# 5

# Matrix Transformations for a Member Element

## 5–1. ROTATION TRANSFORMATION

Suppose we know the scalar components of a vector with respect to a reference frame and we want to determine the components of the vector corresponding to a second reference frame. We can visualize the determination of the second set of components from the point of view of applying a transformation to the column matrix of initial components. We refer to this transformation as a *rotation transformation.* Also, we call the matrix which defines the transformation a *rotation matrix.*

Let $X_j^n$, $\bar{\imath}_j^n$ ($j = 1, 2, 3$ and $n = 1, 2$) be the directions and corresponding unit vectors for reference frame $n$. (See Fig. 5–1.) We will generally use a superscript to indicate the reference frame for directions, unit vectors, and scalar

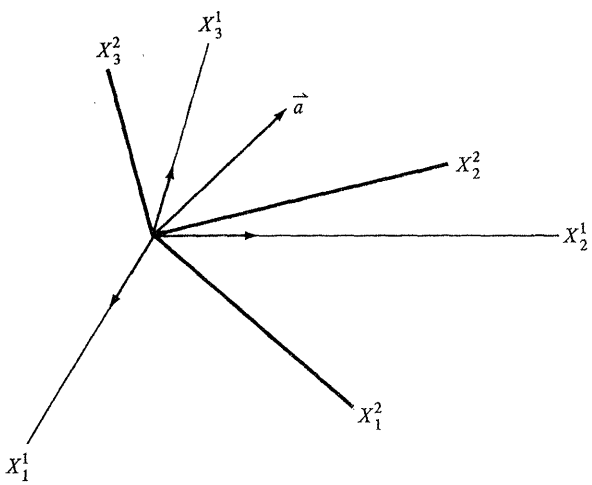



**Fig. 5–1.** Directions for reference frames "1" and "2."





components in this text. We consider a vector, $\bar{a}$. The scalar components of $\bar{a}$ with respect to frame $n$ are $a_j^n$ and we express $\bar{a}$ as

$$\bar{a} = a_1^n \bar{\imath}_1^n + a_2^n \bar{\imath}_2^n + a_3^n \bar{\imath}_3^n = (\mathbf{a}^n)^T \mathbf{i}^n \tag{5–1}$$

Now, $\bar{a}$ is *independent* of the reference frame. Then,

$$\bar{a} = (\mathbf{a}^1)^T \mathbf{i}^1 = (\mathbf{a}^2)^T \mathbf{i}^2 \tag{a}$$

To proceed further, we must relate the two reference frames. We write the relations between the unit vectors as

$$\mathbf{i}^2 = \boldsymbol{\beta} \mathbf{i}^1 \tag{5–2}$$

where $\beta_{jk}$ is the scalar component of $\bar{\imath}_j^2$ with respect to $\bar{\imath}_k^1$. The transformation matrix, $\boldsymbol{\beta}$, is nonsingular when the unit vectors are linearly independent. Substituting for $\mathbf{i}^2$ and equating the coefficients of $\mathbf{i}^1$ leads to

$$\begin{aligned} \mathbf{a}^1 &= \boldsymbol{\beta}^T \mathbf{a}^2 \\ \mathbf{a}^2 &= (\boldsymbol{\beta}^T)^{-1} \mathbf{a}^1 \end{aligned} \tag{b}$$

Finally, we let

$$\begin{aligned} \mathbf{R}^{12} &= (\boldsymbol{\beta}^T)^{-1} \\ \mathbf{R}^{21} &= (\mathbf{R}^{12})^{-1} = \boldsymbol{\beta}^T \end{aligned} \tag{5–3}$$

With this notation, the relations between the component matrices take the form

$$\begin{aligned} \mathbf{a}^2 &= \mathbf{R}^{12} \mathbf{a}^1 \\ \mathbf{a}^1 &= \mathbf{R}^{21} \mathbf{a}^2 \\ \mathbf{i}^2 &= (\mathbf{R}^{12})^{-1,T} \mathbf{i}^1 \end{aligned} \tag{5–4}$$

The order of the superscripts on $\mathbf{R}$ corresponds to the direction of the transformation. For example, $\mathbf{R}^{12}$ is the rotation transformation matrix corresponding to a change *from* frame 1 *to* frame 2. We see that the transformation matrix for the scalar components of a vector is the *inverse transpose* of the transformation matrix governing the unit vectors for the reference frames.

### Example 5–1

We consider the two-dimensional case shown in Fig. E5–1. The relations between the unit vectors are

$$\begin{aligned} \bar{\imath}_1^2 &= \cos\theta \bar{\imath}_1^1 + \sin\theta \bar{\imath}_2^1 \\ \bar{\imath}_2^2 &= -\sin\phi \bar{\imath}_1^1 + \cos\phi \bar{\imath}_2^1 \end{aligned} \tag{a}$$

We write (a) according to (5–2).

$$\mathbf{i}^2 = \boldsymbol{\beta} \mathbf{i}^1$$

$$\boldsymbol{\beta} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\phi & \cos\phi \end{bmatrix} \tag{b}$$

Then,

$$\mathbf{R}^{21} = \boldsymbol{\beta}^T = \begin{bmatrix} \cos\theta & -\sin\phi \\ \sin\theta & \cos\phi \end{bmatrix} \qquad \text{(c)}$$

$$\mathbf{R}^{12} = (\boldsymbol{\beta}^T)^{-1} = \frac{1}{|\boldsymbol{\beta}^T|}\begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\theta & \cos\theta \end{bmatrix} \qquad \text{(d)}$$

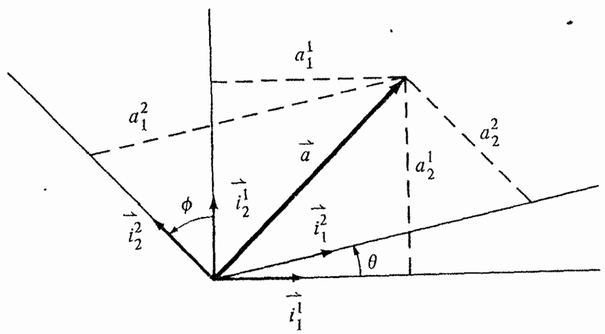
$$|\boldsymbol{\beta}^T| = \cos\theta\cos\phi + \sin\theta\sin\phi$$

and

$$\begin{Bmatrix} a_1^2 \\ a_2^2 \end{Bmatrix} = \frac{1}{|\boldsymbol{\beta}^T|}\begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{Bmatrix} a_1^1 \\ a_2^1 \end{Bmatrix} \qquad \text{(e)}$$

When both frames are orthogonal, $\phi = \theta$ and $\boldsymbol{\beta}^T = \boldsymbol{\beta}^{-1}$.

**Fig. E5–1**




The result obtained in the preceding example can be readily extended to the case of two 3-dimensional orthogonal reference frames. When both frames are orthogonal, the change in reference frames can be visualized as a rigid body rotation of one frame into the other, $\beta_{jk}$ is the direction cosine for $X_j^2$ with respect to $X_k^1$, and the rotation transformation matrix is an orthogonal matrix:

$$\mathbf{R}^{12} = [\beta_{jk}] \qquad (5\text{–}5)$$
$$\beta_{jk} = \cos(X_j^2, X_k^1)$$

In Sec. 4–7, we defined the orientation of the local frame $(\vec{t}_1, \vec{t}_2, \vec{t}_3)$ at a point on the reference axis of a member element with respect to the natural frame $(\vec{t}, \vec{n}, \vec{b})$ at the point. This frame, in turn, was defined with respect to the three fixed cartesian frame $(\vec{\imath}_1, \vec{\imath}_2, \vec{\imath}_3)$. In order to distinguish between the three frames, we use superscripts $p$ and $p'$ for the local and natural frames at $p$ and a superscript 1 for the basic cartesian frame:

$$\mathbf{t}^p = \{\vec{t}_1, \vec{t}_2, \vec{t}_3\}_p$$
$$\mathbf{t}^{p'} = \{\vec{t}, \vec{n}, \vec{b}\} \qquad (5\text{–}6)$$
$$\mathbf{i}^1 = \{\vec{\imath}_1, \vec{\imath}_2, \vec{\imath}_3\}$$



With this notation, the relations between the unit vectors and the various rotation matrices are:

$$\mathbf{t}^p = \mathbf{R}^{p'p}\mathbf{t}^{p'} = \mathbf{R}^{1p}\mathbf{i}^1$$
$$\mathbf{t}^{p'} = \mathbf{R}^{1p'}\mathbf{i}^1$$

1. From (4–21),

$$\mathbf{R}^{1p'} = [\ell_{jk}] \qquad (5\text{–}7)$$

2. From (4–24),

$$\mathbf{R}^{p'p} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix}$$

3.

$$\mathbf{R}^{1p} = \boldsymbol{\beta} = \mathbf{R}^{1p'}\mathbf{R}^{p'p}$$

$\boldsymbol{\beta}$ defined by (4–25).

## 5–2. THREE-DIMENSIONAL FORCE TRANSFORMATIONS

The equilibrium analysis of a member element involves the determination of the internal force and moment vectors at a cross section due to external forces and moments acting on the member. We shall refer to both forces and moments as "forces." Also, we speak of the force and moment at a point, say $P$, as the "force system" at $P$. The relationship between the external force system at $P$ and the statically equivalent internal force system at $Q$ has a simple form when vector notation is used. Consider a force $\vec{F}$ and moment $\vec{M}$ acting at $P$ shown in Fig. 5–2. The statically equivalent force and moment at $Q$ are

$$\vec{F}_{\text{equiv.}} = \vec{F}$$
$$\vec{M}_{\text{equiv.}} = \vec{M} + \vec{r} \times \vec{F} \qquad (5\text{–}8)$$

One can visualize (5–8) as a force transformation in which the force system at $P$ is transformed into the force system at $Q$. This transformation will be

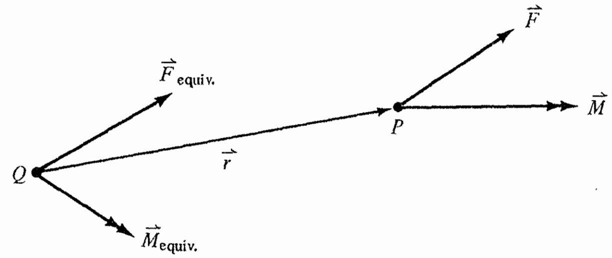



**Fig. 5–2.** Equivalent force system.

linear if $\vec{r}$ is constant, that is, if the geometry of the element does not change appreciably when the external loads are applied. We will write (5–8) in matrix form and treat force transformations as matrix transformations.

We develop first the matrix transformation associated with the moment of a force about a point. Let $\vec{F}_P$ be a force vector acting at point $P$ and $\vec{M}_Q$ the moment vector at point $Q$ corresponding to $\vec{F}_P$. We will *always* indicate the point of application of a force or moment vector with a subscript. The relation between $\vec{M}_Q$ and $\vec{F}_P$ is

$$\vec{M}_Q = \vec{QP} \times \vec{F}_P \tag{5-9}$$

We work with an orthogonal reference frame (frame 1) shown in Fig. 5-3 and write the component expansions as

$$\begin{aligned} \vec{F}_P &= \sum F^1_{Pj}\vec{i}^1_j = (\mathbf{i}^1)^T\mathbf{F}^1_P \\ \vec{M}_P &= \sum M^1_{Pj}\vec{i}^1_j = (\mathbf{i}^1)^T\mathbf{M}^1_P \end{aligned} \tag{5-10}$$

The scalar components of $\vec{QP}$ are $x^1_{Pj} - x^1_{Qj}$. Expanding the vector cross product leads to

$$\mathbf{M}^1_Q = \mathbf{X}^1_{PQ}\mathbf{F}^1_P \tag{5-11}$$

$$\mathbf{X}^1_{PQ} = \left[\begin{array}{c|c|c} 0 & -(x^1_{P3} - x^1_{Q3}) & +(x^1_{P2} - x^1_{Q2}) \\ \hline +(x^1_{P3} - x^1_{Q3}) & 0 & -(x^1_{P1} - x^1_{Q1}) \\ \hline -(x^1_{P2} - x^1_{Q2}) & +(x^1_{P1} - x^1_{Q1}) & 0 \end{array}\right]$$

Note that $\mathbf{X}^1_{PQ}$ is a skew-symmetric matrix. One can interpret it as a force-translation transformation matrix. The force at $P$ is transformed by $\mathbf{X}^1_{PQ}$ into

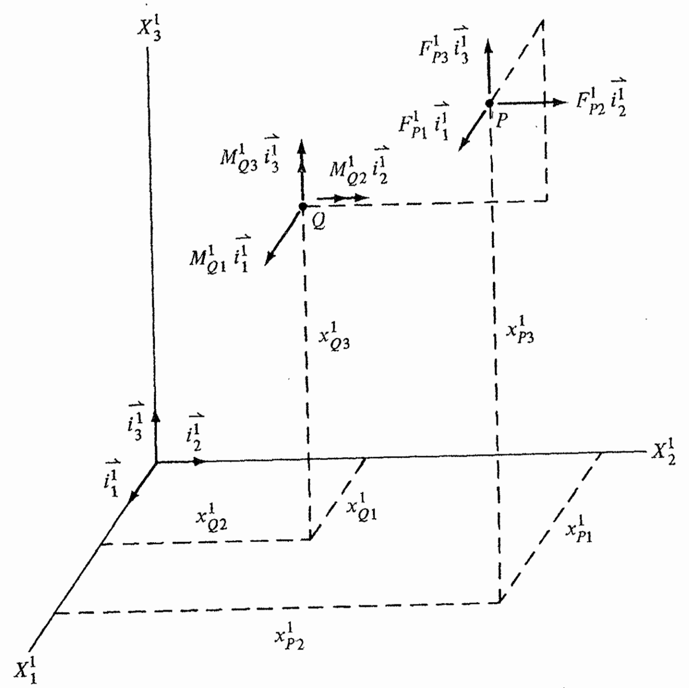



**Fig. 5-3.** Notation for orthogonal reference frame.



a moment at $Q$. *Note* that the order of the subscripts for the translation transformation matrix, $\mathbf{X}^1_{PQ}$, corresponds to the order of the translation (from $P$ to $Q$). Also, $\mathbf{X}_{PQ}$ and $\mathbf{F}_P$ must be referred to the *same* frame, that is, the *superscripts* must be equal.

Up to this point, we have considered only one orthogonal reference frame. In general, there will be a local orthogonal reference frame associated with each point on the axis of the member, and these frames will coincide only when the member is prismatic. To handle the general case we must introduce rotation transformations which transform the components of $\vec{F}$ and $\vec{M}$ from the local frames to the basic frame (frame 1) and vice versa. We use a superscript $p$ to indicate the *local* frame at point $P$ and the rotation matrix corresponding to a transformation *from* the local frame at $P$ *to* frame 1 is denoted by $\mathbf{R}^{p1}$. With this notation,

$$\begin{aligned} \mathbf{F}^1_P &= \mathbf{R}^{p1}\mathbf{F}^p_P \\ \mathbf{M}^q_Q &= \mathbf{R}^{1q}\mathbf{M}^1_Q \end{aligned} \tag{5-12}$$

and the general expression for $\mathbf{M}^q_Q$ takes the form

$$\mathbf{M}^q_Q = (\mathbf{R}^{1q}\mathbf{X}^1_{PQ}\mathbf{R}^{p1})\mathbf{F}^p_P \tag{5-13}$$

We consider next the total force transformation. The statically equivalent force and moment at $Q$ associated with a force and moment at $P$ are given by

$$\begin{aligned} \vec{F}_Q &= \vec{F}_P \\ \vec{M}_Q &= \vec{M}_P + \vec{QP} \times \vec{F}_P \end{aligned} \tag{a}$$

When all the vectors are referred to a common frame, say frame 1, the matrix transformation is

$$\begin{Bmatrix} \mathbf{F}^1_Q \\ \hline \mathbf{M}^1_Q \end{Bmatrix} = \left[\begin{array}{c|c} \mathbf{I}_3 & \mathbf{0} \\ \hline \mathbf{X}^1_{PQ} & \mathbf{I}_3 \end{array}\right] \begin{Bmatrix} \mathbf{F}^1_P \\ \hline \mathbf{M}^1_P \end{Bmatrix} \tag{b}$$

We let

$$\mathscr{X}^1_{PQ} = \left[\begin{array}{c|c} \mathbf{I}_3 & \mathbf{0} \\ \hline \mathbf{X}^1_{PQ} & \mathbf{I}_3 \end{array}\right] \tag{5-14}$$

$$\mathscr{F}^1_Q = \begin{Bmatrix} \mathbf{F}^1_Q \\ \hline \mathbf{M}^1_Q \end{Bmatrix} \qquad \mathscr{F}^1_P = \begin{Bmatrix} \mathbf{F}^1_P \\ \hline \mathbf{M}^1_P \end{Bmatrix}$$

The $6 \times 1$ matrix $\mathscr{F}^1_Q$ is called the force system at $Q$ referred to frame 1. Using this notation, (b) simplifies to

$$\mathscr{F}^1_Q = \mathscr{X}^1_{PQ}\mathscr{F}^1_P \tag{5-15}$$

When the force systems are referred to local frames, we must first transform them to a common frame and then apply (5-15). Utilizing the general rotation matrix,

$$\mathscr{R}^{pn} = \left[\begin{array}{c|c} \mathbf{R}^{pn} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{R}^{pn} \end{array}\right] = (\mathscr{R}^{np})^{-1} = (\mathscr{R}^{np})^T \tag{5-16}$$

and applying

$$\begin{aligned} \mathscr{F}^1_P &= \mathscr{R}^{p1}\mathscr{F}^p_P \\ \mathscr{F}^q_Q &= \mathscr{R}^{1q}\mathscr{F}^1_Q \end{aligned} \tag{a}$$

we obtain

$$\mathscr{F}^q_Q = (\mathscr{R}^{1q}\mathscr{X}^1_{PQ}\mathscr{R}^{p1})\mathscr{F}^p_P \qquad (5\text{–}17)$$
$$= \mathscr{T}^{pq}_{PQ}\mathscr{F}^p_P$$

Equation (5–17) states that when the matrix transformation $\mathscr{T}^{pq}_{PQ}$ is applied to $\mathscr{F}^p_P$ we obtain its statical equivalent at $Q$. Actually, we could leave off the subscripts and superscripts on $\mathscr{T}$ when we write (5–17). However, if $\mathscr{T}$ appears alone, we must include them. Note that the force transformation generally involves *both* translation and rotation. The order of the subscripts corresponds to the direction of the translation, e.g., from $P$ to $Q$. Similarly the order of the superscripts defines the direction of the rotation or change in reference frames, e.g., from frame $p$ to frame $q$.

In general, the geometry of a member element is defined with respect to a basic reference frame which we take as frame 1. To evaluate $\mathscr{T}^{pq}_{PQ}$ we must determine $\mathbf{R}^{p1}$, $\mathbf{R}^{q1}$, and $\mathbf{X}^1_{PQ}$ from the geometrical relations for the member. We have already discussed how one determines $\mathbf{R}^{1p}$ in Secs. 4–7 and 5–1.

When the member is planar* and the geometry is fairly simple (such as a straight or circular member), we can take frame 1 parallel to one of the local frames. This eliminates one rotation transformation. For example, suppose we take frame 1 parallel to frame $p$. Then, $\mathbf{R}^{1p} = \mathbf{I}$ and $\mathscr{T}^{pq}_{PQ}$ reduces to

$$\mathscr{T}^{pq}_{PQ} = \left[\begin{array}{c|c}\mathbf{R}^{1q} & \mathbf{0} \\ \hline \mathbf{R}^{1q}\mathbf{X}^1_{PQ} & \mathbf{R}^{1q}\end{array}\right] = \left[\begin{array}{c|c}\mathbf{R}^{pq} & \mathbf{0} \\ \hline \mathbf{R}^{pq}\mathbf{X}^p_{PQ} & \mathbf{R}^{pq}\end{array}\right] \qquad (5\text{–}18)$$

Similarly, if 1 and $q$ are parallel,

$$\mathscr{T}^{pq}_{PQ} = \left[\begin{array}{c|c}\mathbf{R}^{p1} & \mathbf{0} \\ \hline \mathbf{X}^1_{PQ}\mathbf{R}^{p1} & \mathbf{R}^{p1}\end{array}\right] = \left[\begin{array}{c|c}\mathbf{R}^{pq} & \mathbf{0} \\ \hline \mathbf{X}^q_{PQ}\mathbf{R}^{pq} & \mathbf{R}^{pq}\end{array}\right] \qquad (5\text{–}19)$$

When both $p$ and $q$ are parallel to 1, $\mathscr{T}^{pq}_{PQ}$ reduces to $\mathscr{X}^1_{PQ}$.

$$\mathscr{T}^{11}_{PQ} = \mathscr{X}^1_{PQ} \qquad (5\text{–}20)$$

By transforming from $P$ to $Q$ and back to $P$, we obtain

$$\mathscr{F}^p_P = \mathscr{T}^{qp}_{QP}\mathscr{T}^{pq}_{PQ}\mathscr{F}^p_P \qquad (a)$$

and it follows that

$$\mathscr{T}^{qp}_{QP} = (\mathscr{T}^{pq}_{PQ})^{-1} \qquad (5\text{–}21)$$

If the transformation from $P$ to $Q$ is carried out in the order $P \to S_1$, $S_1 \to S_2, \ldots, S_n \to Q$, where $S_1, S_2, \ldots, S_n$ are intermediate points, the transformation matrix, $\mathscr{T}^{pq}_{PQ}$ is equal to the product of the intermediate transformation matrices.

$$\mathscr{T}^{pq}_{PQ} = \mathscr{T}^{s_nq}_{S_nQ} \cdots \mathscr{T}^{s_1s_2}_{S_1S_2}\mathscr{T}^{ps_1}_{PS_1} \qquad (5\text{–}22)$$

* If the reference axis is a plane curve and the local frame coincides with the natural frame ($\phi = 0$) we say the member is *planar*.



where $s_1, s_2, \ldots, s_N$ are arbitrary reference frames. It is convenient to take a common reference frame for the intermediate transformations.

**Example 5–2**

We consider the plane circular member shown in Fig. E5–2. We take frame 1 parallel to frame $p$. Then,

$$\mathbf{x}^1_P - \mathbf{x}^1_Q = \{a\sin\theta, -a(1-\cos\theta), 0\}$$

$$\mathbf{X}^1_{PQ} = \mathbf{X}^p_{PQ} = \left[\begin{array}{c|c|c} 0 & 0 & -a(1-\cos\theta) \\ 0 & 0 & -a\sin\theta \\ a(1-\cos\theta) & a\sin\theta & 0 \end{array}\right]$$

$$\mathbf{R}^{1q} = \mathbf{R}^{pq} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

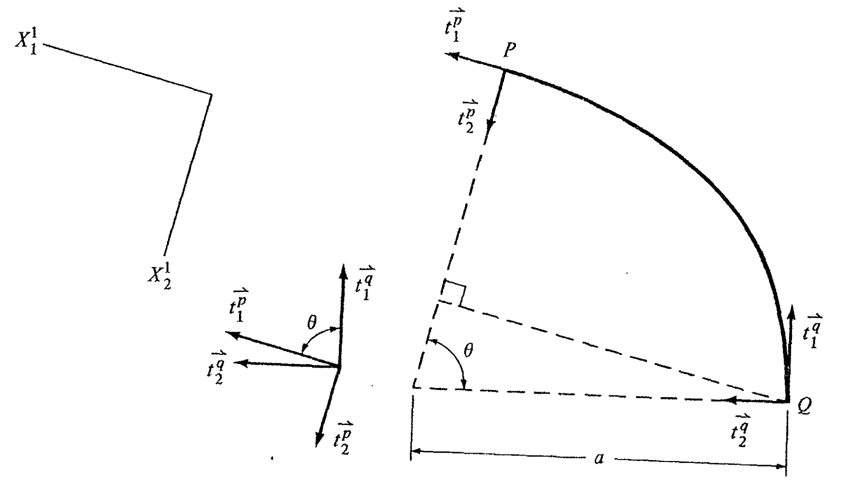
The transformation matrix has the form

$$\mathscr{T}^{pq}_{PQ} = \mathbf{R}^{pq}\mathscr{X}^p_{PQ} = \left[\begin{array}{c|c}\mathbf{R}^{pq} & \mathbf{0} \\ \hline \mathbf{R}^{pq}\mathbf{X}^p_{PQ} & \mathbf{R}^{pq}\end{array}\right]$$

where

$$\mathbf{R}^{pq}\mathbf{X}^p_{PQ} = \left[\begin{array}{c|c|c} 0 & 0 & +a(1-\cos\theta) \\ 0 & 0 & -a\sin\theta \\ a(1-\cos\theta) & a\sin\theta & 0 \end{array}\right]$$

Fig. E5–2




**Example 5–3**

As an illustration of the case where the geometry is defined with respect to a basic cartesian frame, we consider the problem of finding $\mathscr{T}^{pq}_{PQ}$ for a circular helix. The general expansion for $\mathcal{T}^{pq}_{PQ}$ has the form

$$\mathcal{T}^{pq}_{PQ} = \left[\begin{array}{c|c} \mathbf{R}^{1q}\mathbf{R}^{p1} & \mathbf{0} \\ \hline \mathbf{R}^{1q}\mathbf{X}^1_{PQ}\mathbf{R}^{p1} & \mathbf{R}^{1q}\mathbf{R}^{p1} \end{array}\right] = \left[\begin{array}{c|c} \mathbf{R}^{pq} & \mathbf{0} \\ \hline \mathbf{R}^{1q}\mathbf{X}^1_{PQ}\mathbf{R}^{p1} & \mathbf{R}^{pq} \end{array}\right]$$

The parametric representation for a circular helix is given in Sec. 4–1:

$$x^1_1 = a\cos y \qquad x^1_2 = a\sin y \qquad x^1_3 = cy$$

where $x^1_j$ ($j = 1, 2, 3$) are the cartesian coordinates with respect to the basic frame (frame 1). Let $y_P$ and $y_Q$ be the values of $y$ corresponding to points $P$ and $Q$. The coordinate matrices for $P$ and $Q$ are

$$\mathbf{x}^1_P = \{a\cos y_P,\ a\sin y_P,\ cy_P\}$$
$$\mathbf{x}^1_Q = \{a\cos y_Q,\ a\sin y_Q,\ cy_Q\}$$

Then,

$$\mathbf{X}^1_{PQ} = \begin{bmatrix} 0 & -c(y_P - y_Q) & a(\sin y_P - \sin y_Q) \\ c(y_P - y_Q) & 0 & -a(\cos y_P - \cos y_Q) \\ -a(\sin y_P - \sin y_Q) & a(\cos y_P - \cos y_Q) & 0 \end{bmatrix}$$

To simplify the algebra, we suppose the local frame coincides with the natural frame at every point along the reference axis, that is, we take $\phi = 0$. Using the results of Sec. 4–7, the rotation matrices reduce to

$$\mathbf{R}^{1q} = \begin{bmatrix} -\frac{a}{\alpha}\sin y_Q & \frac{a}{\alpha}\cos y_Q & \frac{c}{\alpha} \\ -\cos y_Q & -\sin y_Q & 0 \\ \frac{c}{\alpha}\sin y_Q & -\frac{c}{\alpha}\cos y_Q & \frac{a}{\alpha} \end{bmatrix}$$

$$\mathbf{R}^{p1} = \begin{bmatrix} -\frac{a}{\alpha}\sin y_P & -\cos y_P & \frac{c}{\alpha}\sin y_P \\ \frac{a}{\alpha}\cos y_P & -\sin y_P & -\frac{c}{\alpha}\cos y_P \\ \frac{c}{\alpha} & 0 & \frac{a}{\alpha} \end{bmatrix} = (\mathbf{R}^{1p})^T$$

where $\alpha^2 = a^2 + c^2$.

Evaluating the product, $\mathbf{R}^{1q}\mathbf{R}^{p1}$, we obtain

$$\mathbf{R}^{pq} = \mathbf{R}^{1q}\mathbf{R}^{p1} = \left[\begin{array}{c|c|c} \left(\frac{a}{\alpha}\right)^2\cos\eta + \left(\frac{c}{\alpha}\right)^2 & -\frac{a}{\alpha}\sin\eta & \frac{ac}{\alpha^2}(1 - \cos\eta) \\ \frac{a}{\alpha}\sin\eta & \cos\eta & -\frac{c}{\alpha}\sin\eta \\ \frac{ac}{\alpha^2}(1 - \cos\eta) & \frac{c}{\alpha}\sin\eta & \left(\frac{a}{\alpha}\right)^2 + \left(\frac{c}{\alpha}\right)^2\cos\eta \end{array}\right]$$



where $\eta = y_P - y_Q$. Also,

$$\mathbf{R}^{1q}\mathbf{X}^1_{PQ}\mathbf{R}^{p1} =$$

$$\left[\begin{array}{c|c|c} \frac{a^2c}{\alpha^2}(2 - 2\cos\eta - \eta\sin\eta) & -\frac{ac}{\alpha}(\eta\cos\eta - \sin\eta) & \frac{ac^2}{\alpha^2}\eta\sin\eta + \frac{a}{\alpha^2}(1 - \cos n)(a^2 - c^2) \\ \hline \frac{ac}{\alpha}(\eta\cos\eta - \sin\eta) & -c\eta\sin\eta & -\frac{c^2}{\alpha}\eta\cos\eta - \frac{a^2}{\alpha}\sin\eta \\ \hline \frac{ac^2}{\alpha^2}\eta\sin\eta + \frac{a}{\alpha^2}(1 - \cos\eta)(a^2 - c^2) & \frac{c^2}{\alpha}\eta\cos\eta + \frac{a^2}{\alpha}\sin\eta & -2\frac{a^2c}{\alpha^2}(1 - \cos\eta) - \frac{c^3}{\alpha^2}\eta\sin\eta \end{array}\right]$$

Note that we can specialize the above general results for the case of a plane circular member (Example 5–2) by taking $c = 0$ and $\eta = \theta$.

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## 5–3. THREE-DIMENSIONAL DISPLACEMENT TRANSFORMATIONS

Let $P$ and $Q$ be two points on a rigid body. Suppose that the body experiences a translation and a rotation. We define $\bar{u}_P$ and $\bar{\omega}_P$ as the translation and rotation* vectors for point $P$. The corresponding vectors for point $Q$ are given by

$$\bar{u}_Q = \bar{u}_P + \bar{\omega}_P \times \overline{PQ} \qquad \bar{\omega}_Q = \bar{\omega}_P \tag{5–23}$$

Equation (5–23) is valid *only* when $|\bar{\omega}_P|^2$ is negligible with respect to unity. Since $\overline{PQ} = -\overline{QP}$ and $\bar{\omega}_P \times \overline{PQ} = -\overline{PQ} \times \bar{\omega}_P$, an alternate form for $\bar{u}_Q$ is

$$\bar{u}_Q = \bar{u}_P + \overline{QP} \times \bar{\omega}_P \tag{5–24}$$

We define

$$\mathcal{U}^1_P = \begin{Bmatrix} \mathbf{u}^1_P \\ \boldsymbol{\omega}^1_P \end{Bmatrix} \tag{5–25}$$

as the displacement matrix for $P$ referred to frame 1. The displacement at $Q$ resulting from the rigid body displacement at $P$ is given by

$$\mathcal{U}^1_Q = \left[\begin{array}{c|c} \mathbf{I}_3 & \mathbf{X}^1_{PQ} \\ \hline \mathbf{0} & \mathbf{I}_3 \end{array}\right] \mathcal{U}^1_P \tag{5–26}$$

We consider next the case where the local frames at $P$ and $Q$ do not coincide. The general relation between the displacements has the form

$$\mathcal{U}^q_Q = \mathcal{R}^{1q}\left[\begin{array}{c|c} \mathbf{I}_3 & \mathbf{X}^1_{PQ} \\ \hline \mathbf{0} & \mathbf{I}_3 \end{array}\right]\mathcal{R}^{p1}\mathcal{U}^p_P = \left[\begin{array}{c|c} \mathbf{R}^{1q}\mathbf{R}^{p1} & \mathbf{R}^{1q}\mathbf{X}^1_{PQ}\mathbf{R}^{p1} \\ \hline \mathbf{0} & \mathbf{R}^{1q}\mathbf{R}^{p1} \end{array}\right]\mathcal{U}^p_P \tag{5–27}$$

One can show† that alternate forms of (5–27) are

$$\mathcal{U}^q_Q = (\mathcal{T}^{pq}_{PQ})^{-1,T}\,\mathcal{U}^p_P = (\mathcal{T}^{qp}_{QP})^T\,\mathcal{U}^p_P \tag{5–28}$$

* The units of $|\bar{\omega}|$ are radians.
† See Prob. 5–7.

We see that the displacement transformation matrix is the *inverse transpose* of the corresponding force transformation matrix. This result is quite useful.

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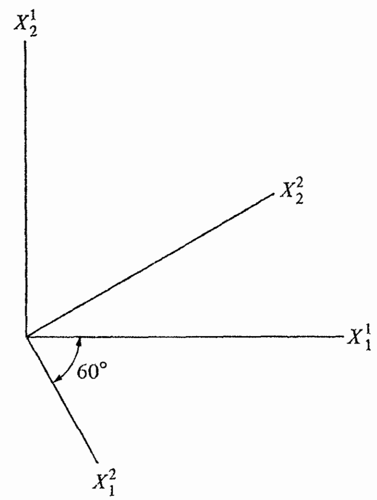
5. Martin, H. C.: *Introduction to Matrix Methods of Structural Analysis*, McGraw-Hill, New York, 1966.


## PROBLEMS

**5–1.** Consider the two-dimensional cartesian reference frames shown. If $\mathbf{a}^1 = \{50, -100\}$, find $\mathbf{a}^2$.

**Prob. 5–1**




**5–2.** The orientation of two orthogonal frames is specified by the direction cosine table listed below.

| | $X_1^1$ | $X_2^1$ | $X_3^1$ |
|---|---|---|---|
| $X_1^2$ | 1/2 | 1/2 | $\sqrt{2}/2$ |
| $X_2^2$ | 1/2 | 1/2 | $-\sqrt{2}/2$ |
| $X_3^2$ | $-\sqrt{2}/2$ | $\sqrt{2}/2$ | 0 |

(a) Determine $\mathbf{R}^{12}$. Verify that $(\mathbf{R}^{12})^T = (\mathbf{R}^{12})^{-1}$.



(b) If $\mathbf{a}^1 = \{10, 5, 10\}$, find $\mathbf{a}^2$.
(c) If $\mathbf{a}^2 = \{5, 10, 10\}$, find $\mathbf{a}^1$.

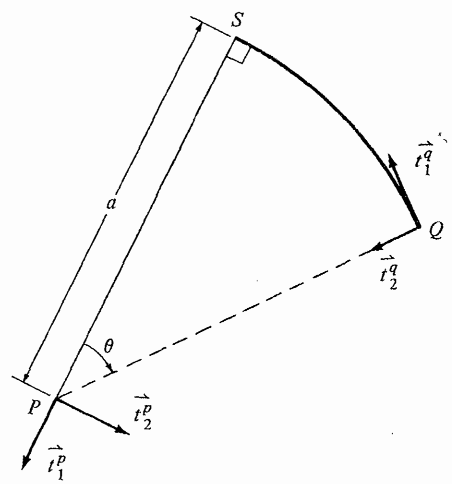
**5–3.** Consider two points, $P$ and $Q$, having coordinates (6, 3, 2) and (−5, 1, 4) with respect to frame 1. The direction cosine tables for the local reference frames are listed below.

| | $\vec{i}_1^1$ | $\vec{i}_2^1$ | $\vec{i}_3^1$ |
|---|---|---|---|
| $\vec{t}_1^p$ | 1/2 | 1/2 | $\sqrt{2}/2$ |
| $\vec{t}_2^p$ | 1/2 | 1/2 | $-\sqrt{2}/2$ |
| $\vec{t}_3^p$ | $-\sqrt{2}/2$ | $\sqrt{2}/2$ | 0 |
| $\vec{t}_1^q$ | $-\sqrt{2}/2$ | 1/2 | 1/2 |
| $\vec{t}_2^q$ | $\sqrt{2}/2$ | 1/2 | 1/2 |
| $\vec{t}_3^q$ | 0 | $-\sqrt{2}/2$ | $\sqrt{2}/2$ |

(a) Determine $\mathscr{X}_{PQ}^1$ and $\mathscr{X}_{PQ}^p$.
(b) Determine $\mathscr{T}_{PQ}^{pq}$.
(c) Suppose $\mathscr{F}_P^p = \{100, -50, 100, 20, -40, +60\}$. Calculate $\mathscr{F}_Q^q$.

**5–4.** Consider the planar member consisting of a circular segment and a straight segment shown in the sketch below. Point $P$ is at the center of the circle.

**Prob. 5–4**




(a) Determine $\mathscr{T}_{PQ}^{pq}$ by transforming directly from $P$ to $Q$. Also find $\mathscr{T}_{QP}^{qp}$.
(b) Determine $\mathscr{T}_{PQ}^{pq}$ by transforming from $P$ to $S$ and then from $S$ to $Q$.
(c) Find $\mathscr{F}_Q^q$ corresponding to $\mathscr{F}_P^p = \{0, 0, 1, 0, 0, 0\}$.

**5–5.** Consider the circular helix,

$$\vec{r} = 2\cos y\vec{i}_1 + 2\sin y\vec{i}_2 + \frac{4y}{\pi}\vec{i}_3.$$

(a) Suppose $\phi(y) = 0$. Determine $\mathscr{T}_{PQ}^{pq}$. Take $y_P = \pi/2$, $y_Q = \pi/4$.
(b) Suppose $\phi(y) = -y$. Determine $\mathscr{T}_{PQ}^{pq}$.

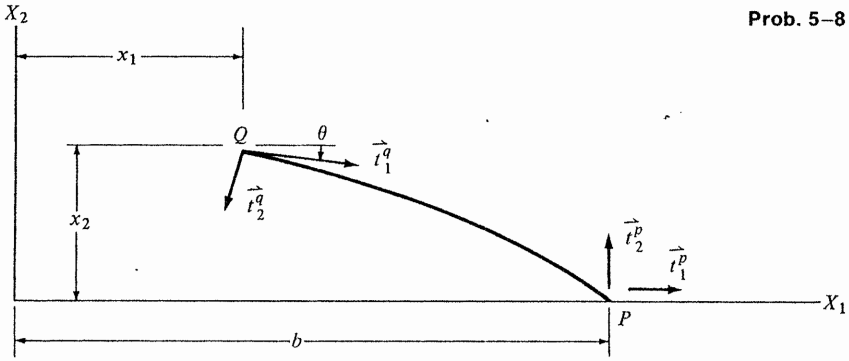
**5–6.** Refer to Problem 5–3. Determine $\mathcal{U}_Q^q$ corresponding to $\mathcal{U}_P^p = \{1/2, -1/4, 1/3, -1/10, 1/10, 0\}$. Verify that

$$\mathcal{F}_Q^{q,T}\mathcal{U}_Q^q = \mathcal{F}_P^{p,T}\mathcal{U}_P^p$$

**5–7.** Verify that (5–27) and (5–28) are equivalent forms. Note that

$$\cdots\left[\begin{array}{c|c} \mathbf{I}_3 & \mathbf{X}_{PQ}^1 \\ \hline \mathbf{0} & \mathbf{I}_3 \end{array}\right] = \left[\begin{array}{c|c} \mathbf{I}_3 & \mathbf{0} \\ \hline \mathbf{X}_{QP} & \mathbf{I}_3 \end{array}\right]^T = (\mathcal{X}_{QP}^1)^T$$

**5–8.** Consider the plane member shown. The reference axis is defined by $x_2 = f(x_1)$.




**Prob. 5–8**

(a) Determine $\mathcal{T}_{PQ}^{pq}$. Note that the local frame at $P$ coincides with the basic frame whereas the local frame at $Q$ coincides with the natural frame at $Q$.

(b) Specialize part (a) for the case where

$$x_2 = \frac{4a}{b^2}(x_1 b - x_1^2)$$

and the $x_1$ coordinate of point $Q$ is equal to $b/4$. Use the results of Prob. 4–2.

# Part II

# ANALYSIS OF AN IDEAL TRUSS