Q,0} + \mathscr{T}^{nq}_{BQ}\bar{\mathscr{F}}^n_B \tag{a}$$

Thus, the analysis of a completely restrained member reduces to a set of matrix multiplications once the member stiffness and initial deformation matrices are established.

When analyzing a system of members by the displacement method, expressions for the end forces in terms of the end displacements are required. In addition to (15–50), we need an expression for $\bar{\mathscr{F}}^n_A$. Now,

$$\bar{\mathscr{F}}^n_A = -\mathscr{F}^n_A = -\mathscr{F}^n_{A,0} - \mathscr{X}^n_{BA}\bar{\mathscr{F}}^n_B \tag{b}$$

Substituting for $\bar{\mathscr{F}}^n_B$, leads to

$$\begin{aligned}\bar{\mathscr{F}}^n_A &= \bar{\mathscr{F}}^n_{A,i} - \mathscr{X}^n_{BA}\mathbf{k}^n\mathscr{U}^n_B + \mathscr{X}^n_{BA}\mathbf{k}^n\mathscr{X}^{n,T}_{BA}\mathscr{U}^n_A\\ \bar{\mathscr{F}}^n_{A,i} &= -\mathscr{F}^n_{A,0} - \mathscr{X}^n_{BA}\bar{\mathscr{F}}^n_{B,i}\end{aligned} \tag{15–52}$$

where $\bar{\mathscr{F}}^n_{A,i}$ represents the initial end forces. In order to express the equations in a more compact form, we let

$$\begin{aligned}\mathbf{k}^n_{BB} &= \mathbf{k}^n\\ \mathbf{k}^n_{BA} &= -\mathbf{k}^n\mathscr{X}^{n,T}_{BA}\\ \mathbf{k}^n_{AB} &= (\mathbf{k}^n_{BA})^T = -\mathscr{X}^n_{BA}\mathbf{k}^n\\ \mathbf{k}^n_{AA} &= \mathscr{X}^n_{BA}\mathbf{k}^n\mathscr{X}^{n,T}_{BA} = -\mathscr{X}^n_{BA}\mathbf{k}^n_{BA}\end{aligned} \tag{15–53}$$

With this notation, the force-displacement relations simplify to

$$\begin{aligned}\bar{\mathscr{F}}^n_B &= \bar{\mathscr{F}}^n_{B,i} + \mathbf{k}^n_{BB}\mathscr{U}^n_B + \mathbf{k}^n_{BA}\mathscr{U}^n_A\\ \bar{\mathscr{F}}^n_A &= \bar{\mathscr{F}}^n_{A,i} + \mathbf{k}^n_{AB}\mathscr{U}^n_B + \mathbf{k}^n_{AA}\mathscr{U}^n_A\end{aligned} \tag{15–54}$$

Note that only $\mathbf{k}^n$ and $\mathscr{X}^n_{BA}$ are required in order to evaluate $\mathbf{k}^n_{BA}$ and $\mathbf{k}^n_{AA}$.

15–8. GENERATION OF MEMBER MATRICES

The member flexibility matrix is defined by

$$\mathbf{f}^n = \int_{S_A}^{S_B}(\mathscr{T}^{nq,T}_{BQ}\mathbf{g}_Q\mathscr{T}^{nq}_{BQ})dS \tag{a}$$

Noting that

$$\mathscr{T}^{nq}_{BQ} = \mathscr{R}^{nq}\mathscr{X}^n_{BQ} \tag{b}$$

and letting

$$\mathbf{g}^n_Q = \mathscr{R}^{nq,T}\mathbf{g}_Q\mathscr{R}^{nq} \tag{15–55}$$

we can write

$$\mathbf{f}^n = \int_{S_A}^{S_B}(\mathscr{X}^{n,T}_{BQ}\mathbf{g}^n_Q\mathscr{X}^n_{BQ})dS \tag{15–56}$$

If numerical integration is used, the values of the integral at intermediate points along the centroidal axis as well as the total integral can be determined in the same operation. This is desirable since, as we shall show later, the intermediate values can be utilized to evaluate the initial deformation matrix.

We consider next the initial deformation matrix:

$$\mathscr{V}^n_0 = \int_{S_A}^{S_B}\mathscr{T}^{nq,T}_{BQ}(\mathscr{E}^0_Q + \mathbf{g}_Q\mathscr{F}_{Q,0})dS \tag{c}$$

We transform $\mathscr{E}$, $\mathbf{g}$, and $\mathscr{F}$ from the local frame to the basic frame, using (15–55) and

$$\mathscr{F}_Q = \mathscr{R}^{nq}\mathscr{F}^n_Q \qquad \mathscr{E}^n_Q = \mathscr{R}^{nq,T}\mathscr{E}_Q \tag{d}$$

The contributions of temperature and external load are

$$(\mathscr{V}^n_0)_{\text{temp.}} = \int_{S_A}^{S_B} (\mathscr{X}^{n,T}_{BQ}\mathscr{R}^{nq,T}\mathscr{E}^0_Q)dS \tag{15–57}$$

$$(\mathscr{V}^n_0)_{\text{load}} = \int_{S_A}^{S_B} (\mathscr{X}^{n,T}_{BQ}\mathbf{g}^n_Q\mathscr{F}^n_{Q,0})dS \tag{15–58}$$

Suppose there is an external force system applied at an intermediate point, say C. Let $\mathbf{P}_C$, $\mathbf{T}_C$ denote the force and moment matrices and $\mathscr{P}_C$ the total force matrix:

$$\mathscr{P}_C = \begin{Bmatrix}\mathbf{P}_C\\ \mathbf{T}_C\end{Bmatrix} \tag{15–59}$$

Normally, the external force quantities are referred to the basic frame for the member, i.e., frame n. The initial force matrix at Q due to this loading is given by

$$\mathscr{F}^n_{Q,0} = \mathscr{X}^n_{CQ}\mathscr{P}^n_C \qquad S_A \leqslant S_Q \leqslant S_C$$
$$\mathscr{F}^n_{Q,0} = \mathbf{0} \qquad S_C \leqslant S_Q \leqslant S_B \tag{15–60}$$

Writing

$$\mathscr{X}^n_{CQ} = \mathscr{X}^n_{BQ}\mathscr{X}^n_{CB} \tag{e}$$

and introducing the above relations in (15–59) result in

$$(\mathscr{V}^n_0)_{\mathscr{P}^n_C} = \left[\int_{S_A}^{S_C} (\mathscr{X}^{n,T}_{BQ}\mathbf{g}^n_Q\mathscr{X}^n_{BQ})dS\right](\mathscr{X}^n_{CB}\mathscr{P}^n_C) \tag{f}$$

The bracketed term is an intermediate value of the integral defining $\mathbf{f}^n$. Finally, we let

$$\mathbf{J}_P = \int_{S_A}^{S_P} (\mathscr{X}^{n,T}_{BQ}\mathbf{g}^n_Q\mathscr{X}^n_{BQ})dS \tag{15–61}$$

With this notation, (f) simplifies to

$$(\mathscr{V}^n_0)_{\mathscr{P}^n_C} = (\mathbf{J}_C\mathscr{X}^n_{CB})\mathscr{P}^n_C \tag{15–62}$$

Also,

$$\mathbf{f}^n = \mathbf{J}_B \tag{15–63}$$

The determination of the member flexibility matrix reduces to evaluating $\mathbf{J}$ defined by (15–61). One can work with unpartitioned matrices, i.e., $\mathscr{X}$, $\mathbf{g}$, but it is more convenient to express the integrand in partitioned form. The partitioning is consistent with the partitioning of $\mathscr{F}$ into $\mathbf{F}$, $\mathbf{M}$. Since the formulation is applicable for arbitrary deformation, it is desirable to maintain this generality when expanding $\mathscr{X}$, $\mathscr{R}$ in partitioned form. Therefore, we define α as the row order of $\mathbf{F}$ and β as the row order of $\mathbf{M}$.

$$\mathscr{F} = \begin{Bmatrix}\mathbf{F}\\ \mathbf{M}\end{Bmatrix} \begin{matrix}(\alpha\times 1)\\ (\beta\times 1)\end{matrix} \tag{15–64}$$

Continuing, we partition $\mathscr{X}$, $\mathscr{R}$ and $\mathbf{g}$ symmetrically, consistent with (15–64),

and simplify the notation somewhat:

$$\mathscr{X}^n_{BQ} = \left[\begin{array}{c|c}\mathbf{I}_\alpha & \mathbf{0}\\ \hline \underset{(\beta\times\alpha)}{\mathbf{X}^n_{BQ}} & \mathbf{I}_\beta\end{array}\right]$$

$$\mathscr{R}^{nq} = \left[\begin{array}{c|c}\overset{(\alpha\times\alpha)}{\mathbf{R}_\alpha} & \mathbf{0}\\ \hline \mathbf{0} & \underset{(\beta\times\beta)}{\mathbf{R}_\beta}\end{array}\right] \tag{15–65}$$

$$\mathbf{g}^{()}_Q = \left[\begin{array}{c|c}\overset{(\alpha\times\alpha)}{\mathbf{g}^{()}_{11}} & \overset{(\alpha\times\beta)}{\mathbf{g}^{()}_{12}}\\ \hline \mathbf{g}^{(),T}_{12} & \underset{(\beta\times\beta)}{\mathbf{g}^{()}_{22}}\end{array}\right]$$

The translation and rotation transformation matrices are developed in Secs. 5–1, 5–2 and the form of $\mathbf{g}$ for a *thin* curved member is given by (15–38).

The local flexibility matrix $\mathbf{g}^n_Q$ is defined by (15–55). Using the above notation, the expressions for the submatrices are

$$\mathbf{g}^n_{11} = \mathbf{R}^T_\alpha\mathbf{g}_{11}\mathbf{R}_\alpha$$
$$\mathbf{g}^n_{12} = \mathbf{R}^T_\alpha\mathbf{g}_{12}\mathbf{R}_\beta \tag{15–66}$$
$$\mathbf{g}^n_{22} = \mathbf{R}^T_\beta\mathbf{g}_{22}\mathbf{R}_\beta$$

Note that $\mathbf{g}_{12} = \mathbf{0}$ and $\mathbf{g}_{11}$, $\mathbf{g}_{22}$ are diagonal matrices when the shear center coincides with the centroid. If, in addition, axial and shear deformation are neglected, $\mathbf{g}_{11} = \mathbf{0}$.

We let

$$\boldsymbol{\psi} = \mathscr{X}^{n,T}_{BQ}\mathbf{g}^n_Q\mathscr{X}^n_{BQ} = \left[\begin{array}{c|c}\overset{(\alpha\times\alpha)}{\boldsymbol{\psi}_{11}} & \overset{(\alpha\times\beta)}{\boldsymbol{\psi}_{12}}\\ \hline \boldsymbol{\psi}^T_{12} & \underset{(\beta\times\beta)}{\boldsymbol{\psi}_{22}}\end{array}\right] \tag{15–67}$$

The submatrices follow from (15–65):

$$\boldsymbol{\psi}_{11} = \mathbf{g}^n_{11} + \mathbf{g}^n_{12}\mathbf{X}^n_{BQ} + (\mathbf{g}^n_{12}\mathbf{X}^n_{BQ})^T + \mathbf{X}^{n,T}_{BQ}\mathbf{g}^n_{22}\mathbf{X}^n_{BQ}$$
$$\boldsymbol{\psi}_{12} = \mathbf{g}^n_{12} + \mathbf{X}^{n,T}_{BQ}\mathbf{g}^n_{22} \tag{15–68}$$
$$\boldsymbol{\psi}_{22} = \mathbf{g}^n_{22}$$

Next, we partition $\mathbf{J}$ consistent with $\boldsymbol{\psi}$:

$$\mathbf{J}_P = \int_{S_A}^{S_P} \boldsymbol{\psi}\,dS = \left[\begin{array}{c|c}\overset{(\alpha\times\alpha)}{\mathbf{J}_{P,11}} & \overset{(\alpha\times\beta)}{\mathbf{J}_{P,12}}\\ \hline \mathbf{J}^T_{P,12} & \underset{(\beta\times\beta)}{\mathbf{J}_{P,22}}\end{array}\right]$$
$$\mathbf{J}_{P,ij} = \int_{S_A}^{S_P} \boldsymbol{\psi}_{ij}\,dS \tag{15–69}$$

Finally, we partition $\mathbf{f}^n$:

$$\mathbf{f}^n = \left[\begin{array}{c|c}\overset{(\alpha\times\alpha)}{\mathbf{f}^n_{11}} & \overset{(\alpha\times\beta)}{\mathbf{f}^n_{12}}\\ \hline \mathbf{f}^{n,T}_{12} & \underset{(\beta\times\beta)}{\mathbf{f}^n_{22}}\end{array}\right] = \mathbf{J}_B$$
$$\mathbf{f}^n_{ij} = \mathbf{J}_{B,ij} = \int_{S_A}^{S_B} \boldsymbol{\psi}_{ij}\,dS \tag{15–70}$$

The initial deformation matrix due to an arbitrary loading at point C can be determined with (15–62). Its partitioned form is

$$(\mathscr{V}^n_0)_{\mathscr{P}^n_C} = (\mathbf{J}_C \mathscr{X}^n_{CB})\mathscr{P}^n_C$$

$$\Downarrow$$

$$\begin{Bmatrix} \mathbf{v}^n_0 \\ \boldsymbol{\theta}^n_0 \end{Bmatrix} = \begin{bmatrix} \overset{(\alpha\times\alpha)}{\mathbf{J}_{C,11} - \mathbf{J}_{C,12}\mathbf{X}^n_{BC}} & \overset{(\alpha\times\beta)}{\mathbf{J}_{C,12}} \\ \mathbf{J}^T_{C,12} - \mathbf{J}_{C,22}\mathbf{X}^n_{BC} & \underset{(\beta\times\beta)}{\mathbf{J}_{C,22}} \end{bmatrix} \begin{Bmatrix} \overset{(\alpha\times 1)}{\mathbf{P}^n_C} \\ \underset{(\beta\times 1)}{\mathbf{T}^n_C} \end{Bmatrix} \qquad (15\text{–}71)$$

where $\mathbf{v}^n_0$, $\boldsymbol{\theta}^n_0$ denote the initial translation and rotation matrices.

The member stiffness matrix, $\mathbf{k}^n$, is obtained by inverting $\mathbf{f}^n$. We write

$$\mathbf{k}^n = (\mathbf{f}^n)^{-1} = \begin{bmatrix} \overset{(\alpha\times\alpha)}{\mathbf{k}^n_{11}} & \overset{(\alpha\times\beta)}{\mathbf{k}^n_{12}} \\ \mathbf{k}^{n,T}_{12} & \underset{(\beta\times\beta)}{\mathbf{k}^n_{22}} \end{bmatrix} \qquad (15\text{–}72)$$

One can easily show that (we drop the frame superscript on $\mathbf{f}^n_{ij}$ for convenience)

$$\begin{aligned} \mathbf{k}^n_{11} &= (\mathbf{f}_{11} - \mathbf{f}_{12}\mathbf{f}^{-1}_{22}\mathbf{f}^T_{12})^{-1} \\ \mathbf{k}^n_{12} &= -\mathbf{k}^n_{11}\mathbf{f}_{12}\mathbf{f}^{-1}_{22} \\ \mathbf{k}^n_{22} &= \mathbf{f}^{-1}_{22}(\mathbf{I}_\beta - \mathbf{f}^T_{12}\mathbf{k}^n_{12}) \end{aligned} \qquad (15\text{–}73)$$

Once $\mathbf{k}^n$ is known, the stiffness matrices $\mathbf{k}^n_{BB}$, $\mathbf{k}^n_{BA}$ and $\mathbf{k}^n_{AA}$ can be generated. Expanding (15–53) leads to the following partitioned forms:

$$\begin{aligned} \mathbf{k}^n_{BB} &= \mathbf{k}^n = \begin{bmatrix} \mathbf{k}^n_{11} & \mathbf{k}^n_{12} \\ \mathbf{k}^{n,T}_{12} & \mathbf{k}^n_{22} \end{bmatrix} \\ \mathbf{k}^n_{BA} &= -\begin{bmatrix} \mathbf{k}^n_{11} & \mathbf{k}^n_{11}\mathbf{X}^{n,T}_{BA} + \mathbf{k}^n_{12} \\ \mathbf{k}^{n,T}_{12} & \mathbf{k}^{n,T}_{12}\mathbf{X}^{n,T}_{BA} + \mathbf{k}^n_{22} \end{bmatrix} = -\begin{bmatrix} \mathbf{k}^n_{11} & \mathbf{A} \\ \mathbf{k}^{n,T}_{12} & \mathbf{B} \end{bmatrix} \\ \mathbf{k}^n_{AA} &= \begin{bmatrix} \mathbf{k}^n_{11} & \mathbf{A} \\ \mathbf{A}^T & \mathbf{X}^n_{BA}\mathbf{A} + \mathbf{B} \end{bmatrix} = \begin{bmatrix} \mathbf{k}^n_{11} & \mathbf{A} \\ \mathbf{A}^T & \mathbf{C} \end{bmatrix} \end{aligned} \qquad (15\text{–}74)$$

15–9. MEMBER MATRICES—PRISMATIC MEMBER

In Chapter 12, we developed the governing equations for a prismatic member and presented a number of examples which illustrate the *displacement* and *force* methods of solution. Actually, we obtained the complete set of force-displacement relations and also the initial end forces for concentrated and uniform loading. Now, in this section, we generate the member flexibility matrix using the matrix formulation. We also list for future reference the various member stiffness matrices.

The notation is summarized in Fig. 15–8. For convenience, we drop the frame reference superscript n, since the basic frame coincides with the local frame, i.e., $\mathbf{R}^{nq} = \mathbf{I}$. The positive sense of a displacement, external force, or end forces coincides with the positive sense of the corresponding coordinate axis.

Starting with (15–66), we have $\mathbf{g}^n_{ij} = \mathbf{g}_{ij}$, since $\mathbf{R}_\alpha$, $\mathbf{R}_\beta$ are identity matrices. Once $\mathbf{X}_{BQ}$ is assembled we can determine the submatrices of $\boldsymbol{\psi}$ from (15–68).

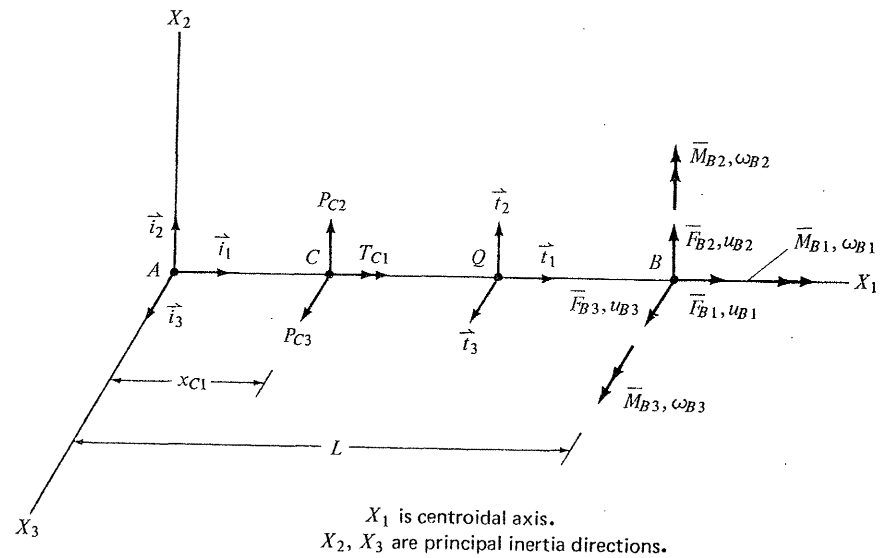


Fig. 15–8. Summary of notation for a prismatic member.

Now,

$$\mathbf{X}_{BQ} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -(L - x_{Q1}) \\ 0 & (L - x_{Q1}) & 0 \end{bmatrix} \qquad \text{(a)}$$

Then, using $\mathbf{g}$ defined by (15–38), we obtain

$$\mathbf{f}_{11} = \begin{bmatrix} L/AE & 0 & 0 \\ & L\left(\dfrac{1}{A_2G} + \dfrac{\bar{x}_3^2}{GJ}\right) + \dfrac{L^3}{3EI_3} & -\dfrac{L}{GJ}\bar{x}_2\bar{x}_3 \\ & \text{Sym} & L\left(\dfrac{1}{A_3G} + \dfrac{\bar{x}_2^2}{GJ}\right) + \dfrac{L^3}{3EI_2} \end{bmatrix}$$

$$\mathbf{f}_{12} = \begin{bmatrix} 0 & 0 & 0 \\ \dfrac{L\bar{x}_3}{GJ} & 0 & \dfrac{L^2}{2EI_3} \\ -\dfrac{L\bar{x}_2}{GJ} & -\dfrac{L^2}{2EI_2} & 0 \end{bmatrix} \qquad (15\text{–}75)$$

$$\mathbf{f}_{22} = \begin{bmatrix} L/GJ & 0 & 0 \\ & L/EI_2 & 0 \\ \text{Sym} & & L/EI_3 \end{bmatrix}$$

The submatrices of $\mathbf{k}$ are generated with (15–73), (15–74) and are listed below for reference. Transverse shear deformation is neglected by setting $a_2 = a_3 = 0$:

$$a_2 = \frac{12EI_2}{GA_3L^2} \qquad a_3 = \frac{12EI_3}{GA_2L^2}$$

$$I_2^* = \frac{I_2}{1 + a_2} \qquad I_3^* = \frac{I_3}{1 + a_3}$$

$$b_1 = \frac{GJ}{L} + \frac{12E}{L^3}(\bar{x}_3^2 I_3^* + \bar{x}_2^2 I_2^*)$$

$$\mathbf{k}_{11} = \begin{bmatrix} \frac{AE}{L} & 0 & 0 \\ & \frac{12EI_3^*}{L^3} & 0 \\ \text{Sym} & & \frac{12EI_2^*}{L^3} \end{bmatrix}$$

$$\mathbf{k}_{12} = \begin{bmatrix} 0 & 0 & 0 \\ -\frac{12EI_3^*\bar{x}_3}{L^3} & 0 & -\frac{6EI_3^*}{L^2} \\ \frac{12EI_2^*\bar{x}_2}{L^3} & \frac{6EI_2^*}{L^2} & 0 \end{bmatrix} \qquad (15\text{–}76)$$

$$\mathbf{k}_{22} = \begin{bmatrix} b_1 & \frac{6EI_2^*\bar{x}_2}{L^2} & \frac{6EI_3^*\bar{x}_3}{L^2} \\ & (4 + a_2)\frac{EI_2^*}{L} & 0 \\ \text{Sym} & & (4 + a_3)\frac{EI_3^*}{L} \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 \\ -\frac{12EI_3^*\bar{x}_3}{L^3} & 0 & \frac{6EI_3^*}{L^2} \\ \frac{12EI_2^*\bar{x}_2}{L^3} & \frac{-6EI_2^*}{L^2} & 0 \end{bmatrix}$$

(change sign of (2, 3) and (3, 2) in $\mathbf{k}_{12}$)

$$\mathbf{B} = \begin{bmatrix} b_1 & \frac{-6EI_2^*\bar{x}_2}{L^2} & \frac{-6EI_3^*\bar{x}_3}{L^2} \\ \frac{6EI_2^*\bar{x}_2}{L^2} & -(2 - a_2)\frac{EI_2^*}{L} & 0 \\ \frac{6EI_3^*\bar{x}_3}{L^2} & 0 & -(2 - a_3)\frac{EI_3^*}{L} \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} b_1 & \frac{-6EI_2^*\bar{x}_2}{L^2} & \frac{-6EI_3^*\bar{x}_3}{L^2} \\ & (4 + a_2)\frac{EI_2^*}{L} & 0 \\ \text{Sym} & & (4 + a_3)\frac{EI_3^*}{L} \end{bmatrix}$$

(change sign of (1, 2), (1, 3) in $\mathbf{k}_{22}$)

Finally, the fixed end forces due to a concentrated transverse force and a uniform transverse loading are summarized below.

Concentrated Force P_{C2}

$$\begin{aligned}
a_3 &= \frac{12EI_3}{GA_2L^3} \\
\bar{x}_C &= \frac{x_{C1}}{L} \\
\bar{M}_{B3} &= LP_{C2}\bar{x}_C(1 - \bar{x}_C)\left(\frac{\bar{x}_C + a_3/2}{1 + a_3}\right) \\
\bar{F}_{B2} &= -(\bar{x}_C)^2 P_{C2} - \frac{2}{L}\bar{M}_{B3} \\
\bar{M}_{B1} &= -\bar{x}_3(\bar{x}_C P_{C2} + \bar{F}_{B2}) \\
\bar{M}_{A1} &= -\bar{M}_{B1} \\
\bar{F}_{A2} &= -P_{C2} - \bar{F}_{B2} \\
\bar{M}_{A3} &= -L\left(\bar{x}_C P_{C2} + \bar{F}_{B2} + \frac{1}{L}\bar{M}_{B3}\right)
\end{aligned} \qquad (15\text{–}77)$$

Concentrated Force P_{C3}

$$\begin{aligned}
a_2 &= \frac{12EI_2}{GA_3L^2} \\
\bar{M}_{B2} &= -LP_{C3}\bar{x}_C(1 - \bar{x}_C)\left(\frac{\bar{x}_C + a_2/2}{1 + a_2}\right) \\
\bar{F}_{B3} &= -P_{C3}(\bar{x}_C)^2 + \frac{2}{L}\bar{M}_{B2} \\
\bar{M}_{B1} &= \bar{x}_2(\bar{x}_C P_{C3} + \bar{F}_{B3}) \\
\bar{M}_{A1} &= -\bar{M}_{B1} \\
\bar{F}_{A3} &= -P_{C3} - \bar{F}_{B3} \\
\bar{M}_{A2} &= L\left(\bar{x}_C P_{C3} + \bar{F}_{B3} - \frac{1}{L}\bar{M}_{B2}\right)
\end{aligned} \qquad (15\text{–}78)$$

Concentrated Torque T_{C1}

$$\begin{aligned} \bar{M}_{B1} &= -T_{C1}\bar{x}_C \\ \bar{M}_{A1} &= -T_{C1}(1 - \bar{x}_C) \end{aligned} \tag{15–79}$$

Uniformly Distributed Load, b_2

$$\begin{aligned} \bar{F}_{B2} &= \bar{F}_{A2} = -\frac{b_2 L}{2} \\ \bar{M}_{B3} &= -\bar{M}_{A3} = \frac{b_2 L^2}{12} \\ \bar{M}_{B1} &= \bar{M}_{A1} = 0 \end{aligned} \tag{15–80}$$

Uniformly Distributed Load, b_3

$$\begin{aligned} \bar{F}_{B3} &= \bar{F}_{A3} = -\frac{b_3 L}{2} \\ \bar{M}_{B2} &= -\bar{M}_{A2} = -\frac{b_3 L^2}{12} \\ \bar{M}_{B1} &= \bar{M}_{A1} = 0 \end{aligned} \tag{15–81}$$

Uniformly Distributed Torque, m_1

$$\bar{M}_{B1} = \bar{M}_{A1} = -\frac{m_1 L}{2} \tag{15–82}$$

15–10. MEMBER MATRICES—THIN PLANAR CIRCULAR MEMBER

In this section, we generate the flexibility and initial deformation matrices for a thin planar circular member, of constant cross section, using matrix operations. We include extensional and transverse shear deformation for the sake of generality. Some of the relations have already been obtained as illustrative examples of the force and displacement methods. In particular, the reader should review Example 14–6, which treats planar deformation, and Examples 15–4, 15–5, 15–10 for out-of-plane deformation.

The notation is summarized in Fig. 15-9. By definition, Y_2 and Y_3 are principal inertia axes and $\bar{y}_3 = 0$, i.e., the shear center lies in the plane containing the centroidal axis. It is convenient to take the basic frame (frame n) to be parallel to the local frame at B. The three-dimensional forms of $\mathbf{R}_\alpha$, $\mathbf{R}_\beta$, and $\mathbf{X}^n_{BQ}$ are

$$\mathbf{R}^{nq} = \mathbf{R}^{bq} = \left[\begin{array}{cc|c} \cos\eta & -\sin\eta & 0 \\ \sin\eta & \cos\eta & 0 \\ \hline 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{c|c} \mathbf{R}_2^{bq} & \mathbf{0} \\ \hline \mathbf{0} & 1 \end{array}\right]$$

$$\mathbf{R}_\alpha = \mathbf{R}_\beta = \mathbf{R}^{nq} \tag{15–83}$$

$$\mathbf{X}^n_{BQ} = \mathbf{X}^b_{BQ} = \left[\begin{array}{c|c|c} 0 & 0 & -R(1-\cos\eta) \\ \hline 0 & 0 & -R\sin\eta \\ \hline R(1-\cos\eta) & R\sin\eta & 0 \end{array}\right]$$

We use

$$\begin{aligned} &\mathbf{R}_\alpha = \mathbf{R}_2^{bq} \qquad \mathbf{R}_\beta = 1 \\ &\mathbf{X}^n_{BQ} = [R(1-\cos\eta) \qquad R\sin\eta] \end{aligned} \tag{a}$$

for planar deformation and

$$\begin{aligned} &\mathbf{R}_\alpha = 1 \qquad \mathbf{R}_\beta = \mathbf{R}_2^{bq} \\ &\mathbf{X}^n_{BQ} = \begin{bmatrix} -R(1-\cos\eta) \\ -R\sin\eta \end{bmatrix} \end{aligned} \tag{b}$$

for out-of-plane deformation. Since the complete flexibility matrix is desired, it is just as convenient to work with submatrices of order 3 as to consider separately the planar and out-of-plane cases.

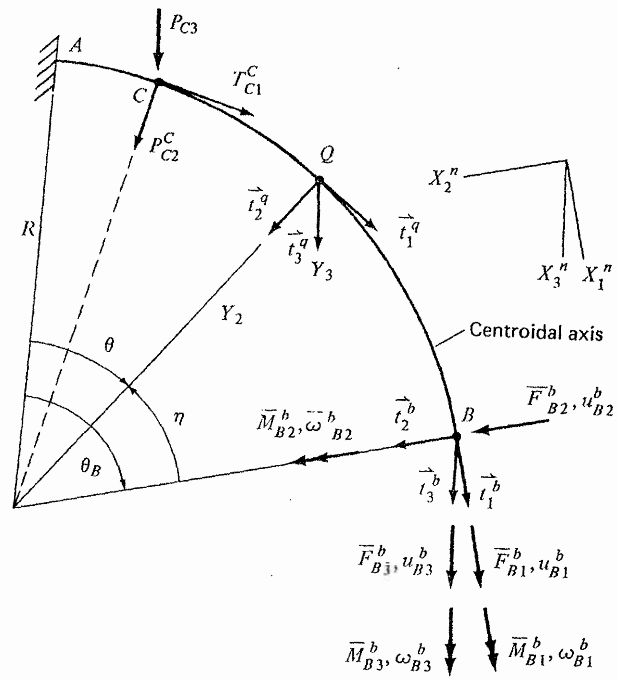


Fig. 15–9. Summary of notation for a planar circular member.

We consider the member to be thin and use the local flexibility matrix defined by (15–38). Expanding (15–66), (15–68) leads to the member flexibility matrix.

$$a_e = \frac{I_3}{AR^2} \qquad a_s = \frac{EI_3}{GA_2R^2}$$

$$c_t = \frac{EI_2}{GJ} \qquad c_s = \frac{EI_2}{GA_3R^2}$$

$$\begin{aligned} a_1 &= a_e + a_s \\ a_2 &= a_e - a_s \\ c_1 &= \tfrac{1}{2}(1 + c_t) \\ c_2 &= \tfrac{1}{2}(1 - c_t) \\ c_3 &= c_t\left(1 - \frac{\bar{y}_2}{R}\right) \\ c_4 &= c_s + c_t\left(1 - \frac{\bar{y}_2}{R}\right)^2 \end{aligned} \tag{15-84}$$

$$\mathbf{f}^b_{11} = \begin{bmatrix} \frac{R^3}{EI_3}\left\{\frac{\theta_B}{2}(3 + a_1) + \sin\theta_B\left[-2 + \frac{\cos\theta_B}{2}(1 + a_2)\right]\right\} & & \text{Symmetrical} \\ \frac{R^3}{EI_3}\left\{1 - \cos\theta_B - \frac{\sin^2\theta_B}{2}(1 + a_2)\right\} & \frac{R^3}{EI_3}\left\{\frac{\theta_B}{2}(1 + a_1) - \frac{1}{2}(1 + a_2)\sin\theta_B\cos\theta_B\right\} & \\ 0 & 0 & \frac{R^3}{EI_2}\{\theta_B(c_1 + c_4) - 2c_3\sin\theta_B - c_2\sin\theta_B\cos\theta_B\} \end{bmatrix}$$

$$\mathbf{f}^b_{12} = (\mathbf{f}^b_{21})^T = \begin{bmatrix} 0 & 0 & \frac{R^2}{EI_3}(\theta_B - \sin\theta_B) \\ 0 & 0 & \frac{R^2}{EI_3}(1 - \cos\theta_B) \\ \frac{R^2}{EI_2}\{-c_1\theta_B + c_3\sin\theta_B + c_2\sin\theta_B\cos\theta_B\} & \frac{R^2}{EI_2}\{-c_2\sin^2\theta_B - c_3(1 - \cos\theta_B)\} & 0 \end{bmatrix}$$

$$\mathbf{f}^b_{22} = \begin{bmatrix} \frac{R}{EI_2}(c_1\theta_B - c_2\sin\theta_B\cos\theta_B) & \text{Symmetrical} & \\ \frac{R}{EI_2}(c_2\sin^2\theta_B) & \frac{R}{EI_2}(c_1\theta_B + c_2\sin\theta_B\cos\theta_B) & \\ 0 & 0 & \frac{R\theta_B}{EI_3} \end{bmatrix}$$

We consider next the determination of the initial deformation matrix due to an arbitrary concentrated load at an interior point, C. Now, the flexibility matrix for the segment AC referred to the local frame at C, which we denote by $\mathbf{f}^c_{AC}$, is known. We just have to change θ_B to θ_C and superscript b to c in (15-84). When the external load is referred to the local frame at C, the displacement at C is given by

$$\mathscr{U}^c_C = \mathbf{f}^c_{AC}\mathscr{P}^c_C \tag{a}$$

The displacement at B due to rigid body motion about C is

$$\mathscr{U}^b_B = \mathscr{X}^{b,T}_{BC}\mathscr{U}^b_C = (\mathscr{X}^{b,T}_{BC}\mathscr{R}^{bc,T})\mathscr{U}^c_C \tag{b}$$

Finally, we can write

$$(\mathscr{V}^b_0)_{\mathscr{P}^c_C} = (\mathscr{U}^b_B)_{\mathscr{P}^c_C} = (\mathscr{X}^{b,T}_{BC}\mathscr{R}^{bc,T}\mathbf{f}^c_{AC})\mathscr{P}^c_C$$

$$\Downarrow$$

$$\begin{aligned} \mathbf{v}^b_0 = \mathbf{u}^b_B &= (\mathbf{R}^T_\alpha\mathbf{f}^c_{AC,11} + \mathbf{X}^{b,T}_{BC}\mathbf{R}^T_\beta\mathbf{f}^{c,T}_{AC,12})\mathbf{P}^c_C \\ &\quad + (\mathbf{R}^T_\alpha\mathbf{f}^c_{AC,12} + \mathbf{X}^{b,T}_{BC}\mathbf{R}^T_\beta\mathbf{f}^c_{AC,22})\mathbf{T}^c_C \\ \boldsymbol{\theta}^b_0 = \boldsymbol{\omega}^b_B &= (\mathbf{R}^T_\beta\mathbf{f}^{c,T}_{AC,12})\mathbf{P}^c_C + (\mathbf{R}^T_\beta\mathbf{f}^c_{AC,22})\mathbf{T}^c_C \end{aligned} \tag{15-85}$$

The uncoupled expressions follow.

Planar Loading

$$\begin{aligned} v^b_{0,1} = u^b_{B1} &= \frac{R^3}{EI_3}\left\{\theta_C\left(1 + \frac{1 + a_1}{2}\cos\eta_C\right) + \sin\eta_C - \sin\theta_B \right. \\ &\quad \left. + \sin\theta_C\left(-1 + \frac{1 + a_2}{2}\cos\theta_B\right)\right\}P^c_{C1} \\ &\quad + \frac{R^3}{EI_3}\left\{1 - \cos\theta_C + \frac{1 + a_1}{2}\theta_C\sin\eta_C \right. \\ &\quad \left. - \frac{1 + a_2}{2}\sin\theta_C\sin\theta_B\right\}P^c_{C2} \\ &\quad + \frac{R^2}{EI_3}\left\{\theta_C + \sin\eta_C - \sin\theta_B\right\}T_{C3} \\ v^b_{0,2} = u^b_{B2} &= \frac{R^3}{EI_3}\left\{-\frac{1 + a_1}{2}\theta_C\sin\eta_C \right. \\ &\quad \left. + \cos\eta_C - \cos\theta_B - \frac{1 + a_2}{2}\sin\theta_C\sin\theta_B\right\}P^c_{C1} \\ &\quad + \frac{R^3}{EI_3}\left\{\frac{1 + a_1}{2}\theta_C\cos\eta_C - \frac{1 + a_2}{2}\sin\theta_C\cos\theta_B\right\}P^c_{C2} \\ &\quad + \frac{R^2}{EI_2}\left\{\cos\eta_C - \cos\theta_B\right\}T_{C3} \\ \theta_{0,3} = \omega_{B3} &= \frac{R^2}{EI_3}\left\{(\theta_C - \sin\theta_C)P^c_{C1} + (1 - \cos\theta_C)P^c_{C2}\right\} + \frac{R\theta_C}{EI_3}T_{C3} \end{aligned} \tag{15-86}$$

Out-of-Plane Loading

$$
\begin{aligned}
v_{0,3} = u_{B3} = {} & \frac{R^3}{EI_2}\Big\{\theta_C(c_1\cos\eta_C + c_4) - c_2\sin\theta_C\cos\theta_B \\
& + c_3(-\sin\theta_C - \sin\theta_B + \sin\eta_C)\Big\}P_{C3} \\
& + \frac{R^2}{EI_2}\Big\{-c_1\theta_C\cos\eta_C + c_2\sin\theta_C\cos\theta_B + c_3\sin\theta_C\Big\}T^c_{C1} \\
& + \frac{R^2}{EI_2}\Big\{-c_1\theta_C\sin\eta_C - c_2\sin\theta_C\sin\theta_B - c_3(1-\cos\theta_C)\Big\}T^c_{C2} \\
\theta^b_{0,1} = \omega^b_{B1} = {} & \frac{R^2}{EI_2}\Big\{-c_1\theta_C\cos\eta_C + c_3(\sin\theta_B - \sin\eta_C) \\
& + c_2\sin\theta_C\cos\theta_B\Big\}P_{C3} \\
& + \frac{R}{EI_2}\Big\{c_1\theta_C\cos\eta_C - c_2\sin\theta_C\cos\theta_B\Big\}T^c_{C1} \\
& + \frac{R}{EI_2}\Big\{c_1\theta_C\sin\eta_C + c_2\sin\theta_C\sin\theta_B\Big\}T^c_{C2} \\
\theta^b_{0,2} = \omega^b_{B2} = {} & \frac{R^2}{EI_2}\Big\{c_1\theta_C\sin\eta_C + c_3(\cos\theta_B - \cos\eta_C) \\
& - c_2\sin\theta_C\sin\theta_B\Big\}P_{C3} \\
& + \frac{R}{EI_2}\Big\{-c_1\theta_C\sin\eta_C + c_2\sin\theta_C\sin\theta_B\Big\}T^c_{C1} \\
& + \frac{R}{EI_2}\Big\{c_1\theta_C\cos\eta_C + c_2\sin\theta_C\cos\theta_B\Big\}T^c_{C2}
\end{aligned}
\tag{15-87}
$$

When the loading is symmetrical, one can utilize symmetry to determine the fixed end forces. The most convenient choice of unknowns is the internal forces at the midpoint, i.e., $\theta = \theta_B/2$; F_1 and M_3 are unknown for the planar case and only M_2 is unknown for the out-of-plane case. Explicit expressions for the fixed end forces due to various loading conditions are listed below.

Planar Loading

Fig. 15–10 defines the notation for the planar case.

We consider two loadings: a concentrated radial force P applied at C, and a uniform distributed radial load b_2 applied per unit arc length over the entire segment. The basic frame is chosen to utilize symmetry. We determine the axial force and moment at C from the symmetry conditions $u_1 = \omega_3 = 0$.

Case 1—Concentrated radial force P

$$F_{C2} = \frac{P}{2} \qquad F_{C1} = \frac{P}{2}\psi$$

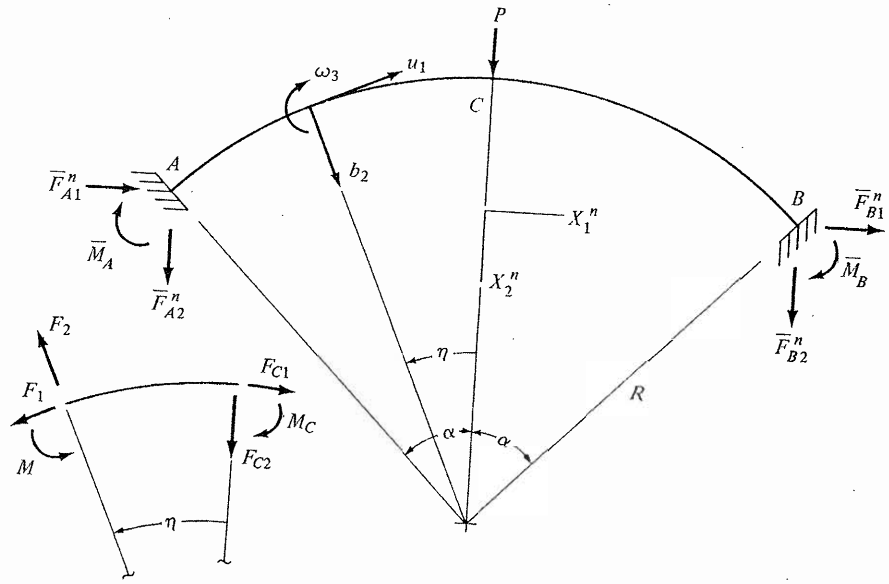


Fig. 15–10. Notation for planar loading.

$$
\begin{aligned}
\psi &= \frac{\dfrac{\sin\alpha}{\alpha}(1-\cos\alpha) + \dfrac{1+a_2}{2}\sin^2\alpha}{\alpha\left(\dfrac{1+a_1}{2}\right) - \dfrac{\sin^2\alpha}{\alpha} + \left(\dfrac{1+a_2}{2}\right)\sin\alpha\cos\alpha} \\
M_C &= -\frac{RP}{2}\left\{\frac{1-\cos\alpha}{\alpha} + \left(1 - \frac{\sin\alpha}{\alpha}\right)\psi\right\} \\
\bar{F}^n_{B1} &= -\bar{F}^n_{A1} = F_{C1} \\
\bar{F}^n_{B2} &= \bar{F}^n_{A2} = -\frac{P}{2} \\
\bar{M}_B &= -\bar{M}_A = \frac{PR}{2}\left\{\sin\alpha - \frac{1-\cos\alpha}{\alpha} + \psi\left(\frac{\sin\alpha}{\alpha} - \cos\alpha\right)\right\}
\end{aligned}
\tag{15-88}
$$

Case 2—Uniform distributed radial load b_2

$$
\begin{aligned}
F_{C2} &= 0 \\
F_{C1} &= -Rb_2(1 - a_e\phi) \\
\phi &= \frac{\sin\alpha}{\alpha\left(\dfrac{1+a_1}{2}\right) - \dfrac{\sin^2\alpha}{\alpha} + \left(\dfrac{1+a_2}{2}\right)\sin\alpha\cos\alpha}
\end{aligned}
$$

$$M_C = R^2 b_2 a_e \phi \left(-1 + \frac{\sin \alpha}{\alpha}\right) \qquad (15\text{-}89)$$

$$\bar{F}^n_{B1} = -\bar{F}^n_{A1} = -Rb_2(\cos\alpha - a_e\phi)$$

$$\bar{F}^n_{B2} = \bar{F}^n_{A2} = -Rb_2 \sin\alpha$$

$$\bar{M}_B = -\bar{M}_A = R^2 b_2 a_e \phi \left[\frac{\sin\alpha}{\alpha} - \cos\alpha\right]$$

Out-of-Plane Loading

Figure 15–11 defines the notation for the out-of-plane case. We consider four loadings: a concentrated force P, and a couple T—both applied at C; a uniform distributed force b_3; and a uniform distributed couple m_1. The bending moment at C is obtained using the symmetry condition $\omega_2 = 0$.

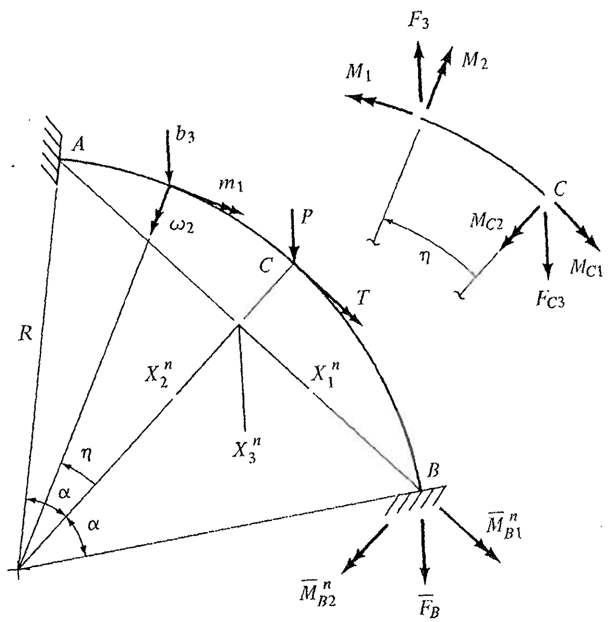


Fig. 15–11. Notation for out-of-plane loading.

CASE 1—CONCENTRATED FORCE P

$$F_{C3} = \frac{P}{2}$$

$$M_{C1} = 0$$

$$M_{C2} = \frac{PR}{2}\,\frac{c_2 \sin^2\alpha + c_3(1 - \cos\alpha)}{c_1\alpha + c_2 \sin\alpha\cos\alpha}$$

$$\bar{M}^n_{B1} = \bar{M}^n_{A1} = \frac{PR}{2}(1 - \cos\alpha) \qquad (15\text{-}90)$$

$$\bar{M}^n_{B2} = -\bar{M}^n_{A2} = M_{C2} - \frac{PR}{2}\sin\alpha$$

$$\bar{F}_B = \bar{F}_A = -\frac{P}{2}$$

CASE 2—CONCENTRATED TORQUE T

$$F_{C3} = 0$$

$$M_{C1} = \frac{T}{2}$$

$$M_{C2} = -\frac{T}{2}\,\frac{c_2\sin^2\alpha}{\alpha c_1 + c_2\sin\alpha\cos\alpha} \qquad (15\text{-}91)$$

$$\bar{M}^n_{B1} = \bar{M}^n_{A1} = -\frac{T}{2}$$

$$\bar{M}^n_{B2} = -\bar{M}^n_{A2} = M_{C1}$$

$$\bar{F}_B = \bar{F}_A = 0$$

CASE 3—UNIFORM DISTRIBUTED LOAD b_3

$$F_{C3} = M_{C1} = 0$$

$$M_{C2} = R^2 b_3 \frac{c_1(\sin\alpha - \alpha) + c_2\sin\alpha(1 - \cos\alpha) + c_3(\sin\alpha - \alpha\cos\alpha)}{\alpha c_1 + c_2\sin\alpha\cos\alpha} \qquad (15\text{-}92)$$

$$\bar{M}^n_{B1} = \bar{M}^n_{A1} = R^2 b_3(\sin\alpha - \alpha\cos\alpha)$$

$$\bar{M}^n_{B2} = -\bar{M}^n_{A2} = M_{C2} - R^2 b_3(\alpha\sin\alpha - 1 + \cos\alpha)$$

$$\bar{F}_B = \bar{F}_A = -PR\alpha$$

CASE 4—UNIFORM DISTRIBUTED COUPLE m_1

$$F_{C3} = M_{C1} = 0$$

$$M_{C2} = m_1 R \frac{c_1(\alpha - \sin\alpha) + c_2\sin\alpha(\cos\alpha - 1)}{\alpha c_1 + c_2\sin\alpha\cos\alpha} \qquad (15\text{-}93)$$

$$\bar{M}^n_{B1} = \bar{M}^n_{A1} = -m_1 R\sin\alpha$$

$$\bar{M}^n_{B2} = -\bar{M}^n_{A2} = M_{C2} - m_1 R(1 - \cos\alpha)$$

15–11. FLEXIBILITY MATRIX—CIRCULAR HELIX

In this section, we develop the flexibility matrix for a member whose centroidal axis is a circular helix. The notation is shown in Fig. 15–12. The principal inertia direction, Y_2, is considered to *coincide* with the normal direction, i.e., the inward radial direction, at each point. We also suppose the cross-sectional properties are constant. For convenience, we summarize the geometrical relations:†

$$
\begin{aligned}
x_1 &= R\cos\theta \\
x_2 &= R\sin\theta \\
x_3 &= C\theta \\
dS &= \alpha\, d\theta \\
\alpha &= [R^2 + C^2)^{1/2} = \text{constant} \\
\vec{t} &= \vec{t}_1 = \frac{1}{\alpha}(-R\sin\theta\vec{\imath}_1 + R\cos\theta\vec{\imath}_2 + C\vec{\imath}_3) \\
\vec{n} &= \vec{t}_2 = -\cos\theta\vec{\imath}_1 - \sin\theta\vec{\imath}_2 \\
\vec{b} &= \vec{t}_3 = \frac{1}{\alpha}(C\sin\theta\vec{\imath}_1 - C\cos\theta\vec{\imath}_2 + R\vec{\imath}_3)
\end{aligned}
\tag{a}
$$

$$
\mathbf{R}_\alpha = \mathbf{R}_\beta = \mathbf{R}^{nq} = \left[\begin{array}{c|c|c}
-\dfrac{R}{\alpha}\sin\theta & \dfrac{R}{\alpha}\cos\theta & \dfrac{C}{\alpha} \\ \hline
-\cos\theta & -\sin\theta & 0 \\ \hline
\dfrac{C}{\alpha}\sin\theta & -\dfrac{C}{\alpha}\cos\theta & \dfrac{R}{\alpha}
\end{array}\right]
$$

$$
\mathbf{X}^n_{BQ} = \left[\begin{array}{c|c|c}
0 & -C(\theta_B - \theta) & R(\sin\theta_B - \sin\theta) \\
C(\theta_B - \theta) & 0 & -R(\cos\theta_B - \cos\theta) \\
-R(\sin\theta_B - \sin\theta) & R(\cos\theta_B - \cos\theta) & 0
\end{array}\right]
$$

The steps involve only algebraic operations and integration. We first determine $\mathbf{g}^n_{ij}$ using (15–66), then $\boldsymbol{\psi}_{ij}$ from (15–68), and finally $\mathbf{f}^n_{ij}$ with (15–70). In what follows, we assume the shear center coincides with the centroid and neglect extensional and transverse shear deformation. With these restrictions,

$$
\mathbf{g}^n_{11} = \mathbf{g}^n_{12} = \mathbf{0} \qquad \mathbf{g}^n_{22} = \mathbf{R}^{nq,\,T}\bar{\mathbf{g}}_2\mathbf{R}^{nq} \tag{b}
$$

$$
\bar{\mathbf{g}}_2 = \begin{bmatrix} \dfrac{1}{GJ} & & \\ & \dfrac{1}{EI_2} & \\ & & \dfrac{1}{EI_3} \end{bmatrix}
$$

and the expressions for $\boldsymbol{\psi}_{ij}$ reduce to

$$
\begin{aligned}
\boldsymbol{\psi}_{22} &= \mathbf{g}^n_{22} \\
\boldsymbol{\psi}_{12} &= \mathbf{X}^{n,\,T}_{BQ}\boldsymbol{\psi}_{22} \\
\boldsymbol{\psi}_{11} &= \boldsymbol{\psi}_{12}\mathbf{X}^n_{BQ}
\end{aligned}
\tag{c}
$$

The flexibility matrix for a constant cross section is given below.

† See Examples 4–6 and 5–3.

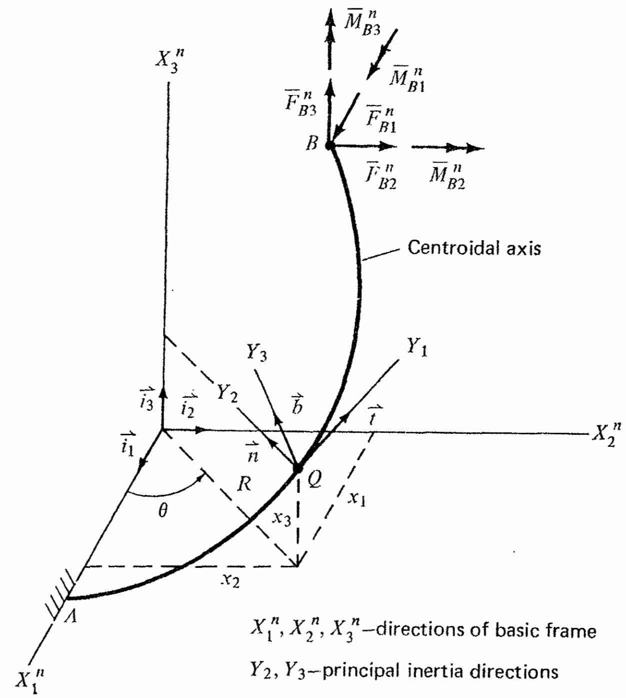


Fig. 15–12. Notation for circular helix.

Notation—Dimensionless Parameters

$$
\begin{aligned}
a_1 &= \frac{R^2}{\alpha^2}\frac{EI_2}{GJ} + \frac{C^2}{\alpha^2}\frac{I_2}{I_3} \\
a_2 &= \frac{R^2}{\alpha^2} + \frac{C^2}{\alpha^2}\left(\frac{EI_2}{GJ}\right)\frac{I_3}{I_2} \\
a_3 &= \frac{RC}{\alpha^2}\left[\frac{I_2}{I_3} - \frac{EI_2}{GJ}\right] \\
a_4 &= \frac{Ra_3}{C} = \frac{R^2}{\alpha^2}\left[\frac{I_2}{I_3} - \frac{EI_2}{GJ}\right]
\end{aligned}
$$

$$
a_5 = \frac{1 + a_1}{2} \qquad a_6 = \frac{1 - a_1}{2}
$$

$$
a_7 = \frac{a_4 + a_6}{2} \qquad a_8 = \frac{a_4 - a_6}{2}
$$

$$
a_9 = \frac{a_6 + 3a_4}{2} \qquad a_{10} = \frac{a_6 - 3a_4}{2}
$$

Elements of $\mathbf{f}^n_{11}$

$$\mathbf{f}^n_{11} = \begin{bmatrix} f_{11} & \text{Sym} & \\ f_{21} & f_{22} & \\ f_{31} & f_{32} & f_{33} \end{bmatrix}$$

$$f_{11} = \frac{C^2\alpha}{EI_2}\left\{-a_7\theta_B + \frac{a_5}{3}\theta_B^3 + 2a_4 \sin\theta_B + a_{10}\sin\theta_B\cos\theta_B\right\} + \frac{R^2 a_2\alpha}{EI_3}\left\{\theta_B\left(\frac{1}{2} + \sin^2\theta_B\right) - 2\sin\theta_B + \frac{3}{2}\sin\theta_B\cos\theta_B\right\}$$

$$f_{21} = \frac{C^2\alpha}{EI_2}\left\{-\frac{a_6}{2}\theta_B^2 - a_8\sin^2\theta_B + a_4(\theta_B\sin\theta_B - \cos\theta_B + \cos^2\theta_B)\right\} - \frac{R^2 a_2\alpha}{EI_3}\left\{\cos\theta_B(\theta_B\sin\theta_B - 1 + \cos\theta_B) - \frac{1}{2}\sin^2\theta_B\right\}$$

$$f_{31} = \frac{RC\alpha}{EI_2}\left\{-\frac{a_5}{2}\theta_B^2\cos\theta_B + (a_1 - a_4)(1 - \cos\theta_B) - a_4\sin^2\theta_B + a_9\theta_B\sin\theta_B\right\}$$

$$f_{22} = \frac{C^2\alpha}{EI_2}\left\{a_7\theta_B + \frac{a_5}{3}\theta_B^3 - a_{10}\sin\theta_B\cos\theta_B - 2a_4\theta_B\cos\theta_B\right\} + \frac{R^2 a_2\alpha}{EI_3}\left\{\theta_B\left(\frac{1}{2} + \cos^2\theta_B\right) - \frac{3}{2}\cos\theta_B\sin\theta_B\right\}$$

$$f_{32} = \frac{RC\alpha}{EI_2}\left\{-\frac{a_5}{2}\theta_B^2\sin\theta_B + a_1(\theta_B - \sin\theta_B) + a_{10}\theta_B\cos\theta_B + a_8\sin\theta_B + a_4\sin\theta_B\cos\theta_B\right\}$$

$$f_{33} = \frac{R^2\alpha}{EI_2}\left\{(a_1 + a_5)\theta_B - 2a_1\sin\theta_B - a_6\sin\theta_B\cos\theta_B\right\} \qquad (15\text{–}94)$$

Elements of $\mathbf{f}^n_{12}$

$$\mathbf{f}^n_{12} = \begin{bmatrix} f_{14} & f_{15} & f_{16} \\ f_{24} & f_{25} & f_{26} \\ f_{34} & f_{35} & f_{36} \end{bmatrix}$$

$$f_{14} = \frac{C\alpha}{EI_2}\left\{-a_4\sin\theta_B + a_7\theta_B + a_8\sin\theta_B\cos\theta_B\right\}$$

$$f_{15} = \frac{C\alpha}{EI_2}\left\{\frac{a_5}{2}\theta_B^2 + a_8\sin^2\theta_B\right\}$$

$$f_{16} = -\frac{Ca_3\alpha}{EI_2}\left\{1 - \cos\theta_B\right\} - \frac{Ra_2\alpha}{EI_3}\left\{\theta_B\sin\theta_B - 1 + \cos\theta_B\right\}$$

$$f_{24} = \frac{C\alpha}{EI_2}\left\{-\frac{a_5}{2}\theta_B^2 + a_8\sin^2\theta_B - a_4(1 - \cos\theta_B)\right\}$$

$$f_{25} = \frac{C\alpha}{EI_2}\left\{a_8(\theta_B - \sin\theta_B\cos\theta_B)\right\}$$

$$f_{26} = -\frac{Ca_3\alpha}{EI_2}\left\{\theta_B - \sin\theta_B\right\} + \frac{Ra_2\alpha}{EI_3}\left\{\theta_B\cos\theta_B - \sin\theta_B\right\}$$

$$f_{34} = \frac{R\alpha}{EI_2}\left\{a_5\theta_B\sin\theta_B - a_1(1 - \cos\theta_B)\right\}$$

$$f_{35} = \frac{R\alpha}{EI_2}\left\{a_5(\sin\theta_B - \theta_B\cos\theta_B)\right\}$$

$$f_{36} = \frac{Ca_4\alpha}{EI_2}\left\{\sin\theta_B - \theta_B\right\}$$

Elements of $\mathbf{f}^n_{22}$

$$\mathbf{f}^n_{22} = \begin{bmatrix} \frac{\alpha}{EI_2}\{a_5\theta_B + a_6\sin\theta_B\cos\theta_B\} & \text{Sym} & \\ \frac{\alpha a_6}{EI_2}\sin^2\theta_B & \frac{\alpha}{EI_2}\{a_5\theta_B - a_6\sin\theta_B\cos\theta_B\} & \\ \frac{\alpha a_3}{EI_2}(1 - \cos\theta_B) & -\frac{\alpha a_3}{EI_2}\sin\theta_B & \frac{\alpha a_2}{EI_3}\theta_B \end{bmatrix}$$

15–12. MEMBER FORCE-DISPLACEMENT RELATIONS—PARTIAL END RESTRAINT

In Sec. 15–7, we considered an arbitrary member which is completely restrained at both ends. This led to the definition of the member flexibility matrix and a set of equations relating the end forces and the end displacements. Now, when the member is only partially restrained, there is a reduction in the number of member force unknowns. For example, if there is no restraint against rotation at B, $\mathbf{M}_B = \mathbf{0}$, and there are only α unknowns (where α is the order of $\bar{\mathbf{F}}_B$), the rotation $\boldsymbol{\omega}_B$ at B has no effect on the end forces. To handle the case of partial restraint, we first determine the compatibility equations corresponding to the reduced set of force unknowns. Inverting these equations and using the equilibrium relations for the end forces results in force-displacement relations which are consistent with the displacement releases.

Let $\mathbf{Z}$ denote the force redundants. Normally, one would work with the primary structure corresponding to $\mathbf{Z} = \mathbf{0}$. However, suppose we first express the force at a point, say Q, in terms of the end forces at B, *using, as a primary structure, the member cantilevered from A*:

$$\begin{aligned} \mathscr{F}_Q &= \mathscr{F}_{Q,0} + \mathscr{T}^{nq}_{BQ}\mathscr{F}^n_B \\ \mathscr{F}^n_A &= -\mathscr{F}_{A,0} - \mathscr{X}^n_{BA}\mathscr{F}_B \end{aligned} \qquad \text{(a)}$$

Next, *using the primary system corresponding to* $\mathbf{Z} = \mathbf{0}$, we express $\mathscr{F}^n_B$ in terms of the applied external load and the force redundants:

$$\mathscr{F}^n_B = \mathbf{EZ} + \mathbf{G} \qquad (15\text{–}95)$$

The elements of $\mathbf{G}$ are the end forces at B (for $\mathbf{Z} = \mathbf{0}$) due to the applied external loads. Note that $\mathbf{G} = \mathbf{0}$ if $\mathbf{Z}$ contains only end forces at B.

Now, the principle of virtual forces requires

$$\int_{S_A}^{S_B} \Delta\mathscr{F}^T(\mathscr{E}^0 + \mathbf{g}\mathscr{F})dS = \Delta\mathscr{F}_B^{n,T}\mathscr{U}_B^n + \Delta\mathscr{F}_A^{n,T}\mathscr{U}_a^n \qquad \text{(b)}$$

for any self-equilibrating virtual-force system. Taking the system due to $\Delta\mathbf{Z}$ results in the compatibility equations for $\mathbf{Z}$. It is convenient to work first with the virtual force system due to $\Delta\mathscr{F}_B^n$. Equation (b) reduces to

$$\begin{aligned} \Delta\mathscr{F}_B^{n,T}(\mathscr{V}_0^n + \mathbf{f}^n\mathscr{F}_B^n) &= \Delta\mathscr{F}_B^{n,T}(\mathscr{U}_B^n - \mathscr{X}_{BA}^{n,T}\mathscr{U}_A^n) \\ &= \Delta\mathscr{F}_B^{n,T}\mathscr{V}^n \end{aligned} \qquad \text{(c)}$$

where $\mathscr{V}_0^n$, $\mathbf{f}^n$ are the initial deformation and flexibility matrices for the *full end restraint* case. Substituting for $\mathscr{F}_B^n$ using (15–95), and requiring the resulting expression to be satisfied for arbitrary $\Delta\mathbf{Z}$, we obtain

$$(\mathbf{E}^T\mathbf{f}^n\mathbf{E})\mathbf{Z} + \mathbf{E}^T(\mathscr{V}_0^n + \mathbf{f}^n\mathbf{G}) = \mathbf{E}^T\mathscr{V}^n = \mathbf{E}^T(\mathscr{U}_B^n - \mathscr{X}_{BA}^{n,T}\mathscr{U}_A^n) \qquad (15\text{–}96)$$

It should be noted that $\mathscr{U}_B^n$, $\mathscr{U}_A^n$ are the displacements of the *supports* at B, A.

We suppose $\mathbf{Z}$ is of order $q \times 1$, i.e., there are q force redundants. Also, we let i be the row order of $\mathscr{F}$ (and $\mathscr{U}$).

$$\underset{(i\times 1)}{\mathscr{F}} = \left\{\begin{array}{c} \underset{(\alpha\times 1)}{\mathbf{F}} \\ \hline \underset{(\beta\times 1)}{\mathbf{M}} \end{array}\right\} \qquad (15\text{–}97)$$

With this notation,

$$\begin{aligned} &\mathbf{E} \text{ is } i \times q \\ &\mathbf{G} \text{ is } i \times 1 \end{aligned} \qquad \text{(d)}$$

and (15–96) represents q equations. For convenience, we let

$$\begin{aligned} \mathbf{f}_r &= \mathbf{E}^T\mathbf{f}^n\mathbf{E} \qquad (q \times q) \\ \mathscr{V}_{0,z}^n &= \mathscr{V}_0^n + \mathbf{f}^n\mathbf{G} \qquad (i \times 1) \end{aligned} \qquad (15\text{–}98)$$

and the member force-deformation relations take the form

$$\begin{aligned} \mathbf{f}_r\mathbf{Z} &= \mathbf{E}^T(\mathscr{V}^n - \mathscr{V}_{0,z}^n) \\ &= \mathbf{E}^T(\mathscr{U}_B^n - \mathscr{X}_{BA}^{n,T}\mathscr{U}_A^n - \mathscr{V}_{0,z}^n) \end{aligned} \qquad (15\text{–}99)$$

We refer to $\mathbf{f}_r$ as the *reduced* flexibility matrix since, in general, $q < i$. Actually, $\mathbf{f}_r$ is the flexibility matrix for $\mathbf{Z}$ and it is *positive definite* since $\mathbf{E}$ must be of rank q, i.e., the force systems corresponding to the redundants must be linearly independent. Note that one can determine $\mathbf{f}_r$ directly by working with the primary system corresponding to $\mathbf{Z} = \mathbf{0}$. This is the normal approach. The approach that we have followed is convenient when the member flexibility matrix is known.

At this point, we summarize the force-displacement relations for partial end restraint:

$$\begin{aligned} \mathbf{Z} &= \text{member force matrix} \\ \mathscr{F}_B^n &= \mathbf{E}\mathbf{Z} + \mathbf{G} \\ \mathscr{F}_A^n &= \mathscr{F}_{A,0}^n - \mathscr{X}_{BA}^n\mathscr{F}_B^n \\ \mathbf{f}_r &= \text{reduced flexibility matrix } (q \times q) = \mathbf{E}^T\mathbf{f}^n\mathbf{E} \\ \mathscr{V}_{0,z}^n &= \mathscr{V}_0^n + \mathbf{f}^n\mathbf{G} \\ \mathbf{f}_r\mathbf{Z} &= \mathbf{E}^T(\mathscr{V}^n - \mathscr{V}_{0,z}^n) \\ &= \mathbf{E}^T(\mathscr{U}_B^n - \mathscr{X}_{BA}^{n,T}\mathscr{U}_A^n - \mathscr{V}_{0,z}^n) \end{aligned} \qquad (15\text{–}100)$$

Note that, for complete end restraint,

$$\begin{aligned} \mathbf{Z} &= \mathscr{F}_B^n \qquad & \mathbf{E} &= \mathbf{I}_i \\ \mathbf{G} &= \mathbf{0} \qquad & \mathscr{V}_{0,z}^n &= \mathscr{V}_0^n \end{aligned} \qquad (15\text{–}101)$$

We will use (15–100) in Chapter 17 when we develop the formulation for a member system.

Continuing, we let

$$\mathbf{k}_r = \mathbf{f}_r^{-1} \qquad (15\text{–}102)$$

The force redundants are obtained by inverting (15–99):

$$\mathbf{Z} = \mathbf{k}_r\mathbf{E}^T(\mathscr{U}_B^n - \mathscr{X}_{BA}^{n,T}\mathscr{U}_A^n - \mathscr{V}_{0,z}^n) \qquad (15\text{–}103)$$

Substituting for $\mathbf{Z}$, the end forces at B are given by

$$\mathscr{F}_B^n = (\mathbf{E}\mathbf{k}_r\mathbf{E}^T)(\mathscr{U}_B^n - \mathscr{X}_{BA}^{n,T}\mathscr{U}_A^n - \mathscr{V}_{0,z}^n) + \mathbf{G} \qquad \text{(e)}$$

We defined $\mathbf{k}_e^n$ as the *effective* member stiffness matrix:

$$\begin{aligned} \mathbf{k}_e^n &= \mathbf{E}\mathbf{k}_r\mathbf{E}^T \\ &= \mathbf{E}(\mathbf{E}^T\mathbf{f}^n\mathbf{E})^{-1}\mathbf{E}^T \end{aligned} \qquad (15\text{–}104)$$

In general, $\mathbf{k}_e^n$ is singular when $q < i$, since $\mathbf{E}$ is only of rank q. Equation (e) takes the form

$$\begin{aligned} \mathscr{F}_B^n &= \mathscr{F}_{B,i}^n + \mathbf{k}_e^n\mathscr{U}_B^n - \mathbf{k}_e^n\mathscr{X}_{BA}^{n,T}\mathscr{U}_A^n \\ \mathscr{F}_{B,i}^n &= -\mathbf{k}_e^n\mathscr{V}_{0,z}^n + \mathbf{G} \\ &= -\mathbf{k}_e^n\mathscr{V}_0^n + (\mathbf{I}_i - \mathbf{k}_e^n\mathbf{f}^n)\mathbf{G} \end{aligned} \qquad (15\text{–}105)$$

The end forces at A are determined from (a):

$$\begin{aligned} \mathscr{F}_A^n &= \mathscr{F}_{A,i}^n - \mathscr{X}_{BA}^n\mathbf{k}_e^n\mathscr{U}_B^n + \mathscr{X}_{BA}^n\mathbf{k}_e^n\mathscr{X}_{BA}^{n,T}\mathscr{U}_A^n \\ \mathscr{F}_{A,i}^n &= -\mathscr{F}_{A,0}^n - \mathscr{X}_{BA}^n\mathscr{F}_{B,i}^n \end{aligned} \qquad (15\text{–}106)$$

Finally, we write the relations in the generalized form

$$\begin{aligned} \mathscr{F}_B^n &= \mathscr{F}_{B,i}^n + \mathbf{k}_{BB}^n\mathscr{U}_B^n + \mathbf{k}_{BA}^n\mathscr{U}_A^n \\ \mathscr{F}_A^n &= \mathscr{F}_{A,i}^n + \mathbf{k}_{AB}^n\mathscr{U}_B^n + \mathbf{k}_{AA}^n\mathscr{U}_A^n \end{aligned}$$

where

$$\begin{aligned}\mathbf{k}^n_{BB} &= \mathbf{k}^n_e \\ \mathbf{k}^n_{BA} &= (\mathbf{k}^n_{AB})^T = -\mathbf{k}^n_e \mathscr{X}^{n,T}_{BA} \\ \mathbf{k}^n_{AA} &= -\mathscr{X}^n_{BA}\mathbf{k}^n_{BA} = \mathscr{X}^n_{BA}\mathbf{k}^n_e\mathscr{X}^{n,T}_{BA}\end{aligned} \tag{15–107}$$

Comparing (15–107) with (15–53), the corresponding expressions for the complete restraint case, we see that one has only to replace $\mathbf{k}^n$ by $\mathbf{k}^n_e$ in the partitioned forms for $\mathbf{k}^n_{BB}$, $\mathbf{k}^n_{BA}$, and $\mathbf{k}^n_{AA}$. The equation for $\bar{\mathscr{F}}^n_{B,i}$ is different, however, due to the presence of the **G** term.

Example 15–12

Suppose there is no restraint against rotation at B. Then, $\bar{\mathbf{M}}^n_B = \mathbf{0}$. We take $\mathbf{Z} = \bar{\mathbf{F}}^n_B$ and generate **E**, **G** with (15–95).

$$\begin{Bmatrix}\bar{\mathbf{F}}^n_B \\ \bar{\mathbf{M}}^n_B\end{Bmatrix} = \begin{bmatrix}\mathbf{I}_\alpha \\ \mathbf{0}\end{bmatrix}\{\bar{\mathbf{F}}^n_B\} = \mathbf{EZ} + \mathbf{G} \tag{a}$$

For this case, $G = 0$. The reduced flexibility and stiffness matrices follow from (15–98), (15–102),

$$\begin{aligned}\mathbf{f}_r &= \mathbf{f}^n_{11} \\ \mathbf{k}_r &= \mathbf{f}^{n,-1}_{11}\end{aligned} \tag{b}$$

and the effective stiffness matrix follows from (15–104):

$$\mathbf{k}^n_e = \left[\begin{array}{c|c}\mathbf{f}^{n,-1}_{11} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0}\end{array}\right] \tag{c}$$

Finally, the force-displacement relations are (see (15–99)):

$$\mathbf{f}^n_{11}\bar{\mathbf{F}}^n_B = \mathbf{u}^n_B - \mathbf{u}^n_A - \mathbf{X}^{n,T}_{BA}\boldsymbol{\omega}^n_A - \mathbf{v}^n_0 \tag{d}$$

Note that premultiplication of $\mathscr{V}^n$ by $\mathbf{E}^T$ eliminates $\boldsymbol{\theta}^n$, the relative rotation at B. There is *no* compatibility requirement for the end rotations in this case; i.e., the support rotation at B, which we have defined as ω^n_B, does not introduce any member deformation.

REFERENCES

1. Reissner, E.: "Variational Considerations for Elastic Beams and Shells," *J. Eng. Mech. Div.* A.S.C.E., Vol. 88, No. EMI, February 1962.
2. Hall, A. S. and R. W. Woodhead: *Frame Analysis*, Wiley, New York, 1967.
3. Gere, J. M. and W. Weaver: *Analysis of Framed Structures*, Van Nostrand, New York, 1965.
4. Rubinstein, M. F.: *Matrix Computer Analysis of Structures*, Prentice-Hall, 1966.
5. Livesley, R. K.: *Matrix Methods of Structural Analysis*, Pergamon Press, London, 1964.
6. Dabrowski, R.: *Gekrümmte dünnwandige Träger* (Curved thin-walled beams), Springer-Verlag, Berlin, 1968.

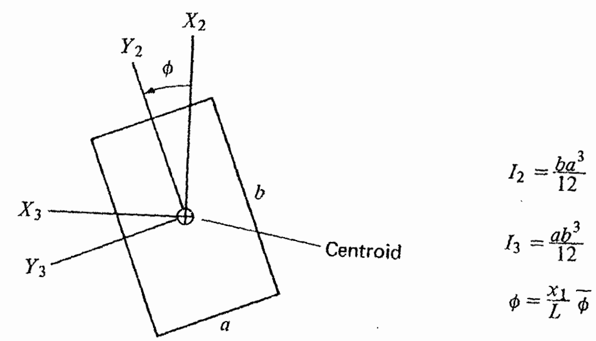
7. Vlasov, V. Z.: *Thin Walled Elastic Beams*, Israel Program for Scientific Translations, Office of Technical Services, U.S. Dept. of Commerce, Washington, D.C., 1961.
8. Bazant, Z. P.: "Nonuniform Torsion of Thin-walled Bars of Variable Section," International Association for Bridge and Structural Engineering Publications, Zurich, Vol. 25, 1965, pp. 17–39.

PROBLEMS

15–1. Refer to Example 15–5. Determine c_t for a typical wide-flange section and a square single cell. Comment on the relative importance of torsional deformation vs. bending deformation (i.e., terms involving c_t in Equation (e)). Distinguish between deep and shallow members.

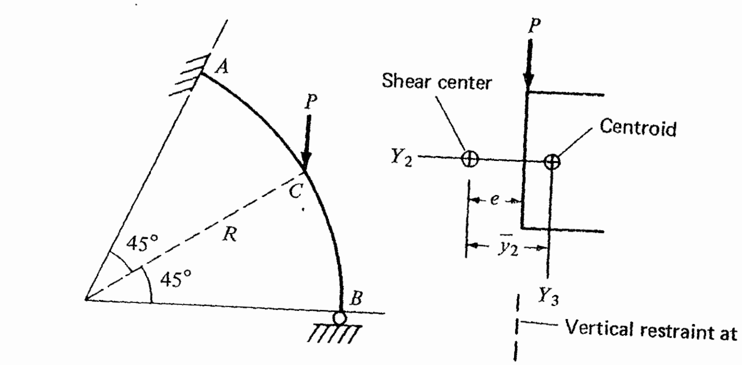
15–2. Refer to Example 15–7. Consider a rectangular cross section and ϕ varying linearly with x_1, as shown in the sketch. Evaluate $v_{B2}\Big/\left(\dfrac{P_2L^3}{3EI_2}\right)$ and $v_{B3}\Big/\left(\dfrac{P_2L^3}{3EI_2}\right)$ for a range of $\bar{\phi}$ and a/b.

Prob. 15–2



15–3. Determine the reaction at B and translation (in the direction of P) at C for the member sketched. Neglect transverse shear deformation.

Prob. 15–3



15–4. Repeat Prob. 15–3, considering complete fixity at B. Utilize symmetry with respect to point C.

15–5. Derive (15–27). Start with the definitions for the strain measures (see Fig. 15–2),

$$\vec{\rho} = \vec{R} + \vec{u}$$

$$(1 + \varepsilon_1)\left|\frac{\partial \vec{R}}{\partial S}\right| = \left|\frac{\partial \vec{\rho}}{\partial S}\right|$$

$$\sin \gamma_{12} = \frac{\dfrac{\partial \vec{\rho}}{\partial S} \cdot \dfrac{\partial \vec{\rho}}{\partial y_2}}{\left|\dfrac{\partial \vec{\rho}}{\partial S}\right| \left|\dfrac{\partial \vec{\rho}}{\partial y_2}\right|}$$

neglect second-order terms, and note (15–26).

15–6. Summarize the governing equations for restrained torsion. Evaluate b_2 and b_3 (see (15–34)) for a symmetrical wide-flange section and a symmetrical rectangular closed cell. Comment on whether one can neglect these terms.

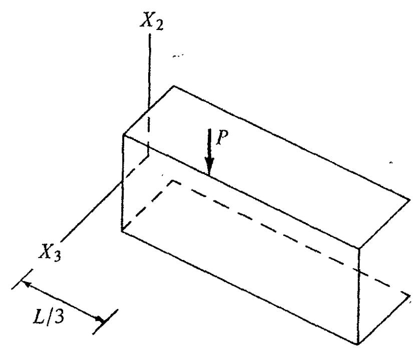
15–7. Refer to Example 15–11. Specialize the solution (Equations f) for $\bar{\lambda}\theta_B = \lambda L \gg 1$. Verify that (g) reduces to the prismatic solution, (13–57), when $\theta_B \to 0$.

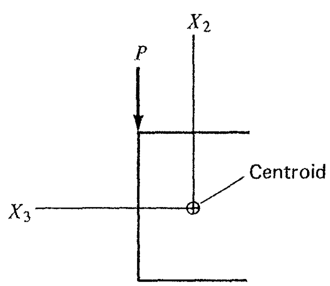
15–8. Consider a member comprising of three segments. Assuming the flexibility matrices for the segments are known, determine an expression for the member flexibility matrix in terms of the segmental flexibility matrices. Generalize for n segments.

15–9. Discuss how you would apply the numerical integration schemes described in Sec. 14–8 to evaluate J_P, defined by (15–69).

15–10. Verify (15–73) and (15–74).

15–11. Determine the fixed end forces for the member shown, using (15–77) and (15–79).





Prob. 15–11

15–12. Solve Prob. 15–3 using (15–84) and (15–87).

15–13. Verify (15–90) and (15–91). Apply them to Prob. 15–4.

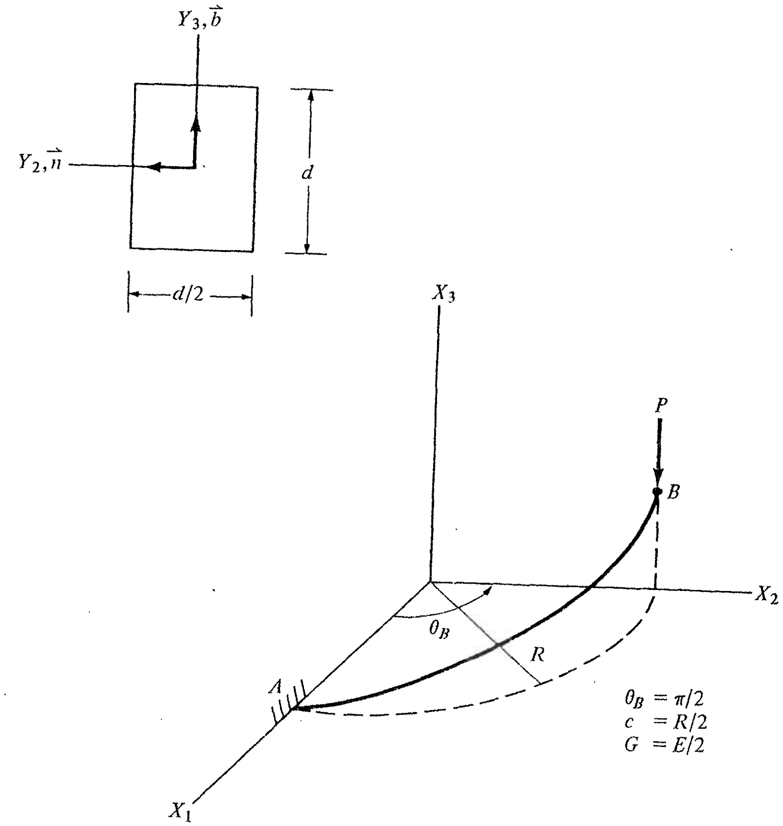
15–14. Starting with (15–87), develop expressions for the initial deformations due to an arbitrary distributed loading, $b_3 = b_3(\theta)$. Specialize for b_3 = constant and verify (15–92).

15–15. Using the geometric relations and flexibility matrix for a circular helix (constant cross section; Y_2 coincides with the normal direction) developed in Sec. 15–11:

(a) Develop a matrix equation for the displacements at B due to a loading referred to the global frame and applied at θ_c. *Hint:* See (15–85).

(b) Evaluate $\mathbf{u}^m_{B3}$ for the loading and geometry shown.

Prob. 15–15



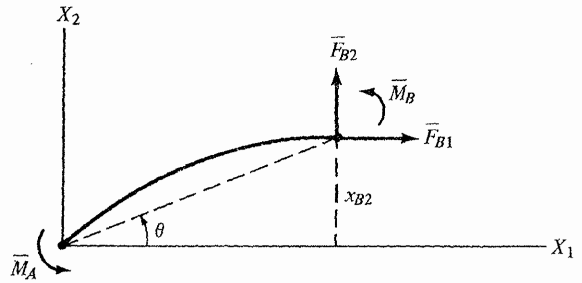
15–16. Determine the reduced member flexibility matrix for no restraint against rotation at an interior point P.

15–17. For the planar member shown, determine $\mathbf{E}$ and $\mathbf{G}$ corresponding to

$$\mathbf{Z} = \{\bar{F}_{B1} \quad \bar{M}_B \quad \bar{M}_A\}$$

Then specialize for rotation releases at A, B and determine $\mathbf{k}_e$.

Prob. 15–17



15–18. Determine **E** and **G** for—

(a) no restraint against translation in a particular direction at B

(b) no restraint against rotation about a particular axis at B

Hint: Review Example 15–12.

Part IV

ANALYSIS OF A MEMBER SYSTEM