*Hint*: One can write

$$\beta_3 = \frac{1}{I_3}\iint (x_2^2\, \nabla^2 \phi_{2r} + x_3^2\, \nabla_{2r}^2) dA$$

Also show that

$$\eta_3 = \tfrac{1}{2}\beta_2$$

**13–12.** Specialize Equations (13–84) and (13–85) for the case where the cross section is symmetrical with respect to the $X_2$ axis. Utilize

$$\iint H_e(x_2, x_3) H_o(x_2, x_3) dA = 0$$

where $H_e$ is an even function and $H_o$ an odd function of $x_3$. Evaluate the coefficients for the channel section of Example 13–5. Finally, specialize the equations for a doubly symmetric section.

**13–13.** Specialize (13–88) for a doubly symmetrical cross section. Then specialize further for negligible transverse shear deformation due to flexure and warping. The symmetry reductions are

$$\bar{x}_2 = \bar{x}_3 = 0 \qquad x_{2r} = x_{3r} = 0$$
$$\beta_2 = \beta_3 = \beta_\phi = 0 \qquad 1/A_{23} = 0$$
$$\eta_2 = \eta_3 = \eta_1^u = \eta_1^r = 0$$

**13–14.** Consider the two following problems involving doubly symmetric cross section.

(a) Establish "linearized" incremental equations by operating on (13–88) and retaining only linear terms in the displacement increments. Specialize for a doubly symmetric cross section (see Prob. 13–12).

(b) Consider the case where the cross section is doubly symmetric and the initial state is pure compression ($F_1 = -P$). Determine the critical load with respect to torsional buckling for the following boundary conditions:

1. $\omega_1 = f = 0$ at $x = 0, L$ (restrained warping)
2. $\omega_1 = \dfrac{df}{dx} = 0$ at $x = 0, L$ (unrestrained warping)

Neutral equilibrium (buckling) is defined as the existence of a *nontrivial* solution of the linearized incremental equations for the same external load. One sets

$$F_1 = -P$$
$$u_2 = u_3 = \omega_1 = \omega_2 = \omega_3 = f = 0$$

and determines the value of $P$ for which a nontrivial solution which satisfies the boundary conditions is possible. Employ the notation introduced in Example 13–7.

**13–15.** Determine the form of $\bar{V}$, the strain energy density function (strain energy per unit length along the centroidal axis), expressed in terms of displacements. Assume no initial strain but allow for geometric nonlinearity. Note that $\bar{V} = \bar{V}^*$ when there is no initial strain.

# 14

# Planar Deformation of a Planar Member

## 14–1. INTRODUCTION: GEOMETRICAL RELATIONS

A member is said to be planar if—

1. The centroidal axis is a plane curve.
2. The plane containing the centroidal axis also contains one of the principal inertia axes for the cross section.
3. The shear center axis coincides with or is parallel to the centroidal axis. However, the present discussion will be limited to the case where the shear center axis lies in the plane containing the centroidal axis.

We consider the centroidal axis to be defined with respect to a global reference frame having directions $X_1$ and $X_2$. This is shown in Fig. 14–1. The orthogonal unit vectors defining the orientation of the local frame ($Y_1$, $Y_2$) at a point are denoted by $\vec{t}_1, \vec{t}_2$, where $\vec{t}_1$ points in the *positive* tangent direction and $\vec{t}_1 \times \vec{t}_2 = \vec{t}_3$. Item 2 requires $Y_2$ to be a principal inertia axis for the cross section.

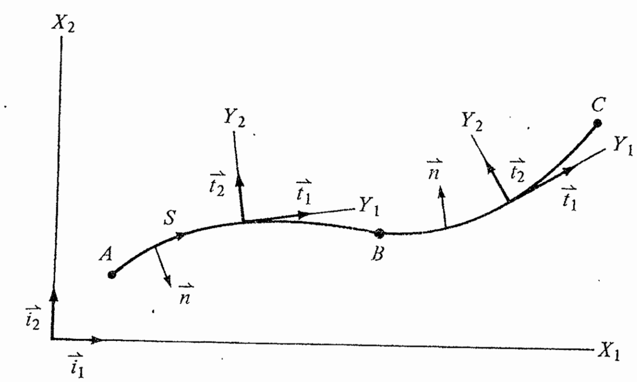



**Fig. 14–1.** Geometrical notation for plane curve.

By definition,†

$$\vec{t}_1 = \frac{d\vec{r}}{dS} = \frac{dx_1}{dS}\vec{\imath}_1 + \frac{dx_2}{dS}\vec{\imath}_2 \tag{14-1}$$

Since we are taking $\vec{t}_2$ according to $\vec{t}_1 \times \vec{t}_2 = \vec{\imath}_3$, it follows that

$$\vec{t}_2 = -\frac{dx_2}{dS}\vec{\imath}_1 + \frac{dx_1}{dS}\vec{\imath}_2 \tag{14-2}$$

The differentiation formulas for the unit vectors are

$$\begin{aligned} \frac{d\vec{t}_1}{dS} &= \frac{1}{R}\vec{t}_2 \\ \frac{d\vec{t}_2}{dS} &= -\frac{1}{R}\vec{t}_1 \end{aligned} \tag{14-3}$$

where

$$\frac{1}{R} = \frac{d\vec{t}_1}{dS}\cdot\vec{t}_2 = -\frac{d^2x_1}{dS^2}\frac{dx_2}{dS} + \frac{d^2x_2}{dS^2}\frac{dx_1}{dS}$$

According to this definition, $R$ is negative when $d\vec{t}_1/dS$ points in the negative $\vec{t}_2$ direction, e.g., for segment $AB$ in Fig. 14–1. One could take $\vec{t}_2 = \vec{n}$, the unit normal vector defined by

$$\vec{n} = \frac{1}{\left|\frac{d\vec{t}_1}{dS}\right|}\frac{d\vec{t}_1}{dS} \tag{14-4}$$

rather than according to $\vec{t}_1 \times \vec{t}_2 = \vec{\imath}_3$ but this choice is inconvenient when there is a reversal in curvature. Also, this definition degenerates at an inflection point, i.e., when $d\vec{t}/dS = \vec{0}$. If the sense of the curvature is constant, one can always orient the $X_1$-$X_2$ frame so that $\vec{t}_2$ coincides with $\vec{n}$, to avoid working with a negative $R$.

To complete the geometrical treatment, we consider the general parametric representation for the curve defining the centroidal axis,

$$\begin{aligned} x_1 &= x_1(y) \\ x_2 &= x_2(y) \end{aligned} \tag{14-5}$$

where $y$ is a parameter. The differential arc length is related to $dy$ by

$$dS = +\left[\left(\frac{dx_1}{dy}\right)^2 + \left(\frac{dx_2}{dy}\right)^2\right]^{1/2} dy = \alpha\, dy \tag{14-6}$$

According to this definition, the $+S$ sense coincides with the direction of

† We summarize here for convenience the essential geometric relations for a plane curve which are developed in Chapter 4.



increasing $y$. Using (14–6), the expressions for $\vec{t}_1$, $\vec{t}_2$, and $1/R$ in terms of $y$ are

$$\begin{aligned} \vec{t}_1 &= \frac{1}{\alpha}\left(\frac{dx_1}{dy}\vec{\imath}_1 + \frac{dx_2}{dy}\vec{\imath}_2\right) \\ \vec{t}_2 &= \frac{1}{\alpha}\left(-\frac{dx_2}{dy}\vec{\imath}_1 + \frac{dx_1}{dy}\vec{\imath}_2\right) \\ \frac{1}{R} &= \frac{1}{\alpha}\left(\vec{t}_2\cdot\frac{d\vec{t}_1}{dy}\right) \\ &= \frac{1}{\alpha^3}\left(-\frac{d^2x_1}{dy^2}\frac{dx_2}{dy} + \frac{d^2x_2}{dy^2}\frac{dx_1}{dy}\right) \end{aligned} \tag{14-7}$$

A planar member subjected to in-plane forces ($X_1$-$X_2$ plane for our notation) will experience only in-plane deformation. In what follows, we develop the governing equations for planar deformation of an arbitrary planar member. This formulation is restricted to the *linear geometric* case. The two basic solution procedures, namely, the displacement and force methods, are described and applied to a circular member.

We also present a simplified formulation (Marguerre's equations) which is valid for a *shallow* member. Finally, we include a discussion of numerical integration techniques, since one must resort to numerical integration when the cross section is not constant.

## 14–2. FORCE-EQUILIBRIUM EQUATIONS

The notation associated with a positive normal cross section, i.e., a cross section whose outward normal points in the $+S$ direction, is shown in Fig. 14–2. We use the same notation as for the prismatic case, except that now the vector

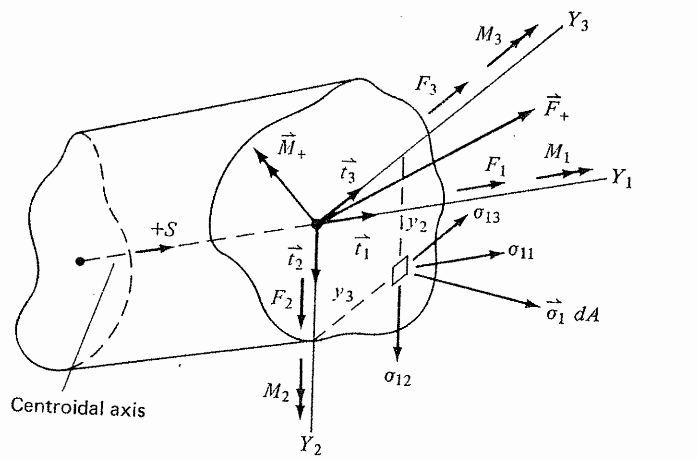



**Fig. 14–2.** Force and moment components acting on a positive cross section.

components are with respect to the *local* frame $(Y_1, Y_2, Y_3)$ rather than the basic frame $(X_1, X_2, X_3)$. The cross-sectional properties are defined by

$$\begin{aligned} A &= \iint dy_2\, dy_2 = \iint dA \\ I_3 &= \iint (y_2)^2\, dA \qquad I_2 = \iint (y_3)^2\, dA \end{aligned} \tag{14-8}$$

Since $Y_2$, $Y_3$ pass through the centroid and are principal directions, it follows that

$$\iint y_2\, dA = \iint y_3\, dA = \iint y_2 y_3\, dA = 0 \tag{14-9}$$

When the member is planar ($X_1$-$X_2$ plane) and is subjected to a planar loading,

$$F_3 = M_1 = M_2 = 0 \tag{14-10}$$

In this case, we work with reduced expressions for $\overline{F}_+$ and $\overline{M}_+$ (see Fig. 14–3) and drop the subscript on $M_3$:

$$\begin{aligned} \overline{F}_+ &= F_1\vec{t}_1 + F_2\vec{t}_2 \\ \overline{M}_+ &= M_3\vec{t}_3 = M\vec{t}_3 \end{aligned} \tag{14-11}$$

Note that $\vec{t}_3$ is *constant* for a planar member.

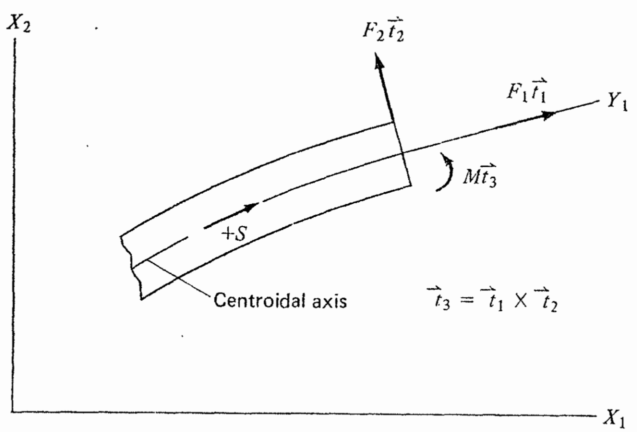



**Fig. 14–3.** Force and moment components in planar behavior.

To establish the force-equilibrium equations, we consider the differential volume element shown in Fig. 14–4. We define $\vec{b}$ and $\vec{m}$ as the statically equivalent external force and moment vectors per unit *arc length* acting at the centroid. For equilibrium, the resultant force and moment vectors must vanish. These conditions lead to the following vector differential equilibrium equations:

$$\begin{aligned} &\frac{d\overline{F}}{dS} + \vec{b} = \vec{0} \\ &\frac{d\overline{M}_+}{dS} + \vec{m} + \vec{t}_1 \times \overline{F}_+ = \vec{0} \end{aligned} \tag{14-12}$$



We expand $\vec{b}$ and $\vec{m}$ in terms of the unit vectors for the local frame:

$$\begin{aligned} \vec{b} &= b_1\vec{t}_1 + b_2\vec{t}_2 \\ \vec{m} &= m\vec{t}_3 \end{aligned} \tag{14-13}$$

Introducing the component expansions in (14–12), and using the differentiation formulas for the unit vectors (14–3), lead to the following scalar differential equilibrium equations:

$$\begin{aligned} &\frac{dF_1}{dS} - \frac{F_2}{R} + b_1 = 0 \\ &\frac{dF_2}{dS} + \frac{F_1}{R} + b_2 = 0 \\ &\frac{dM}{dS} + F_2 + m = 0 \end{aligned} \tag{14-14}$$

Note that the force-equilibrium equations are *coupled* due to the curvature. The moment equilibrium equation has the *same* form as for the prismatic case.

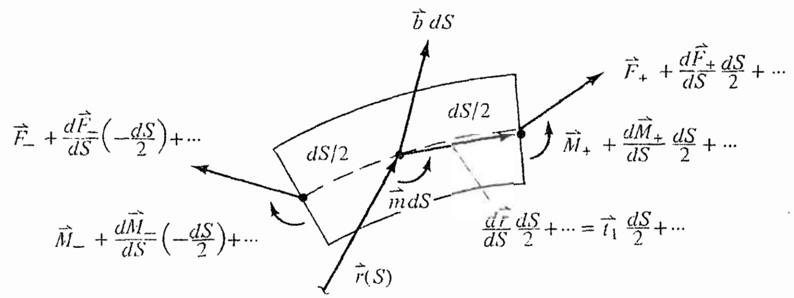



**Fig. 14–4.** Differential element for equilibrium analysis.

The positive sense of the end forces is shown in Fig. 14–5. We work with components referred to the *local* frame at each end. The end forces are related to the stress resultants and stress couples by

$$\begin{aligned} \overline{F}_{Bj} &= F_j|_{S_B} \\ \overline{M}_B &= M|_{S_B} \\ \overline{F}_{Aj} &= -F_j|_{S_A} \\ \overline{M}_A &= -M|_{S_A} \qquad j = 1, 2 \end{aligned} \tag{14-15}$$

## 14–3. FORCE-DISPLACEMENT RELATIONS; PRINCIPLE OF VIRTUAL FORCES

We establish the force-displacement relations by applying the principal of virtual forces to a differential element. The procedure is the same as for the prismatic case described in Sec. 12–3, except that now we work with displacement components referred to the *local* frame at each point. We define $\vec{u}$ and $\vec{\omega}$ as

$$\vec{u} = \sum u_j \vec{t}_j = \text{equivalent† rigid-body translation vector at the centroid.}$$
$$\vec{\omega} = \sum \omega_j \vec{t}_j = \text{equivalent rigid-body rotation vector} \tag{14–16}$$

For planar deformation, only $u_1$, $u_2$ and $\omega_3$ are finite, and the terms involving $u_3$, $\omega_1$, and $\omega_2$ can be deleted:

$$\begin{aligned}\vec{u} &= u_1\vec{t}_1 + u_2\vec{t}_2\\ \vec{\omega} &= \omega_3\vec{t}_3 = \omega\vec{t}_3\end{aligned} \tag{14–17}$$

The positive sense of the displacement components is shown in Fig. 14–6.

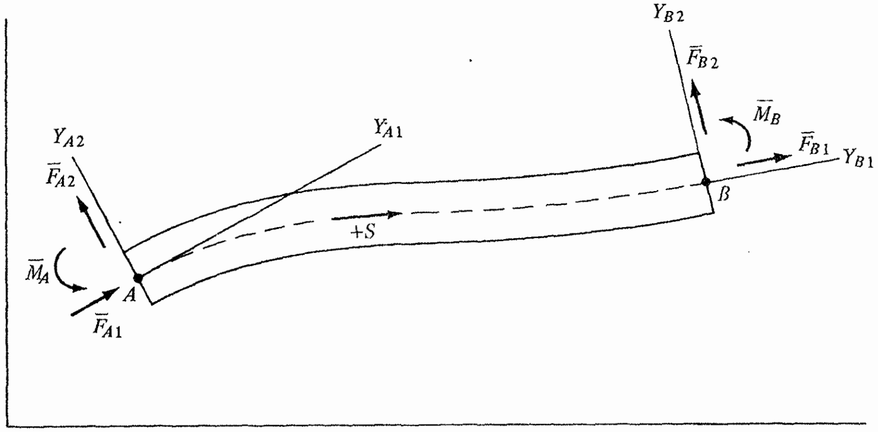



**Fig. 14–5.** Convention for end forces.

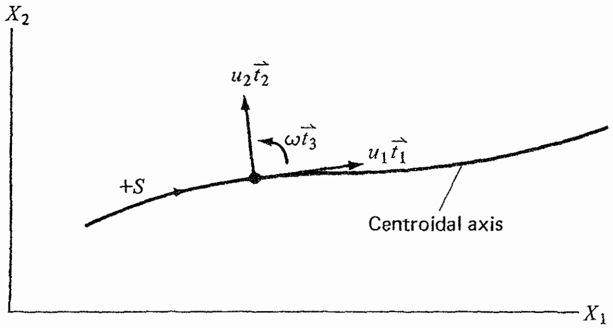



**Fig. 14–6.** Definition of displacement measures.

† "Equivalence" refers to work. See (12–8).



We define $\bar{V}^*$ as the complementary energy per unit *arc length*. For planar deformation, $\bar{V}^* = \bar{V}^*(F_1, F_2, M)$. One determines $\bar{V}^*$ by taking expansions for the stresses in terms of $F_1$, $F_2$, $M$, substituting in the complementary energy density, and integrating with respect to the cross-sectional coordinates $y_2$, $y_3$. We will discuss the determination of $\bar{V}^*$ later.

Specializing the three-dimensional principle of virtual forces for the one-dimensional elastic case, and writing

$$\begin{aligned} d\bar{V}^* &= \frac{\partial \bar{V}^*}{\partial F_1}\Delta F_1 + \frac{\partial \bar{V}^*}{\partial F_2}\Delta F_2 + \frac{\partial \bar{V}^*}{\partial M}\Delta M \\ &= e_1\,\Delta F_1 + e_2\,\Delta F_2 + k\,\Delta M \end{aligned} \tag{14–18}$$

lead to the one-dimensional form

$$\int_S (e_1\,\Delta F_1 + e_2\,\Delta F_2 + k\,\Delta M)\,dS = \sum d_i\,\Delta P_i \tag{14–19}$$

where $d_i$ is a displacement measure and $P_i$ is the force measure corresponding to $d_i$. The virtual-force system ($\Delta F_1$, $\Delta F_2$, $\Delta M$, $\Delta P_i$) must be statically permissible, i.e., it must satisfy the one-dimensional equilibrium equations.

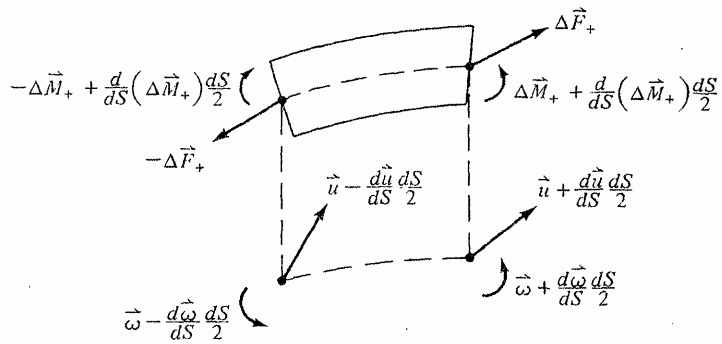



**Fig. 14–7.** Virtual force system

We apply (14–19) to the differential element shown in Fig. 14–7. The virtual force system must satisfy the force-equilibrium equations (14–17),

$$\begin{aligned} \frac{d}{dS}\Delta\vec{F}_+ &= \vec{0} \\ \frac{d}{dS}\Delta\vec{M}_+ + \vec{t}_1 \times \Delta\vec{F}_+ &= \vec{0} \end{aligned} \tag{a}$$

Evaluating $\sum d_i\,\Delta P_i$,

$$\begin{aligned} \sum d_i\,\Delta P_i &= \left\{\Delta\vec{F}_+\cdot\frac{d\vec{u}}{dS} + \Delta\vec{M}_+\cdot\frac{d\vec{\omega}}{dS} + \left(\frac{d}{dS}\Delta\vec{M}_+\right)\cdot\vec{\omega}\right\}dS \\ &= \left\{\Delta F_1\left(\frac{du_1}{dS} - \frac{u_2}{R}\right) + \Delta F_2\left(\frac{du_2}{dS} + \frac{u_1}{R} - \omega\right) + \Delta M\frac{d\omega}{dS}\right\}dS \end{aligned} \tag{b}$$

and then substituting in (14–19) results in the following relations between the force and displacement parameters:

$$e_1 = \frac{\partial \bar{V}^*}{\partial F_1} = \frac{du_1}{dS} - \frac{u_2}{R}$$
$$e_2 = \frac{\partial \bar{V}^*}{\partial F_2} = \frac{du_2}{dS} + \frac{u_1}{R} - \omega \qquad (14\text{–}20)$$
$$k = \frac{\partial \bar{V}^*}{\partial M} = \frac{d\omega}{dS}$$

We interpret $e_1$ as an average extension, $e_2$ as an average transverse shear deformation, and $k$ as a bending deformation. Actually, $k$ is the relative rotation of adjacent cross sections. In what follows, we discuss the determination of $\bar{V}^*$.

Consider the differential volume element shown in Fig. 14–8. The vector defining the arc $QQ_1$ is

$$\overline{QQ}_1 = \frac{\partial \bar{r}_2}{dy} dy = \left(\frac{d\bar{r}}{dy} + y_2 \frac{d\bar{t}_2}{dy} + y_3 \frac{d\bar{t}_3}{dy}\right) dy \qquad \text{(a)}$$

Noting that

$$\frac{d\bar{r}}{dy} = \alpha \bar{t}_1$$
$$\frac{d\bar{t}_2}{dy} = -\frac{\alpha}{R} \bar{t}_1 \qquad \text{(b)}$$
$$\frac{d\bar{t}_3}{dy} = \bar{0}$$

for a planar member, (a) can be written as

$$dS_2 = |\overline{QQ}_1| = \alpha\left(1 - \frac{y_2}{R}\right) dy = \left(1 - \frac{y_2}{R}\right) dS \qquad \text{(c)}$$

By definition, $\bar{V}^*$ is the complementary energy per unit length along the *centroidal axis*. Substituting for $dS_2$ in the general definition, we obtain

$$\bar{V}^* \, dS = \iint_{y_2, y_3} V^* \, dS_2 \, dy_2 \, dy_3$$
$$\Downarrow \qquad (14\text{–}21)$$
$$\bar{V}^* = \iint V^* \left(1 - \frac{y_2}{R}\right) dA$$

In general, $V^* = V^*(\sigma_{11}, \sigma_{12}, \sigma_{13})$. We select suitable expansions for the stress components in terms of $F_1$, $F_2$, $M$, expand $V^*$, and integrate over the cross section. The only restriction on the stress expansions is that they satisfy the definition equations for the stress resultants and couples identically:

$$\iint \sigma_{11} \, dA = F_1 \qquad \iint \sigma_{12} \, dA = F_2 \qquad \iint \sigma_{13} \, dA = 0$$
$$\iint y_3 \sigma_{11} \, dA = 0 \qquad -\iint y_2 \sigma_{11} \, dA = M \qquad \text{(a)}$$
$$\iint (y_2 \sigma_{13} - y_3 \sigma_{12}) dA = 0$$



The most convenient choice for $\sigma_{11}$ is the linear expansion,†

$$\sigma_{11} = \frac{F_1}{A} - y_2 \frac{M}{I} \qquad (14\text{–}22)$$

where $I \equiv I_3$. A logical choice for $\sigma_{1j}$ (when the cross section is thin-walled) is the distribution predicted by the engineering theory of flexural shear stress distribution described in Sec. 11–7:

$$\sigma_{1j} = \frac{1}{t} q(F_2) \qquad q = F_2 \psi \qquad (14\text{–}23)$$

where $t$ denotes the local thickness, and $q$ is the flexural shear flow due to $F_2$. Both expansions satisfy (a).

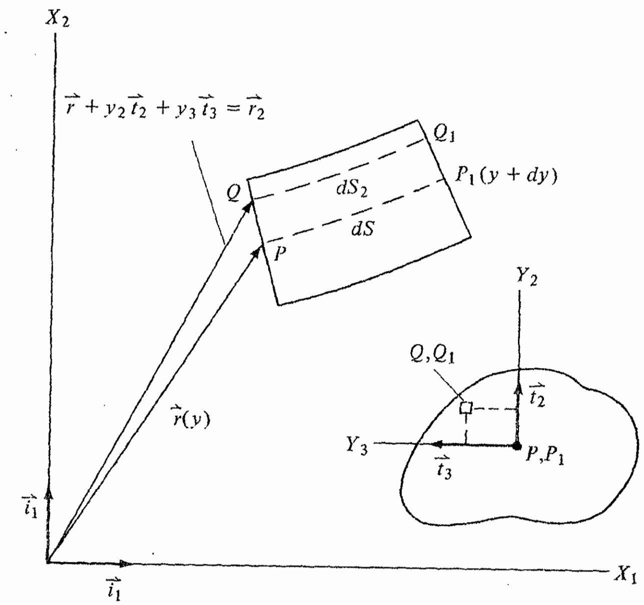



**Fig. 14–8.** Differential volume element.

In what follows, we consider the material to be linearly elastic. The complementary energy density is given by

$$V^* = \varepsilon_1^0 \sigma_{11} + \frac{1}{2E} \sigma_{11}^2 + \frac{1}{2G} (\sigma_{12}^2 + \sigma_{13}^2) \qquad \text{(a)}$$

where $\varepsilon_1^0$ is the initial extensional strain. Substituting (a) in (14–21) and taking the stresses according to (14–22), (14–23) results in the following expression

† This applies for a homogeneous beam. Composite beams are more conveniently treated with the approach described in the next section.

for $\bar{V}^*$:

$$\bar{V}^* = e_1^0 F_1 + k^0 M + \frac{1}{2AE} F_1^2 + \frac{1}{AER} F_1 M + \frac{1}{2EI^*} M^2 + \frac{1}{2GA_2^*} F_2^2 \qquad (14\text{–}24)$$

where

$$e_1^0 = \frac{1}{A} \iint \varepsilon_1^0 \left(1 - \frac{y_2}{R}\right) dA$$
$$k^0 = \frac{-1}{I} \iint y_2 \varepsilon_1^0 \left(1 - \frac{y_2}{R}\right) dA$$
$$\frac{1}{I^*} = \frac{1}{I}\left(1 - \frac{1}{IR} \iint y_2^3 \, dA\right)$$
$$\frac{1}{A_2^*} = \int_S \frac{\psi^2}{t} \left(1 - \frac{y_2}{R}\right) dS$$

If the section is *symmetrical* with respect to the $Y_3$ axis, $I^* = I$ and $A_2^* = A_2$.
The deformation-force relations corresponding to this choice for $\bar{V}^*$ are

$$\begin{aligned} e_1 &= e_1^0 + \frac{F_1}{AE} + \frac{M}{AER} = \frac{du_1}{dS} - \frac{u_2}{R} \\ e_2 &= \frac{F_2}{GA_2^*} = \frac{du_2}{dS} + \frac{u_1}{R} - \omega \\ k &= k^0 + \frac{F_1}{AER} + \frac{M}{EI^*} = \frac{d\omega}{dS} \end{aligned} \qquad (14\text{–}25)$$

Note that the axial force and moment are coupled, due to the curvature. Inverting (14–25) leads to expressions for the forces in terms of the deformations:

$$\begin{aligned} F_1 &= \frac{EA}{1-\delta}(e_1 - e_1^0) - \frac{EI^*}{R(1-\delta)}(k - k^0) \\ M &= -\frac{EI^*}{R(1-\delta)}(e_1 - e_1^0) + \frac{EI^*}{1-\delta}(k - k^0) \\ F_2 &= GA_2^* e_2 \\ \delta &= \frac{I^*}{AR^2} = \frac{I}{AR^2}\left[1 - \frac{1}{IR}\iint y_2^3 \, dA\right]^{-1} \end{aligned} \qquad (14\text{–}26)$$

We observe that

$$\frac{I}{AR^2} = \left(\frac{\rho}{R}\right)^2 = 0\left(\frac{d}{R}\right)^2 \qquad \text{(a)}$$

where $\rho$ is the radius of gyration and $d$ is the depth of the cross section. For example,

$$\frac{I}{AR^2} = \frac{d^2}{12R^2} \qquad \text{(b)}$$

for a rectangular cross section. Then, $\delta$ is of the order of $(d/R^2)$ and can be neglected when $(d/R)^2 \ll 1$.

A curved member is said to be *thin* when $0(d/R) \ll 1$ and *thick* when $0(d/R)^2 \ll 1$. We set $\delta = 0$ for a thick member. The *thinness* assumption is introduced by neglecting $y_2/R$ with respect to unity in the expression for the differential arc length, i.e., by taking



$$\begin{aligned} dS_2 &\approx dS \\ \bar{V}^* &\approx \iint_A V^* \, dA \end{aligned} \qquad (14\text{–}27)$$

Assuming a curved member to be *thin* is equivalent to using the expression for $\bar{V}^*$ developed for a prismatic member. The approximate form of (14–25) for a thin member is

$$\begin{aligned} e_1 &= e_1^0 + \frac{F_1}{AE} = \frac{du_1}{dS} - \frac{u_2}{R} \\ e_2 &= \frac{F_2}{GA_2} = \frac{du_2}{dS} + \frac{u_1}{R} - \omega \\ k &= k^0 + \frac{M}{EI} = \frac{d\omega}{dS} \end{aligned} \qquad (14\text{–}28)$$

To complete the treatment of the linear elastic case, we list the expanded forms of the principle of virtual forces for *thick and thin* members. Note that these expressions are based on a linear variation in normal stress over the cross section.

***Thick Member***

$$\int_S \left\{\left(e_1^0 + \frac{F_1}{AE} + \frac{M}{AER}\right)\Delta F_1 + \frac{F_2}{GA_2^*}\Delta F_2 + \left(k^0 + \frac{F_1}{AER} + \frac{M}{EI^*}\right)\Delta M\right\} dS = \sum d_i \, \Delta P_i \qquad (14\text{–}29)$$

***Thin Member***

$$\int_S \left\{\left(e_1^0 + \frac{F_1}{AE}\right)\Delta F_1 + \frac{F_2}{GA_2}\Delta F_2 + \left(k^0 + \frac{M}{EI}\right)\Delta M\right\} dS = \sum d_i \, \Delta P_i \qquad (14\text{–}30)$$

## 14–4. FORCE-DISPLACEMENT RELATIONS—DISPLACEMENT EXPANSION APPROACH; PRINCIPLE OF VIRTUAL DISPLACEMENTS

In the variational procedure for establishing one-dimensional force-displacement relations, it is not necessary to analyze the deformation, i.e., to determine the strains at a point. One has only to introduce suitable expansions for the stress components in terms of the one-dimensional force parameters. Now, we can also establish force-displacement relations by starting with expansions for the displacement components in terms of one-dimensional displacement parameters and determining the corresponding strain distribution. We express the stresses in terms of the displacement parameters using the stress-strain relations, and then substitute the stress expansions in the definition equations for $F_1$, $F_2$, and $M$. The effect of transverse shear deformation is usually neglected in this approach. To determine the strain distribution, we must first analyze the deformation at a point. This step is described in detail below.

Figure 14–9 shows the initial position of two orthogonal line elements, $QQ_1$ and $QQ_2$, at a point $(y, y_2, y_3)$. The vectors defining these elements are

$$\begin{aligned} \overline{QQ}_1 &= \frac{\partial \bar{r}_2}{\partial y} dy = \alpha_2 \, dy \, \bar{t}_1 \\ \overline{QQ}_2 &= \frac{\partial \bar{r}_2}{\partial y_2} dy_2 = dy_2 \, \bar{t}_2 \\ \alpha_2 &= \left(1 - \frac{y_2}{R}\right) \alpha \end{aligned} \tag{14–31}$$

We use a prime superscript to denote quantities associated with the deformed position of the member, which is shown in Fig. 14–10; for example:

$\bar{r}' = \bar{r}'(y) =$ position vector to point $P(y)$ in the deformed position (point $P'$).

$\bar{t}'_1 =$ tangent vector to the deformed centroidal axis.

$\bar{r}'_2 =$ position vector to $Q(y, y_2, y_3)$ in the deformed position (point $Q'$).

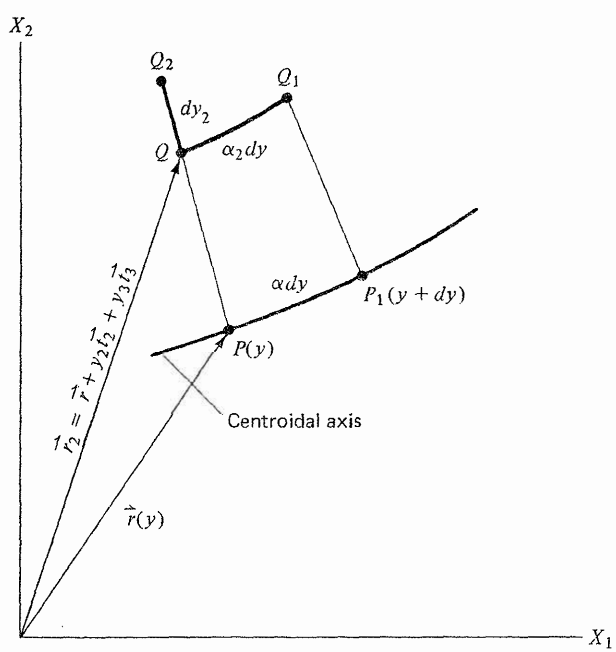



**Fig. 14–9.** Initial geometry for orthogonal curvilinear line elements.



From Fig. 14–10, and noting (14–31):

$$\begin{aligned} \overline{P'P'}_1 &= \frac{d\bar{r}'}{dy} dy = \left(\alpha \bar{t}_1 + \frac{d\bar{u}}{dy}\right) dy = \alpha' \, dy \bar{t}'_1 \\ \overline{Q'Q'}_1 &= \frac{\partial \bar{r}'_2}{\partial y} dy = \left(\alpha_2 \bar{t}_1 + \frac{\partial \bar{u}_2}{\partial y}\right) dy \\ \overline{Q'Q'}_2 &= \frac{\partial \bar{r}'_2}{\partial y_2} dy_2 = \left(\bar{t}_2 + \frac{\partial \bar{u}_2}{\partial y_2}\right) dy_2 \end{aligned} \tag{14–32}$$

The analysis of strain consists of determining the extensions and change in angle between the line elements. We denote the extensional strains by $\varepsilon_j$ $(j = 1, 2)$ and the shearing strain by $\gamma_{12}$. The general expressions are

$$\begin{aligned} \varepsilon_j &= \frac{|\overline{Q'Q'}_j|}{|\overline{QQ}_j|} - 1 \qquad j = 1, 2 \\ \sin \gamma_{12} &= \frac{\overline{Q'Q'}_1 \cdot \overline{Q'Q'}_2}{|\overline{Q'Q'}_1| \, |\overline{Q'Q'}_2|} \end{aligned} \tag{14–33}$$

Now, we restrict this discussion to small strain. Substituting for the deformed vectors and neglecting strains with respect to unity, (14–33) expands to

$$\begin{aligned} \varepsilon_1 &\approx \frac{1}{\alpha_2} \bar{t}_1 \cdot \frac{\partial \bar{u}_2}{\partial y} + \frac{1}{2(\alpha_2)^2} \frac{\partial \bar{u}_2}{\partial y} \cdot \frac{\partial \bar{u}_2}{\partial y} \\ \varepsilon_2 &\approx \bar{t}_2 \cdot \frac{\partial \bar{u}_2}{\partial y_2} + \frac{1}{2} \frac{\partial \bar{u}_2}{\partial y_2} \cdot \frac{\partial \bar{u}_2}{\partial y_2} \\ \gamma_{12} &\approx \bar{t}_1 \cdot \frac{\partial \bar{u}_2}{\partial y_2} + \frac{1}{\alpha_2} \bar{t}_2 \cdot \frac{\partial \bar{u}_2}{\partial y} + \frac{1}{\alpha_2} \frac{\partial \bar{u}_2}{\partial y} \cdot \frac{\partial \bar{u}_2}{\partial y_2} \end{aligned} \tag{14–34}$$

The nonlinear terms are associated with the rotation of the tangent vector. Neglecting these terms corresponds to neglecting the difference between the deformed and undeformed geometry, i.e., to assuming linear geometry.

The next step involves introducing an expansion for $\bar{u}_2$ in terms of $y_2$. We express $\bar{u}_2$ as a linear function of $y_2$.

$$\bar{u}_2 = \bar{u} - \omega y_2 \bar{t}_1 \tag{14–35}$$

where $\omega = \omega(y)$ and

$$\bar{u} = u_1 \bar{t}_1 + u_2 \bar{t}_2 = \bar{u}(y) \tag{14–36}$$

is the displacement vector for a point on the centroidal axis. Equation (14–35) implies that a normal cross section remains a *plane* after deformation. One can interpret $\omega$ as the rotation of the cross section in the direction from $\bar{t}_1$ toward $\bar{t}_2$. This notation is illustrated in Fig. 14–11.

In what follows, we consider only linear geometry. Substituting for $\bar{u}_2$, taking $y = S$, and evaluating the derivatives lead to the following strain expansions:

$$\varepsilon_1 = \frac{1}{1 - y_2/R}(e_1 - y_2 k) \qquad e_1 = \frac{du_1}{dS} - \frac{u_2}{R} = |\varepsilon_1|_{y_2=0}$$

$$\varepsilon_2 = 0 \qquad e_2 = \frac{du_2}{dS} + \frac{u_1}{R} - \omega = |\gamma_{12}|_{y_2=0} \qquad (14\text{–}37)$$

$$\gamma_{12} = \frac{1}{1 - y_2/R} e_2 \qquad k = \frac{d\omega}{dS}$$

The vanishing of $\varepsilon_2$ is due to our choice for $\vec{u}_2$. One could include an additional linear term, $\beta y_2 \vec{t}_2$. This would give $\varepsilon_2 = \beta$ and additional terms in the

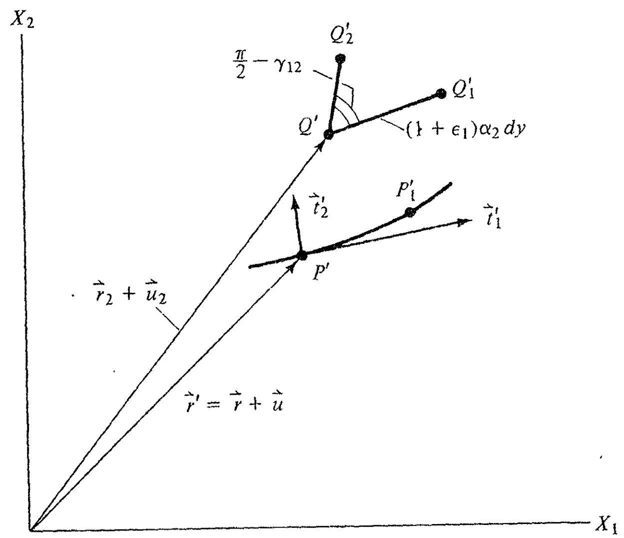



**Fig. 14–10.** Deformed geometry for orthogonal curvilinear line elements.

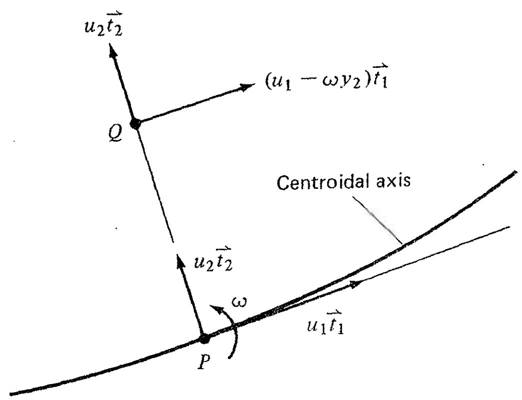



**Fig. 14–11.** Displacement expansion.



expressions for $\varepsilon_1$ and $\gamma_{12}$. Note that the assumption that a normal cross section remains plane does not lead to a linear variation in extensional strain over the depth when the member is curved.

We introduce the assumption of negligible transverse deformation by setting $e_2 = 0$. The resulting expressions for $\omega$ and $k$ in terms of $u_1$ and $u_2$ are

$$e_2 = 0$$
$$\Downarrow$$
$$\omega = \frac{du_2}{dS} + \frac{u_1}{R} \qquad (14\text{–}38)$$
$$k = \frac{d\omega}{dS} = \frac{d^2u_2}{dS^2} + \frac{d}{dS}\left(\frac{u_1}{R}\right)$$

When transverse shear deformation is neglected, one must determine $F_2$ using the moment-equilibrium equation.

The next step involves expressing $F_1$, $F_2$, and $M$ in terms of the one-dimensional deformation parameters $e_1 e_2$ and $k$. In what follows, we consider the material to be linearly elastic and take the stress-strain relations for $\sigma_{11}$, $\sigma_{12}$ as:†

$$\sigma_{11} = E(\varepsilon_1 - \varepsilon_1^0) \qquad \sigma_{12} = G\gamma_{12} \qquad \text{(a)}$$

Substituting for $\varepsilon_1$, $\gamma_{12}$, using (14–37),

$$\sigma_{11} = \frac{E}{1 - y_2/R}(e_1 - y_2 k) - E\varepsilon_1^0$$
$$\sigma_{12} = \frac{G}{1 - y_2/R} e_2 \qquad (14\text{–}39)$$

and then evaluating $F_1$, $F_2$, and $M$, we obtain

$$F_1 = Ee_1 \iint \frac{dA}{1 - y_2/R} - Ek \iint \frac{y_2 \, dA}{1 - y_2/R} - E \iint \varepsilon_1^0 \, dA$$
$$F_2 = Ge_2 \iint \frac{dA}{1 - y_2/R} \qquad (14\text{–}40)$$
$$M = -Ee_1 \iint \frac{y_2 \, dA}{1 - y_2/R} + Ek \iint \frac{y_2^2 \, dA}{1 - y_2/R} + \iint y_2 \varepsilon_1^0 \, dA$$

The various integrals can be expressed in terms of only one integral by using the identity

$$\frac{1}{1 - y_2/R} \equiv 1 + \frac{y_2/R}{1 - y_2/R} \qquad \text{(a)}$$

and noting that $Y_3$ is a centroidal axis:

$$\iint y_2 \, dA = 0 \qquad \text{(b)}$$

† The relation for $\sigma_{11}$ is exact only when $\sigma_{22} = \sigma_{33} = 0$. We generally neglect $\sigma_{22}$, $\sigma_{33}$ for a member.

One can easily show that

$$\iint \frac{dA}{1 - y_2/R} = A + \frac{I'}{R^2}$$
$$\iint \frac{y_2\, dA}{1 - y_2/R} = \frac{I'}{R} \qquad (14\text{–}41)$$
$$I' = \iint \frac{y_2^2\, dA}{1 - y_2/R}$$

For completeness, we list the inverted form of (14–40),

$$e_1 = e_1^0 + \frac{F_1}{EA} + \frac{M}{EAR}$$
$$e_2 = \frac{F_2}{GA_2'}$$
$$k = k^0 + \frac{F_1}{EAR} + \frac{M}{EI''}$$

where

$$I'' = I' \Big/ \left(1 + \frac{I'}{AR^2}\right)$$
$$A_2' = A\left(1 + \frac{I'}{AR^2}\right)$$
$$e_1^0 = \frac{1}{A}\iint \varepsilon_1^0 \left(1 - \frac{y_2}{R}\right) dA \qquad (14\text{–}42)$$
$$k^0 = \frac{-1}{I'}\iint \varepsilon_1^0 \left(y_2 - \frac{I'}{AR}\right) dA$$

The expressions for $e_1$ are identical with the result (see (14–25)) obtained with the variational approach. However, the result for $k$ differs in the coefficient for $M$. This difference ($I'$ or $I''$) is due to the nonlinear expansion used for $\sigma_{11}$.

**Example 14–1**

We determine $I'$ for the rectangular cross section shown in Fig. E14–1.

$$I' = \iint \frac{y_2^2\, dA}{1 - y_2/R} = b\int_{-d/2}^{d/2} \frac{y_2^2\, dy_2}{1 - y_2/R}$$
$$= -R^2 bd + R^3 b \ln\left(\frac{1 + \dfrac{d}{2R}}{1 - \dfrac{d}{2R}}\right) \qquad \text{(a)}$$

To obtain a more tractable form, we expand the log terms, using

$$\ln\left(\frac{1 + x}{1 - x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots\right) \qquad \text{(b)}$$



This series converges for $|x| < 1$. Then

$$\ln\left(\frac{1 + \dfrac{d}{2R}}{1 - \dfrac{d}{2R}}\right) = \frac{d}{R} + \frac{d^3}{12R^3}\left\{1 + \frac{3}{5}\left(\frac{d}{2R}\right)^2 + \frac{3}{7}\left(\frac{d}{2R}\right)^4 + \cdots\right\} \qquad \text{(c)}$$

and

$$I' = I\left\{1 + \frac{3}{20}\left(\frac{d}{R}\right)^2 + \frac{3}{112}\left(\frac{d}{R}\right)^4 + \cdots\right\} \qquad \text{(d)}$$

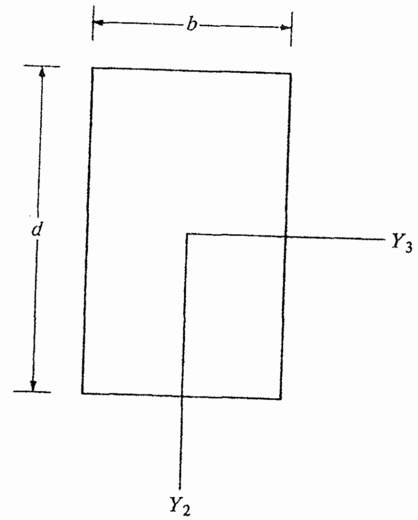



Fig. E14–1

The relations listed above involve exact integrals. Now, when the member is thick, we neglect $(y_2/R)^2$ with respect to unity. This assumption is introduced by taking

$$\frac{1}{1 - y_2/R} = 1 + \frac{y_2}{R} + \left(\frac{y_2}{R}\right)^2 + \cdots \approx 1 + \frac{y_2}{R} \qquad \text{(a)}$$

in the expansions for $\sigma_{11}$, $\sigma_{12}$ and $I'$:

$$\frac{\sigma_{11}}{E} \approx e_1\left(1 + \frac{y_2}{R}\right) - k\left(y_2 + \frac{y_2^2}{R}\right) - \varepsilon_1^0$$
$$\frac{\sigma_{12}}{G} \approx e_2\left(1 + \frac{y_2}{R}\right)$$
$$I' = \iint \frac{y_2^2\, dA}{1 - y_2/R} \approx \iint \left(y_2^2 + \frac{y_2^3}{R}\right) dA \qquad (14\text{–}43)$$
$$= I\left\{1 + \frac{1}{IR}\iint y_2^3\, dA\right\}$$

To be consistent, we must also neglect $I'/AR^2$ with respect to unity in the expression for $A'_2$ and $I''$. When the member is thin, we neglect $y_2/R$ with respect to unity.

$$\frac{1}{1 - y_2/R} \approx 1$$

$$\frac{\sigma_{11}}{E} \approx e_1 - y_2 k - \varepsilon_1^0 \qquad (14\text{–}44)$$

$$\frac{\sigma_{12}}{G} \approx e_2$$

It is of interest to establish the one-dimensional form of the principle of virtual displacements corresponding to the linear displacement expansion used in this development. The general three-dimensional form for an orthogonal coordinate system is (see Sec. 10–6):

$$\iiint(\sigma_{11}\,\delta\varepsilon_1 + \cdots + \sigma_{12}\,\delta\gamma_{12} + \cdots)d(\text{vol.}) = \sum P_i\,\Delta d_i \qquad \text{(a)}$$

where $P_i$ represents an external force quantity and $d_i$ is the displacement quantity corresponding to $P_i$. We consider only $\varepsilon_1$ and $\gamma_{12}$ to be finite, and express the differential volume in terms of the cross-sectional coordinates $y_2$, $y_3$ and arc length along the centroidal axes (see Fig. 14–9):

$$d(\text{vol.}) = dS_2\,dy_2\,dy_3 = \left(1 - \frac{y_2}{R}\right) dS\,dy_2\,dy_3 \qquad \text{(b)}$$

Then (a) reduces to

$$\int_S \left[\iint_A (\sigma_{11}\,\delta\varepsilon_1 + \sigma_{12}\,\delta\gamma_{12})\left(1 - \frac{y_2}{R}\right) dA\right] dS = \sum P_i\,\Delta d_i \qquad (14\text{–}45)$$

We take (14–45) as the form of the principle of virtual displacements for *planar deformation.*

The strains corresponding to a linear expansion for displacements and linear geometry are defined by (14–37), which are listed below for convenience:

$$\varepsilon_1 = \frac{1}{1 - y_2/R}(e_1 - y_2 k) \qquad \gamma_{12} = \frac{1}{1 - y_2/R} e_2 \qquad \text{(c)}$$

$$e_1 = \frac{du_1}{dS} - \frac{u_2}{R} \qquad e_2 = \frac{du_2}{dS} + \frac{u_1}{R} - \omega \qquad k = \frac{d\omega}{dS} \qquad \text{(d)}$$

Substituting for $\varepsilon_1$, $\gamma_{12}$ and using the definition equations for $F_1$, $F_2$, and $M$,



we obtain

$$\int_S [F_1\,\delta e_1 + F_2\,\delta e_2 + M\,\delta k] dS = \sum P_i\,\Delta d_i \qquad (14\text{–}46)$$

This result depends only on the strain expansions, i.e., (c). One can apply it for the geometrically nonlinear case, provided that (c) are taken as defining the strain distribution over the cross section.

We use the principle of virtual displacements to establish consistent force-equilibrium equations. One starts with one-dimensional deformation-displacement relations, substitutes in (14–46), and integrates the left-hand side by parts. Equating coefficients of the displacement parameters leads to a set of force equilibrium equations and boundary conditions that are consistent with the geometrical assumptions introduced in establishing the deformation-displacement relations. The following example illustrates this application.

**Example 14–2**

The assumption of negligible transverse shear deformation is introduced by setting $e_2$ equal to zero. This leads to an expression for the rotation, $\omega$, in terms of the translation components,

$$\omega = \frac{du_2}{dS} + \frac{u_1}{R} \qquad \text{(a)}$$

and the relations for negligible transverse shear deformation reduce to

$$\int_S [F_1\,\delta e_1 + M\,\delta k] dS = \sum P_i\,\Delta d_i \qquad \text{(b)}$$

$$e_1 = \frac{du_1}{dS} - \frac{u_2}{R} \qquad k = \frac{d\omega}{dS} = \frac{d}{dS}\left(\frac{du_2}{dS} + \frac{u_1}{R}\right) \qquad \text{(c)}$$

Substituting for $\Delta\omega$ and the strain variations,

$$\Delta\omega = \frac{\Delta u_1}{R} + \frac{d}{dS}\Delta u_2 \qquad \delta e_1 = \frac{d}{dS}\Delta u_1 - \frac{1}{R}\Delta u_2 \qquad \delta k = \frac{d^2}{dS^2}\Delta u_2 + \frac{1}{R}\frac{d}{dS}\Delta u_1 \qquad \text{(d)}$$

and integrating by parts, the left- and right-hand sides of (b) expand to

$$\int_{S_A}^{S_B} [F_1\,\delta e_1 + M\,\delta k] dS$$

$$= \left|\left(F_1 + \frac{M}{R}\right)\Delta u_1 - \frac{dM}{dS}\Delta u_2 + M\frac{d}{dS}\Delta u_2\right|_{S=S_B} - \left|\left(F_1 + \frac{M}{R}\right)\Delta u_1 - \frac{dM}{dS}\Delta u_2 + M\frac{d}{dS}\Delta u_2\right|_{S=S_A} + \int_{S_A}^{S_B}\left\{\Delta u_1\left[-\frac{dF_1}{dS} - \frac{1}{R}\frac{dM}{dS}\right] + \Delta u_2\left[-\frac{F_1}{R} + \frac{d^2M}{dS^2}\right]\right\} dS \qquad \text{(e)}$$

and

$$\sum P_i\,\Delta d_i = \int_{S_A}^{S_B}\left\{\Delta u_1\left(b_1 + \frac{m}{R}\right) + \Delta u_2\left(b_2 - \frac{dm}{dS}\right)\right\} dS$$
$$+ \left(\bar{F}_{B1} + \frac{\bar{M}_B}{R}\right)\Delta u_{B1} + \left(\bar{F}_{B2} + m_B\right)\Delta u_{B2} + \bar{M}_B\,\Delta\left(\frac{du_2}{dS}\right)_B \qquad \text{(f)}$$
$$+ \left(\bar{F}_{A1} + \frac{\bar{M}_A}{R}\right)\Delta u_{A1} + \left(\bar{F}_{A2} - m_A\right)\Delta u_{A2} + \bar{M}_A\,\Delta\left(\frac{du_2}{dS}\right)_A$$

The consistent equilibrium equations and boundary conditions for negligible transverse shear deformation follow by equating corresponding coefficients of the displacement variations in (e) and (f):

$S_A < S < S_B$

$$+\frac{dF_1}{dS} + \frac{1}{R}\frac{dM}{dS} + b_1 + \frac{m}{R} = 0$$
$$+\frac{F_1}{R} - \frac{d^2M}{dS^2} + b_2 - \frac{dm}{dS} = 0$$

$S = S_A$

| | | |
|---|---|---|
| $u_1$ | prescribed or | $F_1 = -\bar{F}_{A1}$ |
| $u_2$ | prescribed or | $\dfrac{dM}{dS} = \bar{F}_{A2} - m$ |
| $\dfrac{du_2}{dS}$ | prescribed or | $M = -\bar{M}_A$ |

(g)

$S = S_B$

| | | |
|---|---|---|
| $u_1$ | prescribed or | $F_1 = \bar{F}_{B1}$ |
| $u_2$ | prescribed or | $\dfrac{dM}{dS} = -\bar{F}_{B2} - m$ |
| $\dfrac{du_2}{dS}$ | prescribed or | $M = \bar{M}_B$ |

One can obtain (g) by solving the last equation in (14–14) for $F_2$ and substituting in the first two equations.

Suppose we neglect $u_1/R$ in the expression for $\omega$:

$$\omega \approx \frac{du_2}{dS} \qquad \text{(h)}$$
$$k \approx \frac{d^2u_2}{dS^2}$$

This assumption† is generally referred to as *Mushtari's approximation*. The equilibrium equations for the tangential direction reduce to

$$-\frac{dF_1}{dS} = b_1 \qquad \text{(i)}$$

† See Ref. 5.



The other equilibrium equation and the boundary conditions are not changed. Using (h) instead of (a) eliminates the shear term, $F_2/R$, in the tangential force-equilibrium equation.

## 14–5. CARTESIAN FORMULATION

We consider the case where the equation defining the centroidal axis has the form $x_2 = f(x_1)$. The geometrical relations for this parametric representation are obtained by taking $y = x_1$ in (14–7). They are summarized below† for convenience and the notation is shown in Fig. 14–12:

$$dS = \alpha\,dx_1$$
$$\tan\theta = \frac{df}{dx_1}$$
$$\alpha = \left[1 + \left(\frac{df}{dx_1}\right)^2\right]^{1/2} = \frac{1}{\cos\theta}$$
$$\bar{t}_1 = \frac{1}{\alpha}\left[\bar{\imath}_1 + \left(\frac{df}{dx_1}\right)\bar{\imath}_2\right] = \frac{1}{\alpha}\frac{d\bar{r}}{dx_1}$$
$$\bar{t}_2 = \frac{1}{\alpha}\left[-\left(\frac{df}{dx_1}\right)\bar{\imath}_1 + \bar{\imath}_2\right] \qquad \text{(14–47)}$$
$$\bar{t}_3 = \bar{t}_1 \times \bar{t}_2 = \bar{\imath}_3$$
$$\frac{1}{R} = \frac{+\dfrac{d^2f}{dx_1^2}}{\left[1 + \left(\dfrac{df}{dx_1}\right)^2\right]^{3/2}}$$
$$\frac{d\bar{t}_1}{dS} = \frac{1}{R}\bar{t}_2 \qquad \frac{d\bar{t}_2}{dS} = -\frac{1}{R}\bar{t}_1$$

In the previous formulation, we worked with displacement components and external force components referred to the local frame. An alternate approach, originally suggested by Marguerre,‡ involves working with components referred to the *basic* frame rather than the local frame. The resulting expressions differ, and it is therefore of interest to describe this approach in detail. We start with the determination of the force-equilibrium equations.

Consider the differential element shown in Fig. 14–13. The vector equilibrium equations are

$$\frac{d\bar{F}_+}{dx_1} + \bar{p} = \bar{0} \qquad \text{(14–48)}$$
$$\frac{d\bar{M}_+}{dx_1} + \frac{d\bar{r}}{dx_1} \times \bar{F}_+ + \bar{h} = 0$$

† See Prob. 14–1.
‡ See Ref. 6.

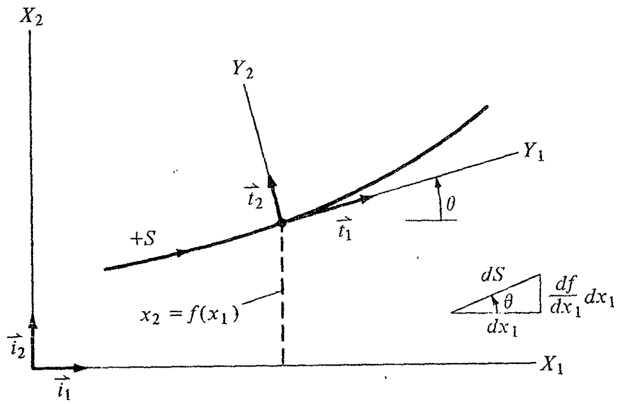



**Fig. 14–12.** Notation for Cartesian formulation.

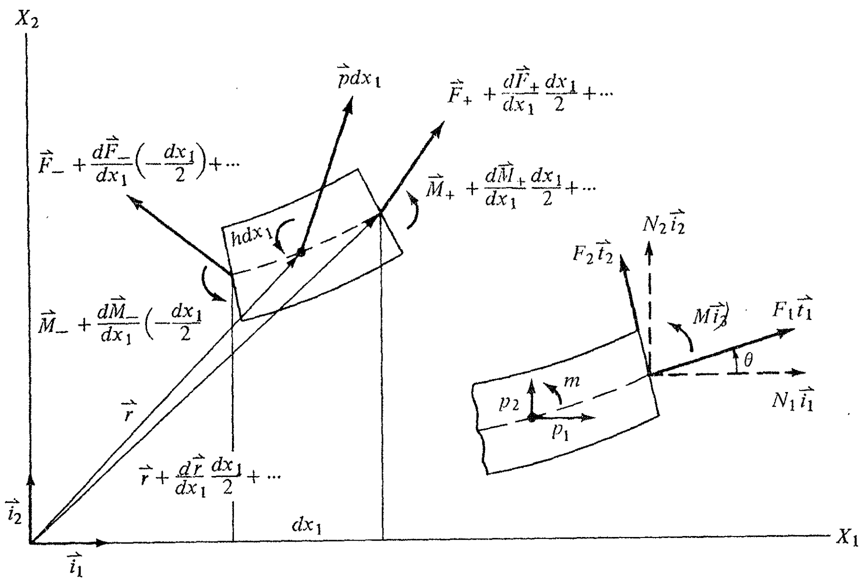



**Fig. 14–13.** Differential element for equilibrium analysis.



where $\vec{p}$, $\vec{h}$ are the external applied force and moment vectors per unit *projected* length, i.e., per unit $x_1$. They are related to $\vec{b}$ and $\vec{m}$ (see Fig. 14–4) by

$$\begin{aligned} \vec{p}\,dx_1 &= \vec{b}\,dS = (\alpha\vec{b})dx_1 \\ \vec{h}\,dx_1 &= \vec{m}\,dS = (\alpha\vec{m})dx_1 \end{aligned} \tag{14–49}$$

Substituting for the force and moment vectors,

$$\begin{aligned} \vec{F}_+ &= F_1\vec{t}_1 + F_2\vec{t}_2 = N_1\vec{i}_1 + N_2\vec{i}_2 \\ \vec{M}_+ &= M\vec{i}_3 \qquad \vec{h} = h\vec{i}_3 \\ \vec{p} &= p_1\vec{i}_1 + p_2\vec{i}_2 \\ N_1 &= F_1\cos\theta - F_2\sin\theta \\ N_2 &= F_1\sin\theta + F_2\cos\theta \end{aligned} \tag{14–50}$$

the equilibrium equations expand to

$$\begin{aligned} \frac{dN_1}{dx_1} &= \frac{d}{dx_1}(F_1\cos\theta - F_2\sin\theta) = -p_1 \\ \frac{dN_2}{dx_1} &= \frac{d}{dx_1}(F_1\sin\theta + F_2\cos\theta) = -p_2 \\ \frac{-1}{\alpha}\left(\frac{dM}{dx_1} + h\right) &= F_2 = -N_1\sin\theta + N_2\cos\theta \end{aligned} \tag{14–51}$$

We restrict this treatment to an elastic material and establish the force-displacement relations, using the principle of virtual forces,

$$\int_{x_1} \alpha\, d\bar{V}^*\, dx_1 = \int_{x_1} [e_1\,\Delta F_1 + e_2\,\Delta F_2 + k\,\Delta M]\alpha\, dx_1 = \sum d_i\,\Delta P_i \tag{a}$$

where $\bar{V}^* = \bar{V}^*(F_1, F_2\ M)$ is the complementary energy per unit arc length. Consider the differential element shown in Fig. 14–14. The virtual-force system is statically permissible, i.e., it satisfies the force-equilibrium equations identically:

$$\begin{aligned} \frac{d}{dx_1}\Delta\vec{F}_+ &= \vec{0} \\ \frac{d}{dx_1}\Delta\vec{M} + \alpha\vec{t}_1 \times \Delta\vec{F}_+ &= \vec{0} \end{aligned} \tag{b}$$

Expanding $\sum d_i\,\Delta P_i$,

$$\sum d_i\,\Delta P_i = \left[\Delta\vec{F}_+ \cdot \left(\frac{d\vec{u}}{dx_1} + \alpha\vec{t}_1 \times \vec{\omega}\right) + \Delta\vec{M}_+ \cdot \frac{d\vec{\omega}}{dx_1}\right] dx_1 \tag{c}$$

and then substituting for the displacement and rotation vectors,

$$\begin{aligned} \vec{u} &= v_1\vec{i}_1 + v_2\vec{i}_2 \\ \omega &= \omega\vec{i}_3 = \omega\vec{t}_3 \end{aligned} \tag{14–52}$$

we obtain

$$\sum d_i \, \Delta P_i = \left( \Delta N_1 \frac{dv_1}{dx_1} + \Delta N_2 \frac{dv_2}{dx_1} - \alpha \, \Delta F_2 \omega + \Delta M \frac{d\omega}{dx_1} \right) dx_1 \qquad \text{(d)}$$

Finally, substituting for $N_1$, $N_2$ in terms of $F_1$, $F_2$ and equating coefficients of the force increments result in

$$\begin{aligned} e_1 &= \frac{\partial \bar{V}^*}{\partial F_1} = \cos^2 \theta \frac{dv_1}{dx_1} + \sin \theta \cos \theta \frac{dv_2}{dx_1} \\ e_2 &= \frac{\partial \bar{V}^*}{\partial F_2} = -\sin \theta \cos \theta \frac{dv_1}{dx_1} + \cos^2 \theta \frac{dv_2}{dx_1} - \omega \\ k &= \frac{\partial \bar{V}^*}{\partial M} = \frac{d\omega}{dx_1} \cos \theta \end{aligned} \qquad (14\text{–}53)$$

The member is said to be shallow when $\theta^2 \ll 1$. One introduces this assumption by setting

$$\cos \theta \approx 1 \qquad \sin \theta \approx \tan \theta = \frac{df}{dx_1} \qquad (14\text{–}54)$$

in (14–50), which relate the *cartesian* and *local* forces.

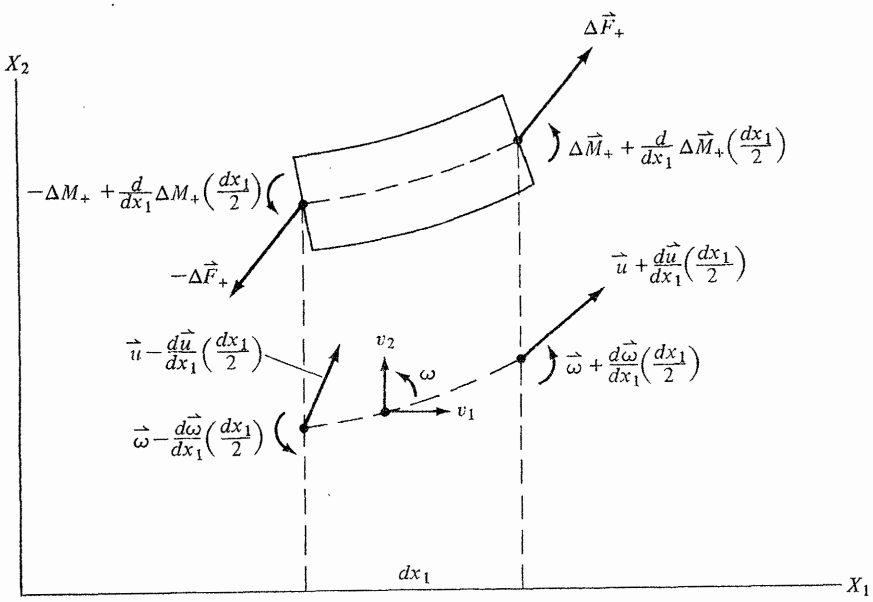



**Fig. 14–14.** Virtual force system.

Marguerre's equations are obtained by assuming the member is shallow and, in addition, neglecting the contribution of $F_2$ in the expression for $N_1$.



Marguerre starts with

$$\begin{aligned} N_1 &\approx F_1 \\ N_2 &\approx F_2 + \left(\frac{df}{dx_1}\right) F_1 \end{aligned} \qquad \text{(a)}$$

and the resulting equations are

$$\begin{aligned} \frac{dF_1}{dx_1} + p_1 &= 0 \\ \frac{dF_2}{dx_1} + \frac{d}{dx_1}\left(F_1 \frac{df}{dx_1}\right) + p_2 &= 0 \\ F_2 &= -\frac{dM}{dx_1} - m \\ e_1 &= \frac{\partial \bar{V}^*}{\partial F_1} = \frac{dv^1}{dx_1} + \frac{df}{dx_1}\frac{dv_2}{dx_1} \\ e_2 &= \frac{\partial \bar{V}^*}{\partial F_2} = \frac{dv_2}{dx_1} - \omega \\ k &= \frac{\partial \bar{V}^*}{\partial M} = \frac{d\omega}{dx_1} \end{aligned} \qquad (14\text{–}55)$$

One step remains, namely, to establish the boundary conditions. The general conditions are

$$\left.\begin{matrix} v_1 & \text{or} & N_1 \\ v_2 & \text{or} & N_2 \\ M & \text{or} & \omega \end{matrix}\right\} \text{prescribed at each end} \qquad (14\text{–}56)$$

We obtain the appropriate boundary conditions for the various cases considered above by substituting for $N_1$, $N_2$ and $\omega$. For example, the boundary conditions for the Marguerre formulation are

$$\left.\begin{matrix} v_1 & \text{or} & F_1 \\ v_2 & \text{or} & F_2 + \dfrac{df}{dx_1} F_1 \\ \omega & \text{or} & M \end{matrix}\right\} \text{prescribed at each end} \qquad (14\text{–}57)$$

## 14–6. DISPLACEMENT METHOD OF SOLUTION—CIRCULAR MEMBER

The displacement method involves solving the system of governing differential equations which, for the planar case, consist of three force-equilibrium equations and three force-displacement equations. If the applied loads are independent of the displacements, we can first solve the force equilibrium equations and then integrate the force-displacement relations. This method is quite straightforward for the prismatic case since stretching and flexure are uncoupled. However, it is usually quite difficult to apply when the member is curved (except when it is circular) or the cross section varies. In what follows, we illustrate the application of the displacement method to a circular *member* having a *constant* cross section, starting with—

1. the *exact* equations (based on stress expansions) for a thick member
2. Marguerre's equations for a thin member

The results obtained for this simple geometry provide us with some insight as to the relative importance of transverse shear deformation and stretching deformation versus bending deformation.

When the centroidal axis is a circular segment, $R$ = const, and the equations simplify somewhat. It is convenient to take the polar angle $\theta$ as the independent variable in this case. We list the governing equations below for convenience and summarize the notation in Fig. 14–15:

$$\text{Eq. (14–19)} \Rightarrow \left\{ \begin{aligned} & R\frac{dF_1}{d\theta} + \frac{dM}{d\theta} = -R^2\left(b_1 + \frac{m}{R}\right) \\ & \frac{d^2M}{d\theta^2} - RF_1 = R^2 b_2 - R\frac{dm}{d\theta} \\ & F_2 = -\frac{1}{R}\frac{dM}{d\theta} - m \end{aligned} \right\} \qquad (14\text{–}58)$$

$$\text{Eq. (14–25)} \Rightarrow \left\{ \begin{aligned} & e_1 = e_1^0 + \frac{F_1}{AE} + \frac{M}{AER} = \frac{1}{R}\left(\frac{du_1}{d\theta} - u_2\right) \\ & e_2 = \frac{F_2}{GA_2^*} = \frac{1}{R}\left(\frac{du_2}{d\theta} + u_1\right) - \omega \\ & k = k^0 + \frac{F_1}{AER} + \frac{M}{EI^*} = \frac{1}{R}\frac{d\omega}{d\theta} \end{aligned} \right\} \qquad (14\text{–}59)$$

### *Solution of the Force-Equilibrium Equations*

We consider the external forces to be independent of the displacements. Integrating the first equilibrium equation, we have

$$RF_1 = -M - R^2\int_\theta \left(b_1 + \frac{m}{R}\right) d\theta + C_1 \qquad \text{(a)}$$

where $C_1$ is an integration constant. Substituting for $F_1$ in the second equation results in a second-order differential equation for $M$:

$$\frac{d^2M}{d\theta^2} + M = C_1 + R^2\left[b_2 - \frac{1}{R}\frac{dm}{d\theta} - \int_\theta \left(b_1 + \frac{m}{R}\right) d\theta\right] \qquad \text{(b)}$$

The general solution of (b) is

$$M = C_1 + C_2\cos\theta + C_3\sin\theta + M_p \qquad (14\text{–}60)$$



where $M_p$ denotes the particular solution due to the external distributed loading and $C_2$, $C_3$ are constants. Once $M$ is known, we find $F_1$ using (a) and $F_2$ from the moment equilibrium equation. The resulting expressions are

$$\begin{aligned} F_1 &= \frac{-1}{R}(C_2\cos\theta + C_3\sin\theta + M_p) - R\int_\theta \left(b_1 + \frac{m}{R}\right) d\theta \\ F_2 &= \frac{-1}{R}\left(-C_2\sin\theta + C_3\cos\theta + \frac{dM_p}{d\theta}\right) - m \end{aligned} \qquad (14\text{–}61)$$

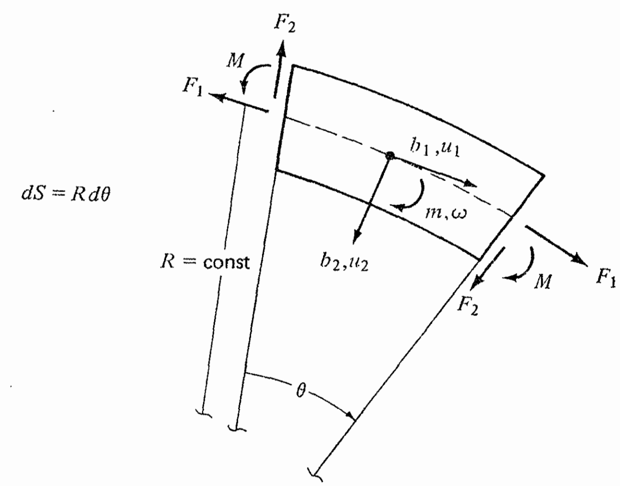



**Fig. 14–15.** Notation for circular member.

### *Integration of the Force-Displacement Relations*

We start with (14–59) written in a slightly rearranged form:

$$\begin{aligned} \frac{du_1}{d\theta} - u_2 &= Re_1^0 + \frac{1}{AE}(M + RF_1) \\ \frac{du_2}{d\theta} + u_1 &= \frac{RF_2}{GA_2^*} + R\omega \\ \frac{d\omega}{d\theta} &= Rk^0 + \frac{R}{EI^*}\left[M + \frac{I^*}{AR^2}(RF_1)\right] \end{aligned} \qquad \text{(a)}$$

To determine $u_1$ and $u_2$, we transform the first two equations to

$$\frac{du_1}{d\theta} = u_2 + Re_1^0 + \frac{1}{AE}(M + RF_1) \qquad (14\text{–}62)$$

and

$$\frac{d^2u_2}{d\theta^2} + u_2 = \psi$$

$$\begin{aligned}\psi &= \frac{R}{GA_2^*}\frac{dF_2}{d\theta} + R\frac{d\omega}{d\theta} - Re_1^0 - \frac{1}{AE}(M + RF_1)\\ &= \frac{R}{GA_2^*}\frac{dF_2}{d\theta} + R^2k^0 - Re_1^0 + \frac{a_1R^2}{EI^*}M \qquad (14\text{–}63)\\ \delta_e &= \frac{I^*}{AR^2} = 0\left(\frac{d}{R}\right)^2\\ a_1 &= 1 - \delta_e\end{aligned}$$

We have previously shown† that $\delta_e$ is of the order of $(d/R)^2$. It is reasonable to neglect $\delta_e$ with respect to 1 but we will retain it in order to keep track of the influence of *extensional* deformation. We solve (14–63) for $u_2$, determine $u_1$ from (14–62), and $\omega$ from the second equation in (a),

$$\omega = -\frac{F_2}{GA_2^*} + \frac{1}{R}\left(\frac{du_2}{d\theta} + u_1\right) \qquad (14\text{–}64)$$

This leads to three additional integration constants. The six constants are determined by enforcing the three boundary conditions at each end. Various loading conditions are treated in the following examples.

**Example 14–3**

Consider a member (Fig. E14–3) fixed at the negative end ($A$) and subjected only to $\bar{F}_{B1}$ at the right end ($B$). The boundary conditions for this case are

$$\begin{aligned}F_1 &= \bar{F}_{B1}; \quad F_2 = M = 0 \quad \text{at } \theta = \theta_B\\ u_1 &= u_2 = \omega = 0 \quad \text{at } \theta = 0\end{aligned} \qquad \text{(a)}$$

Specializing the force solution for no external distributed loading and enforcing the boundary conditions at $B$, we obtain

$$\begin{aligned}F_1 &= \bar{F}_{B1}\cos(\theta_B - \theta)\\ F_2 &= \bar{F}_{B1}\sin(\theta_B - \theta)\\ M &= R\bar{F}_{B1}(1 - \cos(\theta_B - \theta))\end{aligned} \qquad \text{(b)}$$

To simplify the analysis, we suppose there is no initial deformation. Using (b), $\psi$ takes the form

$$\psi = \frac{\bar{F}_{B1}R^3}{EI^*}[a_1 - a_2\cos(\theta_B - \theta)] \qquad \text{(c)}$$

where

$$\begin{aligned}\delta_s &= \frac{EI^*}{GA_2^*R^2} = 0\left(\frac{d}{R}\right)^2\\ a_2 &= a_1 + \delta_s = 1 - \delta_e + \delta_s\end{aligned} \qquad \text{(d)}$$

† See Sec. 14–3, Eq. (14–26).



Note that $\delta_s$ is associated with *transverse shear deformation*. Substituting for $\psi$ in (14–63) and integrating, we obtain

$$u_2 = C_4\cos\theta + C_5\sin\theta + \frac{\bar{F}_{B1}R^3}{EI^*}\left[a_1 + \frac{a_2}{2}\theta\sin(\theta_B - \theta)\right] \qquad \text{(e)}$$

The solution for $u_1$ follows from (14–62):

$$\begin{aligned}u_1 = {} & C_4\sin\theta - C_5\cos\theta + C_6\\ & + \frac{\bar{F}_{B1}R^3}{EI^*}\left\{\theta + \frac{a_2}{2}[\theta\cos(\theta_B - \theta) + \sin(\theta_B - \theta)]\right\}\end{aligned} \qquad \text{(f)}$$

Next, we determine $\omega$ using (14–64),

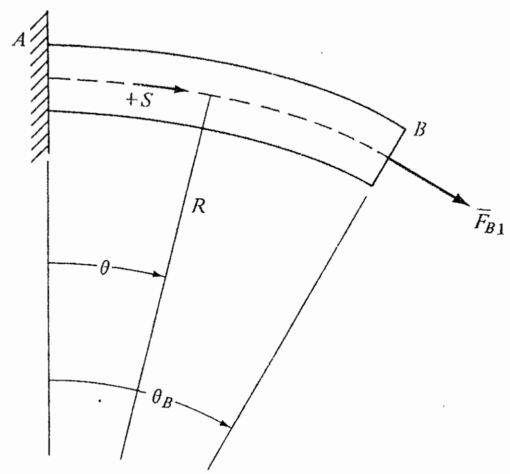
$$\omega = \frac{C_6}{R} + \frac{\bar{F}_{B1}R^2}{EI^*}\{\theta + a_1\sin(\theta_B - \theta)\} \qquad \text{(g)}$$

Finally, the constants are found by enforcing the displacement boundary conditions at $\theta = 0$:

$$\begin{aligned}C_4 &= -a_1\frac{\bar{F}_{B1}R^3}{EI^*}\\ C_5 &= \left(\frac{a_2}{2} - a_1\right)\frac{\bar{F}_{B1}R^3}{EI^*}\sin\theta_B\\ C_6 &= -a_1\frac{\bar{F}_{B1}R^3}{EI^*}\sin\theta_B\end{aligned} \qquad \text{(h)}$$

To determine the relative importance of stretching and shear deformation versus bending deformation, we evaluate the displacements at $\theta = \theta_B$ and write the resulting expressions

Fig. E14–3




Constant cross section

in the following form:

$$
\left.
\begin{aligned}
\omega_B &= \frac{\bar{F}_{B1}R^2}{EI^*}(\theta_B - \sin\theta_B)(1 + b_1\,\delta_e) \\
u_{B1} &= \frac{\bar{F}_{B1}R^3}{EI^*}(\tfrac{3}{2}\theta_B - 2\sin\theta_B + \tfrac{1}{2}\sin\theta_B\cos\theta_B)(1 + b_2\,\delta_e + b_3\,\delta_s) \\
u_{B2} &= \frac{\bar{F}_{B1}R^3}{EI^*}(1 - \cos\theta_B - \tfrac{1}{2}\sin^2\theta_B)(1 - \delta_e + b_4\,\delta_s) \\
b_1 &= \frac{\sin\theta_B}{\theta_B - \sin\theta_B} \\
b_2 &= \frac{-\tfrac{1}{2}\theta_B + 2\sin\theta_B - \tfrac{1}{2}\sin\theta_B\cos\theta_B}{\tfrac{3}{2}\theta_B - 2\sin\theta_B + \tfrac{1}{2}\sin\theta_B\cos\theta_B} \\
b_3 &= \frac{\tfrac{1}{2}(\theta_B - \sin\theta_B\cos\theta_B)}{\tfrac{3}{2}\theta_B - 2\sin\theta_B + \tfrac{1}{2}\sin\theta_B\cos\theta_B} \\
b_4 &= \frac{\tfrac{1}{2}\sin^2\theta_B}{1 - \cos\theta_B - \tfrac{1}{2}\sin^2\theta_B} = \frac{1 + \cos\theta_B}{1 - \cos\theta_B}
\end{aligned}
\right\} \qquad \text{(i)}
$$

The coefficients $(b_1, \ldots, b_4)$ are of order unity or less when $\theta_B$ is not small with respect to unity, i.e., when the segment is *not* shallow. Also, $\delta_e$ and $\delta_s$ are of order $(d/R)^2$. It follows that the displacements due to stretching and shear deformation are of order $(d/R)^2$ times the displacement due to bending deformation for a *nonshallow* member.

To investigate the shallow case, we replace the trigometric terms in (i) by their Taylor series expansions,

$$
\begin{aligned}
\sin\theta &= \theta\left(1 - \frac{\theta^2}{6} + \frac{\theta^4}{120} - \cdots\right) \\
\cos\theta &= 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24} - \cdots \\
\sin\theta\cos\theta &= \theta\left(1 - \frac{2}{3}\theta^2 + \frac{2}{15}\theta^4 - \cdots\right) \\
\sin^2\theta &= \theta^2\left(1 - \frac{\theta^2}{3} + \frac{2}{45}\theta^4 - \cdots\right)
\end{aligned}
\qquad \text{(j)}
$$

and neglect $\theta_B^2$ with respect to unity. The resulting expressions are

$$
\begin{aligned}
\omega_B &= \frac{\bar{F}_{B1}S^2}{EI^*}\left\{\theta_B\left[\frac{1}{6} + \frac{I^*}{AS^2}\right]\right\} \\
u_{B1} &= \frac{\bar{F}_{B1}S^3}{EI^*}\left\{\theta_B^2\left[\frac{1}{20} + \frac{1}{3}\frac{EI^*}{GA_2^*S^2}\right] + \frac{I^*}{AS^2}\right\} \\
u_{B2} &= \frac{\bar{F}_{B1}S^3}{EI^*}\left\{\frac{\theta_B}{2}\left[\frac{1}{4} + \frac{EI^*}{GA_2^*S^2}\right]\right\}
\end{aligned}
\qquad \text{(k)}
$$

Now,

$$
\begin{aligned}
\frac{I^*}{AS^2} &= 0\left(\frac{d}{S}\right)^2 \\
\frac{EI^*}{GA_2^*S^2} &= 0\left(\frac{d}{S}\right)^2
\end{aligned}
\qquad \text{(l)}
$$



For example,

$$
\begin{aligned}
\frac{I^*}{AS^2} &= \frac{1}{12}\left(\frac{d}{S}\right)^2 \\
\frac{EI^*}{GA_2^*S^2} &= \frac{E}{10G}\left(\frac{d}{S}\right)^2 = 0.26\left(\frac{d}{S}\right)^2
\end{aligned}
\qquad \text{(m)}
$$

for a rectangular section and $\nu = 0.3$. Since $(d/S)^2 \ll 1$ for a member, we can neglect the transverse shear terms in $u_{B1}$, $u_{B2}$ and the stretching term in $\omega_B$. However, we must *retain* the stretching term in $u_{B1}$ since it is of the same order as the bending term. The appropriate expression for $u_{B1}$ is

$$
u_{B1} = \frac{\bar{F}_{B1}S^3}{EI^*}\left(\frac{\theta_B^2}{20} + \frac{I^*}{AS^2}\right) \qquad \text{(n)}
$$

In sum, we have shown that the percentage of error due to neglecting stretching and transverse shear deformation is of the order of $(d/R)^2$ for a *nonshallow* circular member. If the member is shallow ($\theta_B < \approx 15°$), we *cannot* neglect stretching deformation. Actually, the stretching term dominates when the member is quite shallow. The error due to neglecting transverse shear deformation for the shallow case is still only of the order of $(d/R)^2$.

## Example 14–4

The internal force distributions due to $\bar{F}_{B2}$ acting on the cantilever member shown in Fig. E14–4 are given by

$$
\begin{aligned}
F_1 &= -\bar{F}_{B2}\sin(\theta_B - \theta) \\
F_2 &= \bar{F}_{B2}\cos(\theta_B - \theta) \\
M &= \bar{F}_{B2}\,R\sin(\theta_B - \theta)
\end{aligned}
\qquad \text{(a)}
$$

We suppose the member is *not shallow* and neglect stretching and shear deformation. The force-displacement relations reduce to (we set $A = A_2^* = \infty$ in (14–59))

$$
\begin{aligned}
\frac{du_1}{d\theta} - u_2 &= Re_1^0 \\
\frac{du_2}{d\theta} + u_1 &= R\omega \\
\frac{d\omega}{d\theta} &= Rk^0 + \frac{RM}{EI^*}
\end{aligned}
\qquad \text{(b)}
$$

Eliminating $u_1$ from the first two equations, we obtain

$$
\begin{aligned}
\frac{d^2u_2}{d\theta^2} + u_2 &= R^2k^0 - Re_1^0 + \frac{R^2}{EI^*}M \\
\frac{du_1}{d\theta} &= u_2 + Re_1^0 \\
\omega &= \frac{1}{R}\left(\frac{du_2}{d\theta} + u_1\right)
\end{aligned}
\qquad \text{(c)}
$$

We determine $u_2$, then $u_1$, and finally $\omega$. Note that (c) corresponds to (14–62), (14–63) and (14–64) with $A = A_2 = \infty$. The final expressions (for no initial deformation or support movement) are

$$u_2 = \frac{\bar{F}_{B2}R^3}{2EI^*}\{(\theta\cos(\theta_B - \theta) - \sin\theta\cos\theta_B\}$$

$$u_1 = \frac{\bar{F}_{B2}R^3}{2EI^*}\{-2\cos\theta_B + \cos\theta_B\cos\theta - \theta\sin(\theta_B - \theta) + \cos(\theta_B - \theta)\} \qquad \text{(d)}$$

$$\omega = \frac{\bar{F}_{B2}R^2}{EI^*}\{\cos(\theta_B - \theta) - \cos\theta_B\}$$

### Example 14–5

We analyze the shallow parabolic member shown in Fig. E14–5 using Marguerre's equations. We consider the member to be thin and neglect transverse shear deformation. Taking $f = ax_1^2/2$ and $p_1 = m = 0$, the governing equations (see (14–55) and (14–57)) reduce to

$$\left.\begin{aligned} &\frac{dF_1}{dx_1} = 0 \\ &\frac{d^2M}{dx_1^2} - aF_1 - p_2 = 0 \\ &F_2 = -\frac{dM}{dx_1} \end{aligned}\right\} \qquad \text{(a)}$$

$$\left.\begin{aligned} e_1 &= e_1^0 + \frac{F_1}{AE} = \frac{dv_1}{dx_1} + ax_1\frac{dv_2}{dx_1} \\ \omega &= \frac{dv_2}{dx_1} \\ k &= k^0 + \frac{M}{EI} = \frac{d^2v_2}{dx_1^2} \end{aligned}\right\} \qquad \text{(b)}$$

$$\left.\begin{aligned} &v_1, v_2, \omega \text{ prescribed at } x_1 = 0 \\ &\left.\begin{aligned} N_1 &= F_1 = \bar{N}_{B1} \\ N_2 &= -\frac{dM}{dx_1} + ax_1F_1 = 0 \\ M &= 0 \end{aligned}\right\} \text{at } x_1 = L \end{aligned}\right\} \qquad \text{(c)}$$

Integrating (a) and using the boundary conditions at $x_1 = L$, we obtain

$$\begin{aligned} F_1 &= \bar{N}_{B1} \\ M &= \frac{p_2}{2}(L - x_1)^2 - \frac{a\bar{N}_{B1}}{2}(L^2 - x_1^2) \qquad \text{(d)} \\ F_2 &= p_2(L - x_1) - ax_1\bar{N}_{B1} \end{aligned}$$

We suppose $e_1^0 = k^\circ = 0$ to simplify the discussion. Integrating the moment-curvature relation,

$$EI\frac{d^2v_2}{dx_1^2} = M = \frac{p_2}{2}(L - x_1)^2 - \frac{a\bar{N}_{B1}}{2}(L^2 - x_1^2) \qquad \text{(e)}$$



Fig. E14–4

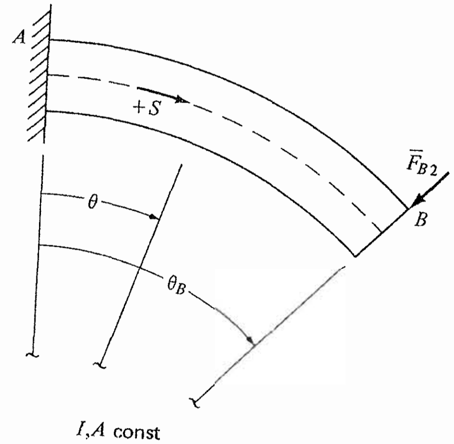
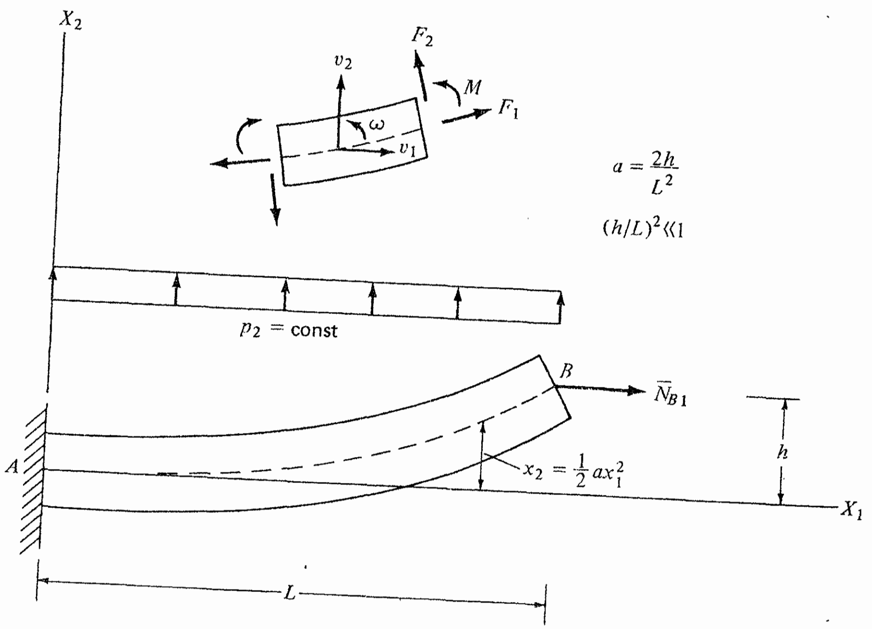



Fig. E14–5



and noting that $v_2 = dv_2/dx_1 = 0$ at $x_1 = 0$ lead to the solution for $v_2$,

$$EIv_2 = \frac{p_2}{2}\left(\frac{1}{2}L^2x_1^2 - \frac{L}{3}x_1^3 + \frac{1}{12}x_1^4\right) - \frac{a\bar{N}_{B1}}{4}\left(L^2x_1^2 - \frac{1}{6}x_1^4\right) \tag{f}$$

The axial displacement is determined by integrating the extensional strain displacement relation,

$$\frac{dv_1}{dx_1} = \frac{F_1}{AE} - ax_1\frac{dv_2}{dx_1}$$
$$\Downarrow \tag{g}$$
$$v_1 = -\frac{ap_2}{2EI}\left(\frac{1}{3}L^2x_1^3 - \frac{1}{4}Lx_1^4 + \frac{1}{15}x_1^5\right) + \bar{N}_{B1}\left[\frac{x_1}{AE} + \frac{a^2}{6EI}\left(L^2x_1^3 - \frac{1}{5}x_1^5\right)\right]$$

We express the last term in (g) as

$$\frac{\bar{N}_{B1}x_1}{AE}\left\{1 + \left(\frac{a^2L^4}{6}\right)\left(\frac{A}{I}\right)\left[\left(\frac{x_1}{L}\right)^2 - \frac{1}{5}\left(\frac{x_1}{L}\right)^4\right]\right\} \tag{h}$$

Now,

$$a = 2h/L^2 \tag{i}$$

Then

$$\frac{a^2L^4}{6}\left(\frac{A}{I}\right) = \frac{2}{3}\left(\frac{h}{\rho}\right)^2 \tag{j}$$

and we see that this term dominates when $h$ is larger with respect to the cross-sectional depth.

---

## 14–7. FORCE METHOD OF SOLUTION

Our starting point is the principle of virtual forces restricted to planar deformation,

$$\int_S (e_1\,\Delta F_1 + e_2\,\Delta F_2 + k\,\Delta M)dS - \sum \bar{d}_k\,\Delta R_k = \sum d_i\,\Delta P_i \tag{14–65}$$

where the virtual-force system is statically permissible, $\bar{d}_k$ represents a support movement, and $\Delta R_k$ is the corresponding reaction increment. The relations between the deformation measures $(e_1, e_2, k)$ and the internal forces $(F_1, F_2, M)$ depend on the material properties and on whether one employs stress or displacement *expansions*. This discussion is limited to a linearly elastic material but one should note that (14–65) is valid for *arbitrary* material. For convenience, we list the force-deformation relations below. The notation for internal force quantities is shown in Fig. 14–3.

### *Arbitrary Linearly Elastic Member*

$$\begin{aligned} e_1 &= \bar{e}_1^0 + \frac{F_1}{AE} + \frac{M}{EAR} \\ e_2 &= \frac{F_2}{G\bar{A}_2} \\ k &= \bar{k}^0 + \frac{F_1}{AER} + \frac{M}{E\bar{I}} \end{aligned} \tag{14–66}$$



where $\bar{e}_1^0$, $\bar{k}^0$, $\bar{A}_2$, and $\bar{I}$ are defined by (14–24) for the stress-expansion approach and (14–42) for the displacement-expansion approach.

### *Thin Linearly Elastic Member*

$$\begin{aligned} e_1 &= e_1^0 + \frac{F_1}{AE} \\ e_2 &= \frac{F_2}{GA_2} \\ k &= k^0 + \frac{M}{EI} \end{aligned} \tag{14–67}$$

where $A_2$, $e_1^0$, $k^0$ are the same as for a prismatic member.

When the member is *not shallow*, it is reasonable to neglect stretching and transverse shear deformation. As shown in Example 14–3, this approximation introduces a percentage error of $0(d/R)^2$. Formally, one sets $A = A_2 = \infty$. If the member is *shallow*, we can still neglect transverse shear deformation but we must include stretching deformation.

The basic steps involved in applying the force method to a curved member are the same as for the prismatic case. However, the algebra is usually more complicated, due to the geometry. We will discuss first the determination of the displacement at a point.

To determine the displacement at $Q$ in the direction defined by $\bar{t}_Q$, we apply an external virtual force, $\Delta P_Q \bar{t}_Q$, generate a statically determinate system of internal forces and reactions corresponding to $\Delta P_Q$,

$$\begin{aligned} \Delta F_j &= F_{j,Q}\,\Delta P_Q \qquad (j = 1, 2) \\ \Delta M &= M_{,Q}\,\Delta P_Q \\ \Delta R_k &= R_{k,Q}\,\Delta P_Q \end{aligned} \tag{14–68}$$

and substitute in (14–65):

$$d_Q = \int_S (e_1F_{1,Q} + e_2F_{2,Q} + kM_{,Q})dS - \sum R_{k,Q}\,\bar{d}_k \tag{14–69}$$

This expression is valid for an *arbitrary* material. We set $e_2 = 0$ if transverse shear deformation is negligible and $e_1 = e_1^0$ if stretching deformation is negligible.

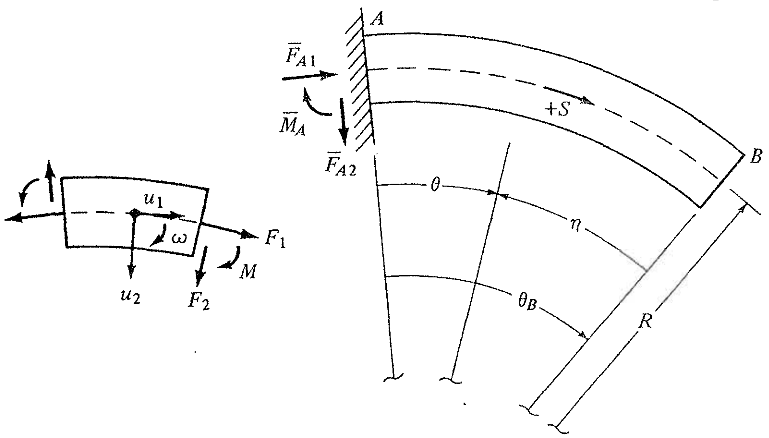
**Example 14–6**

We consider the thin linearly elastic circular segment shown in Fig. E14–6A. We suppose the member is *not* shallow and *neglect stretching* and *transverse* shear deformation. The reactions are the end forces at A for this example, and (14–69) expands to

$$d_Q = \int_S \left(e_1^0F_{1,Q} + \left(k^0 + \frac{M}{EI}\right)M_{,Q}\right)dS + \bar{u}_{A1}F_{A1,Q} + \bar{u}_{A2}F_{A2,Q} + \bar{\omega}_A M_{A,Q} \tag{a}$$

In what follows, we illustrate the application of (a)

Fig. E14–6A




### *Expressions for Displacements at B*

To determine $u_{B1}$, we take $\Delta P_Q = \Delta F_{B1}$. The internal virtual-force system corresponds to $F_{B1} = +1$. It is convenient to work with $\eta = \theta_B - \theta$ as the independent variable rather than $\theta$.

The force-influence coefficients $(F_{1,Q}, F_{2,Q}, M_{,Q})$ follow directly from Fig. E14–6B:

$$\begin{aligned} F_{1,Q} &= F_1|_{\Delta P_Q=+1} = F_1|_{F_{B1}=+1} = \cos\eta \\ F_{2,Q} &= \sin\eta \\ M_{,Q} &= R(1-\cos\eta) \end{aligned} \tag{b}$$

Substituting (b) in (a) results in the following general expression for $u_{B1}$:

$$\begin{aligned} u_{B1} = R\int_0^{\theta_B}\left\{e_1^0\cos\eta + R\left(k^0 + \frac{M}{EI}\right)(1-\cos\eta)\right\}d\eta \\ +\bar{u}_{A1}\cos\theta_B + \bar{u}_{A2}\sin\theta_B + \bar{\omega}_A R(1-\cos\theta_B) \end{aligned} \tag{c}$$

Once the loading is specified, we can evaluate the integral. Terms involving the support displacements define the rigid body displacement at $B$.

Taking $\Delta P_Q = \Delta F_{B2}$, $\Delta M_B$ leads to expression for $u_{B2}$ and $\omega_B$. We list them below for future reference:

$$u_{B2} = R\int_0^{\theta_B}\left\{-e_1^0\sin\eta + R\left(k^0+\frac{M}{EI}\right)\sin\eta\right\}d\eta - \bar{u}_{A1}\sin\theta_B + \bar{u}_{A2}\cos\theta_B + \bar{\omega}_A R\sin\theta_B \tag{d}$$

$$\omega_B = \bar{\omega}_A + R\int_0^{\theta_B}\left(k^0 + \frac{M}{EI}\right)d\eta$$

### *Solution for a Concentrated Loading at an Arbitrary Interior Point*

We consider an arbitrary force vector, $P_C$, and moment, $M_C$, applied at point $C$ as shown in Fig. E14–6C.



Fig. E14–6B

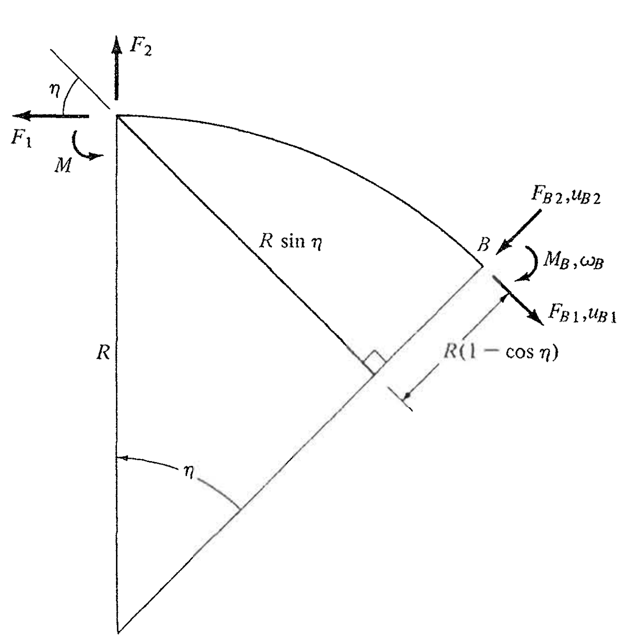
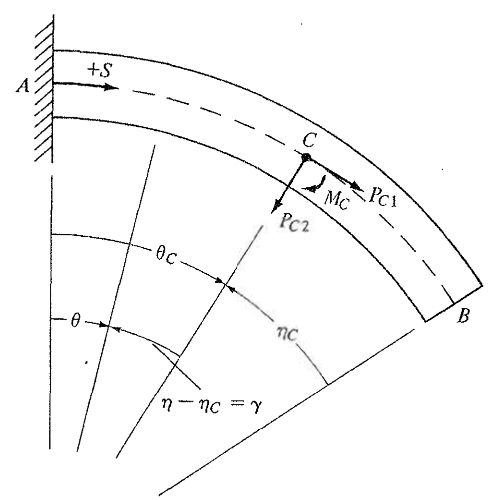



Fig. E14–6C



$$\vec{P}_C = P_{C1}\vec{t}_1 + P_{C2}\vec{t}_2 \qquad M_C = M_C\vec{t}_3 \tag{e}$$

The expressions for the displacements at $B$ due to an external loading are obtained by specializing (c) and (d) for no initial deformation or support movement and noting that

$$\begin{aligned} M &= 0 & \eta < \eta_C \\ M &= RP_{C1}[1 - \cos(\eta - \eta_C)] + RP_{C2}\sin(\eta - \eta_C) + M_C & \eta \geqslant \eta_C \end{aligned} \tag{f}$$

The solution for constant $I$ is

$$\begin{aligned} u_{B1} &= \frac{P_{C1}R^3}{EI}\left(\theta_C - \sin\theta_C - \sin\theta_B + \sin\eta_C + \frac{\theta_C}{2}\cos\eta_C + \frac{1}{2}\sin\theta_C\cos\theta_B\right) \\ &\quad + \frac{P_{C2}R^3}{EI}\left(1 - \cos\theta_C + \frac{\theta_C}{2}\sin\eta_C - \frac{1}{2}\sin\theta_C\sin\theta_B\right) \\ &\quad + \frac{M_C R^2}{EI}(\theta_C + \sin\eta_C - \sin\theta_B) \\ u_{B2} &= \frac{P_{C1}R^3}{EI}\left(-\cos\theta_B + \cos\eta_C - \frac{1}{2}\theta_C\sin\eta_C - \frac{1}{2}\sin\theta_C\sin\theta_B\right) \\ &\quad + \frac{P_{C2}R^3}{EI}\left(\frac{1}{2}\theta_C\cos\eta_C - \frac{1}{2}\sin\theta_C\cos\theta_B\right) \\ &\quad + \frac{R^2 M_C}{EI}(\cos\eta_C - \cos\theta_B) \\ \omega_B &= \frac{R^2P_{C1}}{EI}(\theta_C - \sin\theta_C) + \frac{R^2P_{C2}}{EI}(1 - \cos\theta_C) + \frac{RM_C}{EI}\theta_C \end{aligned} \tag{g}$$

If we take point $C$ to coincide with $B$, $\eta_C = 0$ and $\theta_C = \theta_B$. The resulting equations relate the displacement at $B$ due to forces applied at $B$ in the directions of the local frame at $B$ and can be interpreted as member force-deformation relations. It is convenient to express these relations in matrix form:

$$\mathscr{U}_B = \mathbf{f}_B \mathscr{F}_B \tag{h}$$

$$\begin{Bmatrix} u_{B1} \\ u_{B2} \\ \omega_B \end{Bmatrix} = \frac{R}{EI}\left[\begin{array}{c|c|c} R^2[\frac{3}{2}\theta_B - 2\sin\theta_B + \frac{1}{2}\sin\theta_B\cos\theta_B] & R^2[1 - \cos\theta_B - \frac{1}{2}\sin^2\theta_B] & R[\theta_B - \sin\theta_B] \\ \hline \text{Symmetrical} & \frac{R^2}{2}[\theta_B - \sin\theta_B\cos\theta_B] & R[1 - \cos\theta_B] \\ \hline & & \theta_B \end{array}\right]\begin{Bmatrix} \bar{F}_{B1} \\ \bar{F}_{B2} \\ \bar{M}_B \end{Bmatrix}$$

We call $\mathbf{f}_B$ the *member flexibility matrix*.

---

We describe next the application of the principle of virtual forces in the analysis of a statically indeterminate planar member. Let the member be indeterminate to the $r$th degree and let $Z_1, \ldots, Z_r$ represent the force redundants. Using the equilibrium equations, we express the internal forces and reactions in



terms of the applied loads and the force redundants:

$$\begin{aligned} F_1 &= F_{1,0} + \sum_{k=1}^{r} F_{1,k}Z_k \\ F_2 &= F_{2,0} + \sum_{k=1}^{r} F_{2,k}Z_k \\ M &= M_{,0} + \sum_{k=1}^{r} M_{,k}Z_k \\ R_i &= R_{i,0} + \sum_{k=1}^{r} R_{i,k}Z_k \end{aligned} \tag{14-70}$$

Substituting the virtual force system corresponding to $\Delta Z_j$ (which is statically permissible and self-equilibrating) in (14–65) and letting $j$ range from 1 to $r$ lead to the compatibility equations relating the actual deformations:

$$\int_S(e_1F_{1,j} + e_2F_{2,j} + kM_{,j})dS - \sum_i \bar{d}_iR_{i,j} = 0 \qquad j = 1, \ldots, r \tag{a}$$

When the material is linearly elastic, the compatibility equations take the form

$$\sum_{k=1}^{r} f_{jk}Z_k = \Delta_j \qquad (j = 1, \ldots, r) \tag{14-71}$$

where

$$\begin{aligned} f_{jk} = f_{kj} &= \int_S\left(\frac{1}{AE}F_{1,j}F_{1,k} + \frac{1}{EAR}(F_{1,j}M_{,k} + F_{1,k}M_{,j}) + \frac{1}{G\bar{A}_2}F_{2,j}F_{2,k} + \frac{1}{E\bar{I}}M_{,j}M_{,k}\right)dS \\ \Delta_j &= \sum_i \bar{d}_iR_{i,j} - \int_S\left(F_{1,j}\bar{e}_1^0 + M_{,j}\bar{k}^0 + \frac{1}{AE}F_{1,j}F_{1,0} + \frac{1}{EAR}(F_{1,j}M_{,0} + F_{1,0}M_{,j}) + \frac{1}{G\bar{A}_2}F_{2,0}F_{2,j} + \frac{1}{E\bar{I}}M_{,0}M_{,j}\right)dS \end{aligned}$$

We set $\bar{I} = I$, $\bar{A}_2 = A_2$, and $1/AR = 0$ for a thin member.

Note that $f_{jk}$ is the displacement of the primary structure in the direction of $Z_j$ due to a unit value of $Z_k$. Also, $\Delta_j$ is the *actual* displacement of the point of application of $Z_j$ *minus* the displacement of the primary structure in the direction of $Z_j$ due to support movement, initial deformation, and the prescribed external forces.

**Example 14–7** ---

Consider the symmetrical closed ring shown in Fig. E14–7. From symmetry,

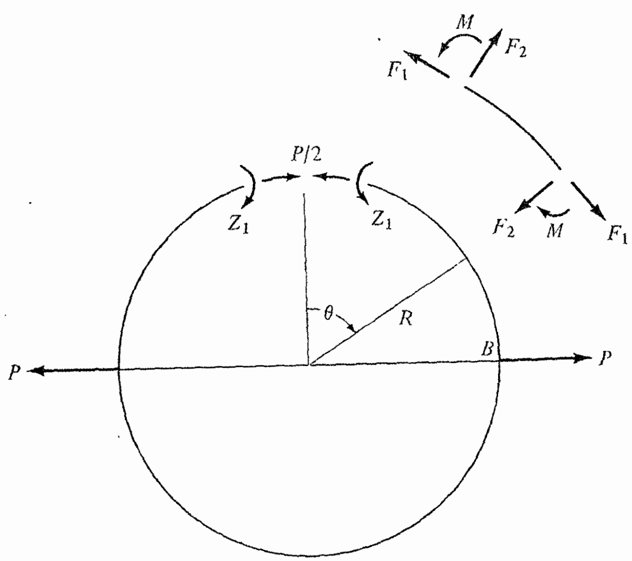
$$\left.\begin{aligned} F_1 &= \frac{P}{2} \\ F_2 &= 0 \end{aligned}\right\} \quad \text{at } \theta = 0 \tag{a}$$

We take the moment at $\theta = 0$ as the force redundant. To simplify the algebra, we suppose the member is thin and neglect stretching and shear deformation. The compatibility equations reduces to

$$f_{11}Z_1 = \Delta_1$$
$$f_{11} = \int_S \frac{1}{EI} M_{,1}^2 \, dS \qquad \text{(b)}$$
$$\Delta_1 = -\int_S \frac{1}{EI} M_{,0} M_{,1} \, dS$$

Note that $f_{11}$ is the relative rotation (⟩⟨) due to a unit value of $Z_1$ and $\Delta_1$ is the relative rotation (⟨⟩) due to the applied load. Equation (b) states that the *net* relative rotation must vanish.

**Fig. E14–7**




Now,

$$M_{,1} = M|_{Z_1=1} = +1 \qquad \text{(c)}$$
$$M_{,0} = \frac{P}{2} R(1 - \cos\theta) \qquad 0 \leqslant \theta \leqslant \frac{\pi}{2}$$

We consider $I$ to be constant. Then, (b) reduces to

$$Z_1 = \frac{-\int M_{,0} M_{,1} \, dS}{\int M_{,1}^2 \, dS} = \frac{-\int_0^{\pi/2} (1 - \cos\theta) d\theta}{\pi/2} \left(\frac{PR}{2}\right) \qquad \text{(d)}$$
$$= -\frac{PR}{2}\left(1 - \frac{2}{\pi}\right)$$



Because of symmetry, we need to integrate over only a quarter of the ring. Finally, the total moment is

$$M = M_{,0} + Z_1 M_{,1} = PR\left(\frac{1}{\pi} - \frac{\cos\theta}{2}\right) \qquad \text{(e)}$$

The axial and shear force variations are given by

$$F_1 = \frac{P}{2}\cos\theta$$
$$F_2 = -\frac{P}{2}\sin\theta \qquad \text{(f)}$$

---

When the equation defining the centroidal axis is expressed in the form $x_2 = f(x_1)$, it is more convenient to work with force and displacement quantities referred to the basic frame rather than to the local frame, i.e., to use the cartesian formulation developed in Sec. (14–5). The cartesian notation is summarized in Fig. 14–16.

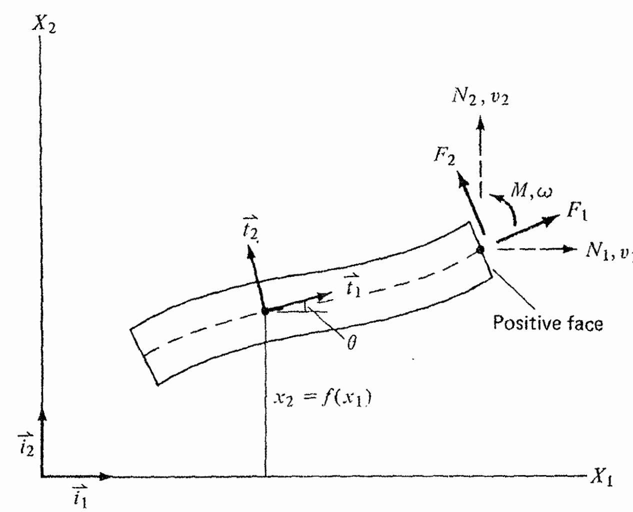



**Fig. 14–16.** Notation for Cartesian formulation.

The geometrical quantities and relations between the internal force components are

$$\tan\theta = \frac{df}{dx_1} \equiv f'$$
$$dS = \frac{dx_1}{\cos\theta}$$
$$F_1 = N_1\cos\theta + N_2\sin\theta \qquad (14\text{–}72)$$
$$F_2 = -N_1\sin\theta + N_2\cos\theta$$

We first find $N_1$, $N_2$ and then determine $F_1$, $F_2$. To obtain the equations for the cartesian case, we just have to replace $dS$ by $dx_1/\cos\theta$ in the general expressions ((14–69) and (14–71)). In what follows, we suppose the member is *thin* and *linearly elastic.*

When the member is *not* shallow, we can neglect the stretching and transverse shear deformation terms. The equations for this case reduce to:

***Displacement at Point Q***

$$d_Q = \int_{x_1} \left(e_1^0 F_{1,Q} + \left(k^0 + \frac{M}{EI}\right) M_{,Q}\right) \frac{dx_1}{\cos\theta} - \sum_i R_{i,Q}\bar{d}_i \qquad (14\text{–}73)$$

***Compatibility Equations***

$$\sum_k f_{jk} Z_k = \Delta_j$$

$$f_{jk} = \int_{x_1} \left(\frac{1}{EI} M_{,j} M_{,k}\right) \frac{dx_1}{\cos\theta}$$

$$\Delta_j = \sum_i R_{i,j}\bar{d}_i - \int_{x_1} \left[e_1^0 F_{1,j} + \left(k^0 + \frac{M_{,0}}{EI}\right) M_{,j}\right] \frac{dx_1}{\cos\theta} \qquad (14\text{–}74)$$

We can express the terms involving $F_{1,()}$ in terms of $N_{1,()}$ and $N_{2,()}$ since

$$F_{1,()} = \cos\theta \, N_{1,()} + \sin\theta \, N_{2,()} \qquad \text{(a)}$$

Then,

$$\int_{x_1} [e_1^0 F_{1,()}] \frac{dx_1}{\cos\theta} = \int_{x_1} e_1^0 [N_{1,()} + f' N_{2,()}] dx_1 \qquad (14\text{–}75)$$

One must generally resort to numerical integration in order to evaluate the integrals, due to the presence of the term $1/\cos\theta$.

When the member is shallow, $\theta^2 \ll 1$, and we can approximate (14–72) with

$$\begin{aligned} \cos\theta &\approx 1 \\ \sin\theta &\approx \tan\theta = f' \\ ds &\approx dx_1 \\ F_1 &\approx N_1 + f' N_2 \\ F_2 &\approx -f' N_1 + N_2 \end{aligned} \qquad (14\text{–}76)$$

We cannot neglect the stretching deformation term in this case. However, it is reasonable to take $F_1 \approx N_1$. We also introduced this assumption in the development of Marguerre's equations. The equations for the shallow case with negligible transverse shear deformation and $F_1 \approx N_1$ have the forms listed below:

***Displacement at Point Q***

$$d_Q = \int_{x_1} \left[\left(e_1^0 + \frac{N_1}{AE}\right) N_{1,Q} + \left(k^0 + \frac{M}{EI}\right) M_{,Q}\right] dx_1 - \sum_i R_{i,Q}\bar{d}_i \qquad (14\text{–}77)$$



***Compatibility Equation***

$$\sum_k f_{jk} Z_k = \Delta_j$$

$$f_{jk} = \int_{x_1} \left[\frac{1}{AE} N_{1,j} N_{1,k} + \frac{1}{EI} M_{,j} M_{,k}\right] dx_1 \qquad (14\text{–}78)$$

$$\Delta_j = \sum_i R_{i,j}\bar{d}_i - \int_{x_1} \left[\left(e_1^0 + \frac{N_{1,0}}{AE}\right) N_{1,j} + \left(k^0 + \frac{M_{,0}}{EI}\right) M_{,j}\right] dx_1$$

**Example 14–8**

Consider the two-hinged arch shown in Fig. E14–8A. We work with reaction components referred to the basic frame and take the horizontal reaction at $B$ as the force redundant.

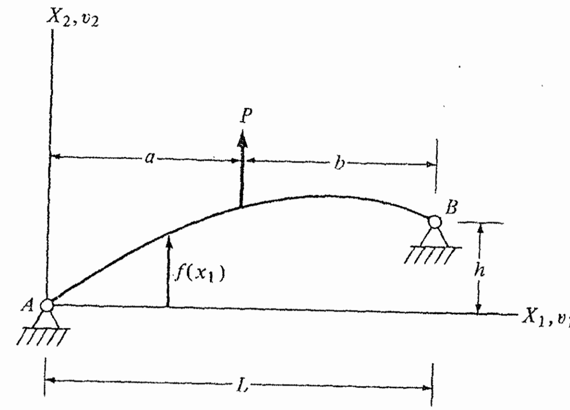



**Fig. E14–8A**

***Primary Structure***

We must carry out two force analyses on the primary structure (Fig. E14–8B), one for the external forces (condition $Z_1 = 0$) and the other for $Z_1 = 1$. The results are displayed in Figs. E14–8C and D, respectively.

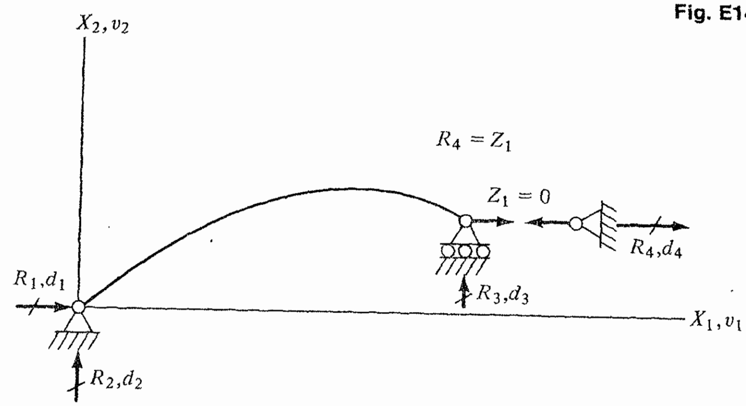



**Fig. E14–8B**

**Fig. E14-8C**

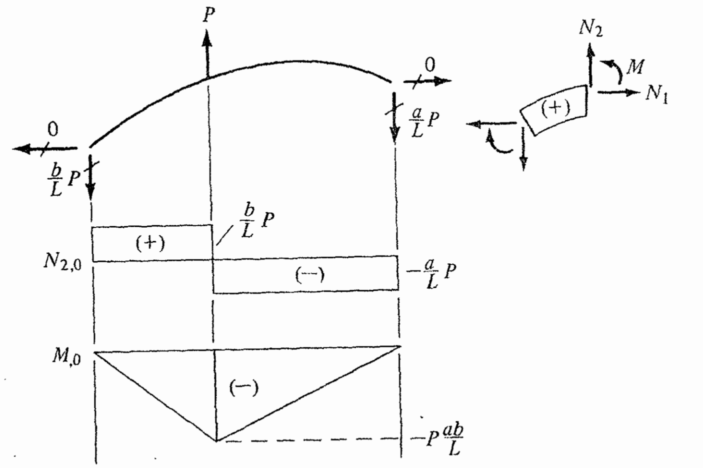
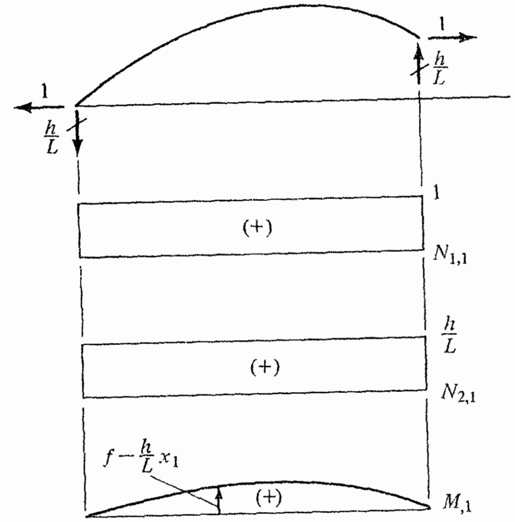



**Fig. E14-8D**




### *Compatibility Equation*

We suppose the member is *not* shallow. The compatibility equations for $Z_1$ follow from (14–74):

$$f_{11}Z_1 = \Delta_1$$

$$f_{11} = \int_0^L \frac{1}{EI} M_{,1}^2 \frac{dx_1}{\cos\theta} \qquad \text{(a)}$$

$$\Delta_1 = \sum R_{i,1}\bar{d}_i - \int_0^L \left[ e_1^0(N_{1,1} + f'N_{2,1}) + \left(k^0 + \frac{M_0}{EI}\right)\frac{M_{,1}}{\cos\theta}\right] dx_1$$



Using the results listed above, the various terms in (a) expand to

$$f_{11} = \int_0^L \frac{1}{EI}\left[f - \frac{h}{L}x_1\right]^2 \frac{dx_1}{\cos\theta}$$

$$\sum R_{i,1}\bar{d}_i = -\bar{v}_{A1} - \frac{h}{L}\bar{v}_{A2} + \bar{v}_{B1} + \frac{h}{L}\bar{v}_{B2}$$

$$\int_0^L \left[e_1^0(N_{1,1} + f'N_{2,1}) + k^0 \frac{M_{,1}}{\cos\theta}\right] dx_1 = \int_0^L \left[e_1^0\left(1 + \frac{h}{L}f'\right) + k^0\left(f - \frac{h}{L}x_1\right)\frac{1}{\cos\theta}\right] dx_1 \qquad \text{(b)}$$

$$\int_0^L \frac{1}{EI\cos\theta} M_{,0}M_{,1}\, dx_1 = \int_0^L \frac{1}{EI\cos\theta}\left(f - \frac{h}{L}x_1\right)\left(-\frac{b}{L}Px_1\right) dx_1 + \int_a^b \frac{1}{EI\cos\theta}\left(f - \frac{h}{L}x_1\right)\left(+P(x_1 - a)\right) dx_1$$

Once the integrals are evaluated, we can determine $Z_1$ from

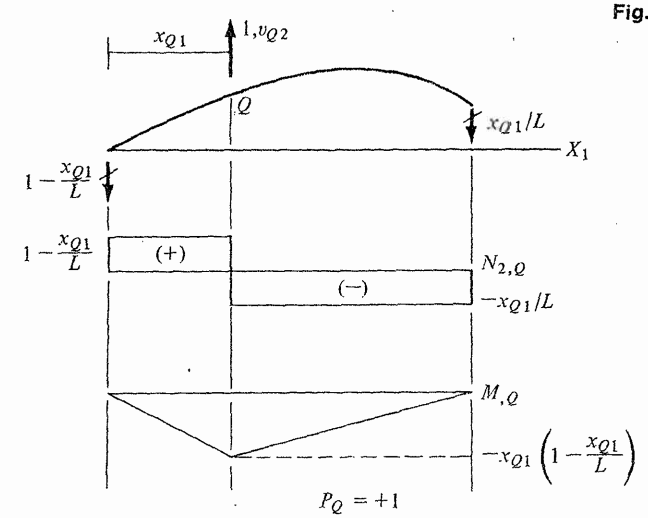
$$Z_1 = \frac{\Delta_1}{f_{11}} \qquad \text{(c)}$$

Finally, the total forces are obtained by superposition of the two loadings:

$$\begin{aligned} N_j &= N_{j,0} + Z_1 N_{j,1} \qquad j = 1, 2 \\ M &= M_{,0} + Z_1 M_{,1} \\ R_i &= R_{i,0} + Z_1 R_{i,1} \qquad i = 1, 2, 3 \\ R_4 &= Z_1 \end{aligned} \qquad \text{(d)}$$

To evaluate the vertical displacement at point $Q$, we apply a unit vertical load at $Q$ on the *primary* structure and determine the required internal forces and reactions plotted in Fig. E14-8E.

**Fig. E14-8E**



Applying (14–73), we obtain

$$
\begin{aligned}
v_{Q2} = \bar{v}_{A2} &+ (\bar{v}_{B2} - \bar{v}_{A2})\frac{x_{Q1}}{L} \\
&+ \int_0^{x_{Q1}} e_1^0 f'\, dx_1 - \frac{x_{Q1}}{L}\int_0^L e_1^0 f'\, dx_1 \\
&- \int_0^{x_{Q1}} x_1\left(k^0 + \frac{M}{EI}\right)\frac{dx_1}{\cos\theta} - x_{Q1}\int_{x_{Q1}}^L \left(k^0 + \frac{M}{EI}\right)\frac{dx_1}{\cos\theta} \\
&+ \frac{x_{Q1}}{L}\int_0^L x_1\left(k^0 + \frac{M}{EI}\right)\frac{dx_1}{\cos\theta}
\end{aligned} \qquad \text{(e)}
$$

A numerical procedure for evaluating these integrals is described in the next section.

## Example 14–9

The symmetrical nonshallow two-hinged parabolic arch shown in Fig. E14–9A is subjected to a uniform load per unit *horizontal* length, that is, per unit $x_1$. The equation for the centroidal axis is

$$x_2 = f = \frac{4h}{L}\left(x_1 - \frac{x_1^2}{L}\right) \qquad \text{(a)}$$

where $h$ is the elevation at mid-span ($x_1 = L/2$). We take the horizontal reaction at the right end as the force redundant and consider only bending deformation. Figures E14–9B and C carry through an analysis parallel to that of the preceding example.

### *Determination of $Z_1$ and Total Forces*

The equation for $Z_1$ follows from (14–74):

$$Z_1 = \frac{\Delta_1}{f_{11}} = -\frac{\displaystyle\int_0^L M_{,0}M_{,1}\frac{dx_1}{EI\cos\theta}}{\displaystyle\int_0^L (M_{,1})^2\frac{dx_1}{EI\cos\theta}} = -\frac{pL^2}{8h} \qquad \text{(b)}$$

Note that this result is valid for an arbitrary variation of $EI$. Finally, the total forces are

$$
\begin{aligned}
N_1 &= N_{1,0} + Z_1 N_{1,1} = -\frac{pL^2}{8h} \\
N_2 &= N_{2,0} + Z_1 N_{2,1} = p\left(x_1 - \frac{L}{2}\right) \\
M &= M_{,0} + Z_1 M_{,1} = 0
\end{aligned} \qquad \text{(c)}
$$

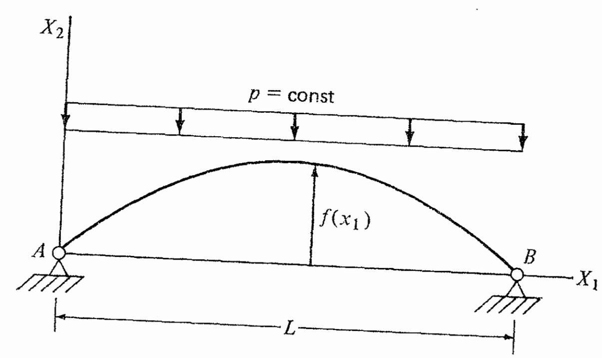
Since $M = 0$, the deformed shape of the arch coincides with the initial shape when *axial* deformation is neglected. It follows that (c) also apply for the *fixed nonshallow* case.

When the arch is *shallow*, the effect of axial deformation cannot be neglected. The expression for $Z_1$ follows from (14–78):

$$Z_1 = -\frac{\displaystyle\int_0^L \left(\left(\frac{1}{AE}\right)N_{1,0}N_{1,1} + \frac{1}{EI}M_{,0}M_{,1}\right)dx_1}{\displaystyle\int_0^L \left(\frac{1}{AE}N_{1,1}^2 + \frac{1}{EI}M_{,1}^2\right)dx_1} \qquad \text{(d)}$$



Fig. E14–9A



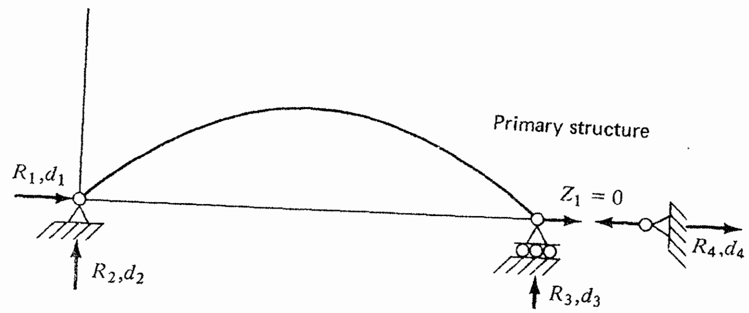
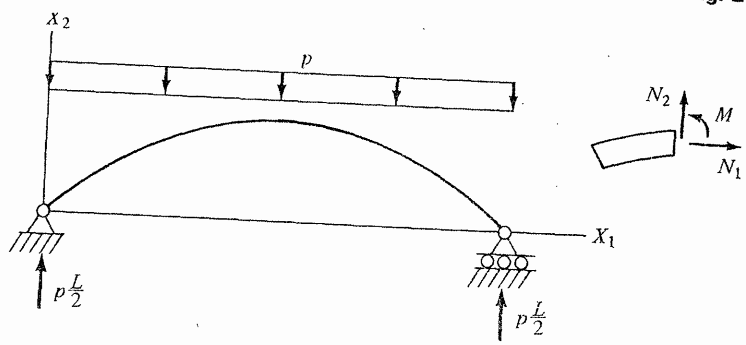




Fig. E14–9B




Force System Due to $p$

If $E$ is constant, (d) reduces to

$$Z_1 = -\frac{pL^2}{8h}\,\frac{\int_0^L \frac{1}{I} f^2\, dx_1}{\int_0^L \frac{1}{A}\, dx_1 + \int_0^L \frac{1}{I} f^2\, dx_1} = -\frac{pL^2}{8h}\,\frac{1}{1+\delta} \qquad \text{(e)}$$

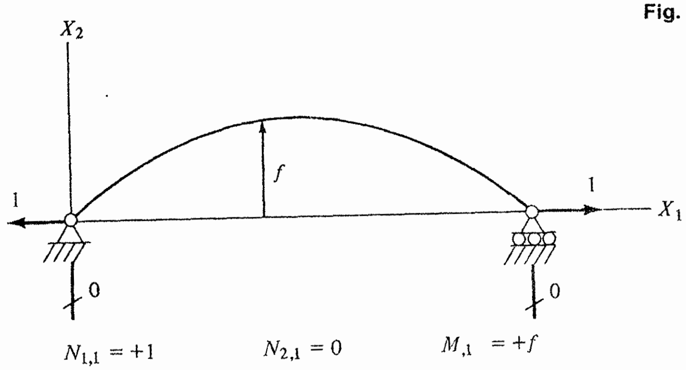
$$\delta = \frac{\int_0^L \frac{1}{A}\, dx_1}{\int_0^L \frac{1}{I} f^2\, dx_1}$$

The parameter $\delta$ is a measure of the influence of axial deformation. As an illustration, we consider $A$ and $I$ to be constant and evaluate $\delta$ for this geometry. The result is

$$\delta = \frac{15}{8}\,\frac{I}{Ah^2} = \frac{15}{8}\left(\frac{\rho}{h}\right)^2 \qquad \text{(f)}$$

where $\rho$ is the radius of gyration for the cross section.

Fig. E14–9C




Force System Due to $Z_1 = +1$

One should note that (e) applies only for the shallow case, i.e., for $(f')^2 \ll 1$. Now,

$$f' = \frac{4h}{L}\left(1 - \frac{2x_1}{L}\right) \qquad \text{(g)}$$

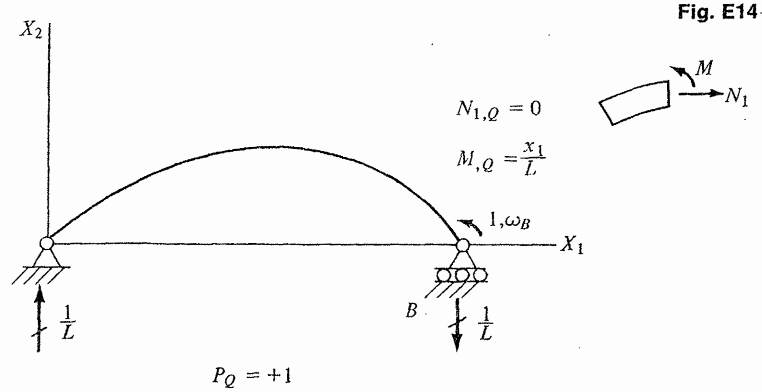
For the assumption of shallowness to be valid, $16(h/L)^2$ must be small with respect to unity. The total forces for the shallow case are

$$\begin{aligned} N_1 &= Z_1 = -\frac{pL^2}{8h}\,\frac{1}{1+\delta} \\ N_2 &= p\left(x_1 - \frac{L}{2}\right) \\ M &= \frac{pL^2}{8h} f\left(\frac{\delta}{1+\delta}\right) = \frac{PL}{2}\left(x_1 - \frac{x_1^2}{L}\right)\left(\frac{\delta}{1+\delta}\right) \end{aligned} \qquad \text{(h)}$$



It is of interest to determine the rotation at $B$. The "$Q$" loading consists of a unit moment applied at $B$ to the primary structure (see Fig. E14–9D). Applying (14–77) (note that

Fig. E14–9D




the stretching terms vanish since $N_{1,Q} = 0$), we obtain

$$\omega_B = \int_0^L \frac{M}{EI}\,\frac{x_1}{L}\, dx_1 = \frac{P}{2}\left(\frac{\delta}{1+\delta}\right)\int_0^L \frac{1}{EI}\left(x_1^2 - \frac{x_1^3}{L}\right) dx_1 \qquad \text{(i)}$$

When $EI$ is constant, (i) reduces to

$$\omega_B = \frac{pL^3}{24EI}\left(\frac{\delta}{1+\delta}\right) \qquad \text{(j)}$$

Since $\omega_B \neq 0$, the results for the fixed end shallow case will differ slightly from (h).

---

## 14–8. NUMERICAL INTEGRATION PROCEDURES

One of the steps in the force method involves evaluating certain integrals which depend on the member geometry and the cross-sectional properties. Closed-form solutions can be obtained for only simple geometries, and one usually must resort to a numerical integration procedure. In what follows, we describe two procedures† which can be conveniently automated and illustrate their application in deflection computations.

We consider the problem of evaluating

$$J = \int_{x_A}^{x_B} f(x)\, dx \qquad \text{(14–79)}$$

† See Ref. 8 for a more detailed treatment of numerical integration schemes.

where $f(x)$ is a reasonably smooth function in the interval $x_A \leqslant x \leqslant x_B$. We divide the total interval into $n$ equal segments, of length $h$:

$$h = \frac{x_B - x_A}{n} \tag{14–80}$$

If $f(x)$ is discontinuous, we work with subintervals and use a different spacing for each subinterval. For convenience, we let $x_0, x_1, \ldots, x_n$ represent the coordinates of the equally spaced points on the $x$ axis, and $f_0, f_1, \ldots, f_n$ the corresponding values of the function. This notation is shown in Figure 14–17.

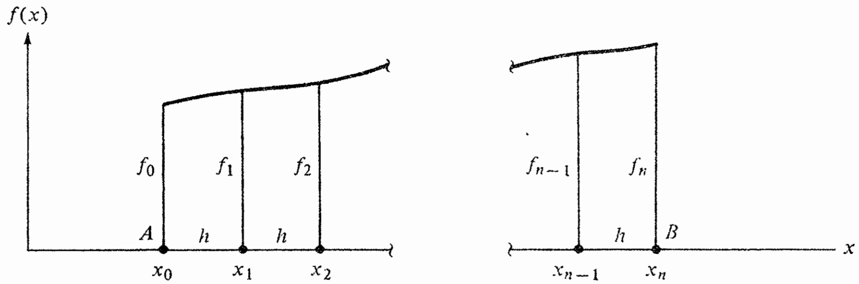



**Fig. 14–17.** Coordinate discretization for numerical integration.

The simplest approach consists in approximating the actual curve by a set of straight lines connecting $(f_0, f_1), (f_1, f_2)$, etc., as shown in Fig. 14–18. With this approximation,

$$\Delta J_{k-1,k} = \int_{x_{k-1}}^{x_k} f(x)dx \approx \frac{h}{2}(f_{k-1} + f_k) \tag{14–81}$$

$$J_k = \int_{x_0}^{x_k} f\,dx = J_{k-1} + \Delta J_{k-1,k}$$

If only the total integral is desired, we use,

$$J_n = \int_{x_0}^{x_n} f(x)dx = \sum_{i=1}^{n} \Delta J_{i-1,i} \approx h\left\{\tfrac{1}{2}(f_0 + f_n) + \sum_{i=1}^{n-1} f_i\right\} \tag{14–82}$$

which is called the *trapezoidal* rule.

A more accurate formula is obtained by approximating the curve connecting three consecutive points with a second-degree polynomial, as shown in Fig. 14–19. This leads to

$$\Delta J_{k,k+2} = \int_{x_k}^{x_{k+2}} f\,dx \approx \frac{h}{3}[f_k + 4f_{k+1} + f_{k+2}] \tag{14–83}$$

$$J_{k+2} = J_k + \Delta J_{k,k+2}$$

To apply (14–83), we must take an even number of segments, that is, $n$ must be an *even* integer. If the values of $J$ at odd points are also desired, they can



be determined using

$$\Delta J_{k,k+1} = \int_0^h f\,d\eta \approx \frac{h}{12}[5f_k + 8f_{k+1} - f_{k+2}] \tag{14–84}$$

Finally, one can express $J_n$ as

$$J_n = \frac{h}{3}[f_0 + f_n + 4(f_1 + f_3 + \cdots + f_{n-1}) + 2(f_2 + f_4 + \cdots + f_{n-2})] \tag{14–85}$$

Equation (14–85) is called *Simpson's rule.*

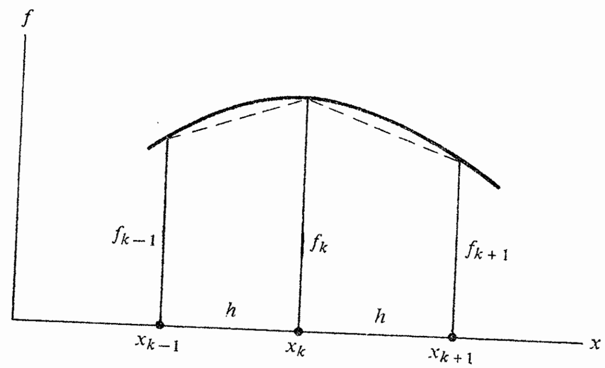



**Fig. 14–18.** Linear approximation.

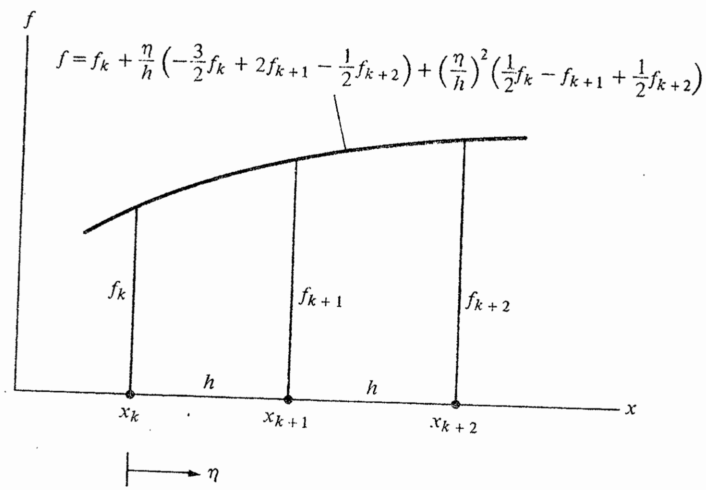
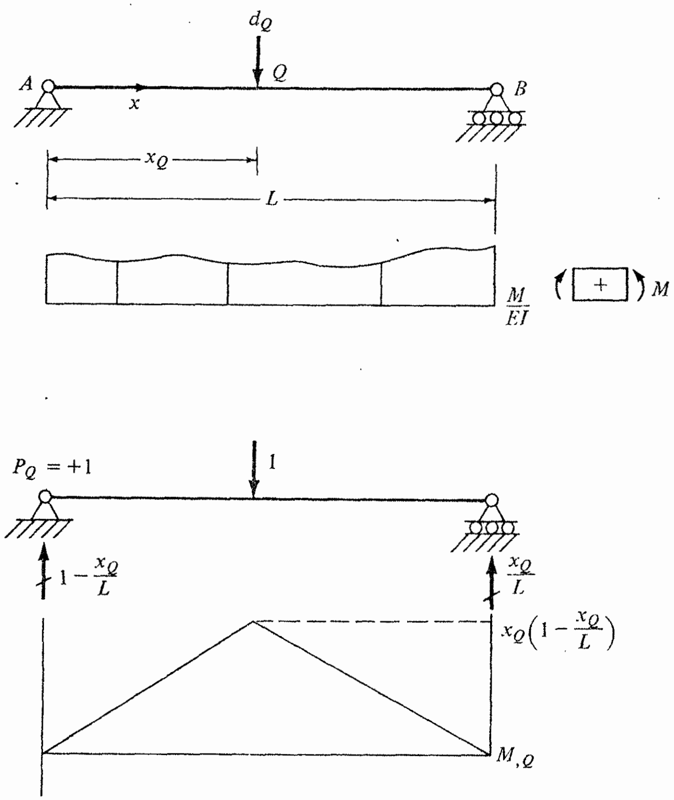



**Fig. 14–19.** Parabolic approximation.

## Example 14–10

Consider the problem of determining the vertical displacement at $Q$ for the straight member of Fig. E14–10. We suppose shear deformation is negligible. The deflection due

**Fig. E14–10**




to bending deformation (we consider the material to be linear elastic) is given by

$$d_Q = \int_0^L \frac{M}{EI} M_Q \, dx \qquad \text{(a)}$$

where $M$ is the actual moment and $M_Q$ is due to the "$Q$" loading. Substituting for $M_{,Q}$, (a) expands to

$$d_Q = x_Q\left(\int_0^L \frac{M}{EI}dx - \int_0^{x_Q}\frac{M}{EI}dx\right) + \int_0^{x_Q} x\frac{M}{EI}dx - \frac{x_Q}{L}\int_0^L x\frac{M}{EI}dx \qquad \text{(b)}$$



To evaluate (b), we divide the total length into $n$ equal segments of length $h$, number the points from 0 to $n$, and let

$$J_k = \int_0^{x_k} \frac{M}{EI}dx \qquad H_k = \int_0^{x_k} x\frac{M}{EI}dx \qquad \text{(c)}$$

With this notation, (b) takes the form

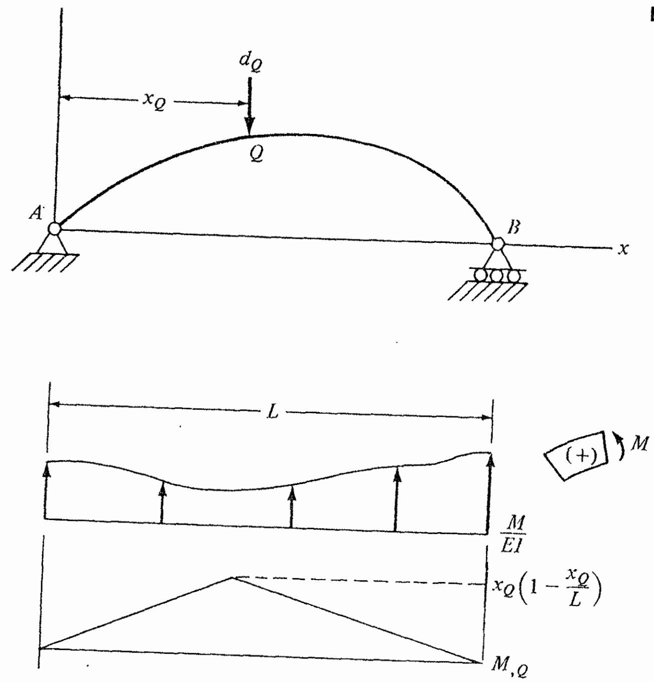
$$d_k = x_k\left[J_n - \frac{1}{L}H_n\right] + H_k - x_k J_k \qquad \text{(d)}$$

If, in determining $J_n$, $H_n$, we also evaluate the integrals at the interior points, then we can readily determine the displacement distribution using (d).

## Example 14–11

Consider the simply supported nonshallow arch shown. We suppose there is some distribution of $M$ and we want to determine the vertical deflection at $Q$. Considering

**Fig. E14–11**




only bending deformation, $d_Q$ is given by

$$d_Q = \int_s \frac{M}{EI} M_{,Q}\, ds = \int_0^L \left(\frac{M}{EI\cos\theta}\right) M_{,Q}\, dx \qquad \text{(a)}$$

Now, the distribution of $M_{,Q}$ is the same as for the straight member. Then, the procedure followed in Example 14–10 is also applicable here. We just have to replace $M/EI$ with $M/EI \cos\theta$ in Equation (c) of Example 14–10.

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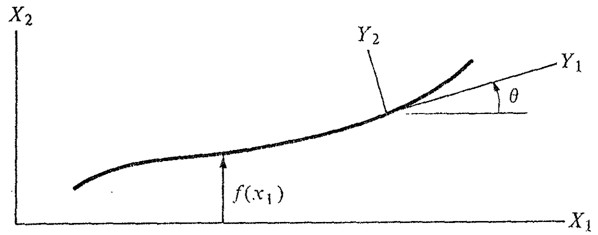
## PROBLEMS

**14–1.** Specialize (14–7) for the case where $y_1 = x_1$. Let $x_2 = f(x_1)$ and let $\theta$ be the angle from $X_1$ to $Y_1$ as shown below. Evaluate the various terms for a parabola

$$f = C_1 x_1 + C_2 x_1^2$$

Finally, specialize the relations for a shallow curve, i.e., where $\theta^2 \ll 1$.

**Prob. 14–1**




**14–2.** Evaluate $I^*$ and $\delta$ (see Equation 14–24) for the section defined by the sketch.

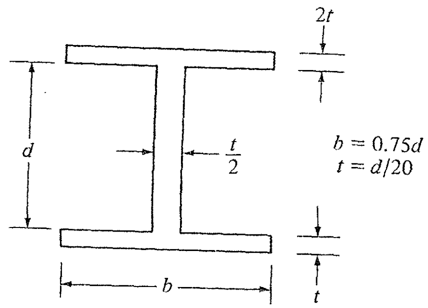
**14–3.** Verify (14–34).

**14–4.** Verify (14–41) and (14–42).

**14–5.** Discuss the difference between the deformation-force relations based on stress and displacement expansions (Equations (14–25) and (14–42)). Illustrate for the rectangular section treated in Example 14–1. Which set of relations would you employ?

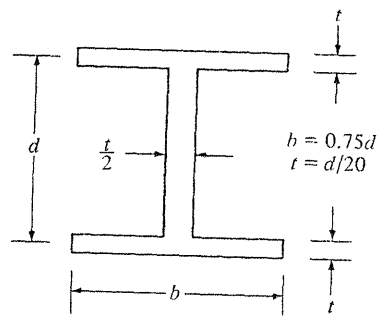


**Prob. 14–2**



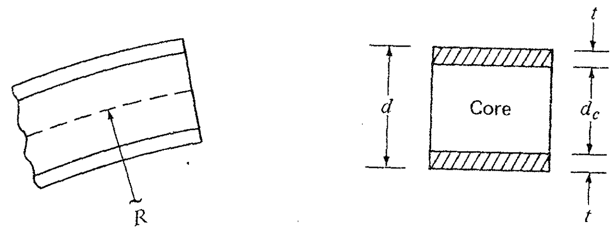

**14–6.** Evaluate $I'$ and $I''$ for the symmetrical section shown.

**Prob. 14–6**




**14–7.** Consider a circular sandwich member comprised of three layers, as shown. The *core* layer is soft ($E \ll 0$), and the face thickness is small in comparison to the depth ($d_c \approx d$). Establish force-deformation relations based on strain expansions (see (14–37)).

**Prob. 14–7**




**14–8.** Starting with (14–34) and (14–35), derive a set of nonlinear strain displacement relations for a thin member. Assume *small finite* rotation, and linearize the expressions with respect to $y_2$, *i.e.*, take

$$\varepsilon_1 = e_1 - y_2 k$$
$$\gamma_{12} \approx e_2$$

Determine the corresponding force-equilibrium equations with the principle of virtual displacements.

**14–9.** Refer to Fig. 14–10 and Equation (14–31). If we neglect transverse shear deformation, $\vec{t}'_2$ is orthogonal to $\vec{t}'_1$ and we can write

$$\begin{aligned}
(1 + \varepsilon_1)\alpha_2 &= \left(1 - \frac{y_2}{R'}\right)\alpha' \\
\vec{t}'_1 &= \frac{1}{\alpha'}\frac{d\vec{r}'}{dy} = \beta_1\vec{t}_1 + \beta_2\vec{t}_2 \\
\vec{t}'_2 &= -\beta_2\vec{t}_1 + \beta_1\vec{t}_2 \\
\frac{d\vec{t}'_1}{dS} &= \frac{1 + e_1}{R'}\vec{t}'_2 \qquad \frac{d\vec{t}'_2}{dS} = -\frac{1 + e_1}{R'}\vec{t}'_1 \\
\alpha' &= \alpha\left|\vec{t}_1 + \frac{d\vec{u}}{dS}\right| = (1 + e_1)\alpha
\end{aligned} \qquad \text{(a)}$$

(a) Verify that $\varepsilon_1$ can be expressed as

$$\begin{aligned}
\varepsilon_1 &= \frac{1}{1 - y_2/R}\left\{e_1 - y_2\left(\frac{1 + e_1}{R'} - \frac{1}{R}\right)\right\} \\
&= \frac{1}{1 - y_2/R}\{e_1 - y_2 k\}
\end{aligned} \qquad \text{(b)}$$

Also determine $e_1$ and $R'$ for *small strain*. Express $\vec{u}$ in terms of the initial tangent vectors,

$$\vec{u} = u_1\vec{t}_1 + u_2\vec{t}_2$$

and take $y \equiv S$ (i.e., $\alpha \equiv 1$).

(b) Derive the force-equilibrium equations, starting with the vector equations (see (14–12) and Fig. 14–4),

$$\frac{d\vec{F}_+}{dS} + \vec{b} = 0$$

$$\frac{d\vec{M}_+}{dS} + \vec{m} + \vec{t}'_1 \times \vec{F}_+ = \vec{0}$$

and expanding the force vectors in terms of components referred to the deformed frame:

$$\vec{F}_+ = F_1\vec{t}'_1 + F_2\vec{t}'_2 \qquad \vec{b} = b_1\vec{t}'_1 + b_2\vec{t}'_2$$
$$\vec{M}_+ = M\vec{t}_3$$

Assume small strain.

(c) Derive the force-equilibrium equations with the principle of virtual displacements. Take the strain distribution according to Equation (b).



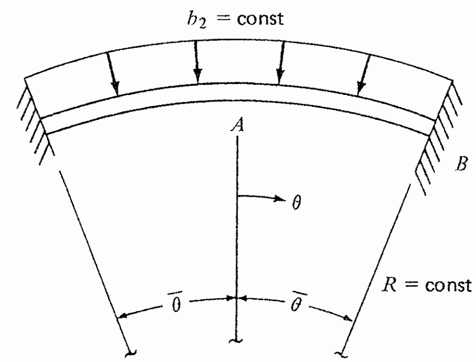
(d) Derive the nonlinear deformation-displacement and equilibrium equations for the cartesian formulation. Refer the translations and loading to the basic frame, i.e., take

$$\vec{u} = v_1\vec{i}_1 + v_2\vec{i}_2$$
$$\vec{p} = p_1\vec{i}_1 + p_2\vec{i}_2$$

Specialize the equations for the case of a shallow member.

**14–10.** The accompanying sketch applies to both phases of this problem.

**Prob. 14–10**

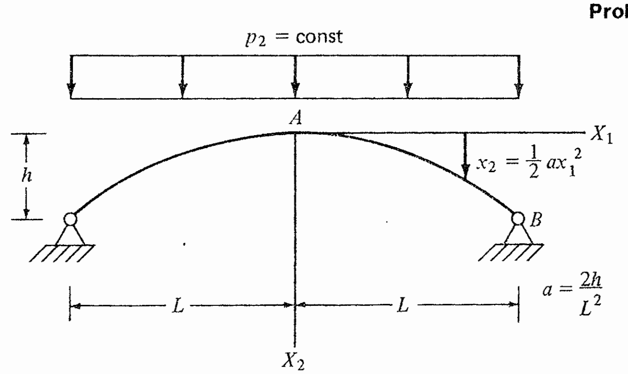



(a) Determine the complete solution for the circular member shown. Utilize symmetry at point $A$ ($u_1 = \omega = F_2 = 0$) and work with (14–58), (14–59). Discuss the effect of neglecting extensional and shear deformation, i.e., setting $(1/A) = (1/A_2) = 0$.

(b) Repeat (a), using Mushtari's equations for a *thin* member with no transverse shear deformation, which are developed in Example 14–2. Show that Mushtari's approximation ($u_1 \ll du_2/d\theta$) is valid when the segment is shallow.

**14–11.** The sketch presents the information relevant to the problem:

**Prob. 14–11**

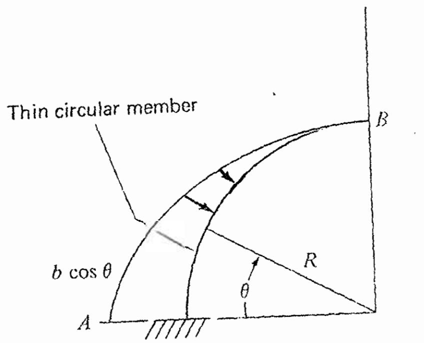


(a) Apply the cartesian formulation to the symmetrical parabolic arch shown. Consider the member to be thin and neglect transverse shear deformation.
(b) Specialize (a) for negligible extensional deformation (set $1/A = 0$).
(c) Specialize (a) for the shallow case and investigate the validity of Marquerre's approximation.

**14–12.** Refer to Example 14–6. Determine $u_{B2}$ due to a uniform distributed loading, $b_2$ = constant.

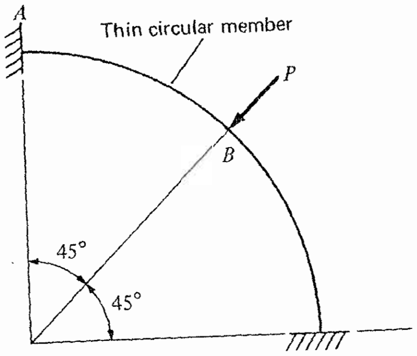
**14–13.** Determine the displacement measures at $B$ (see sketch). Consider only bending deformation. *Note*: It may be more convenient to integrate the governing equations rather than apply (14–69).

**Prob. 14–13**




**14–14.** Solve two problems with the information sketched:

**Prob. 14–14**

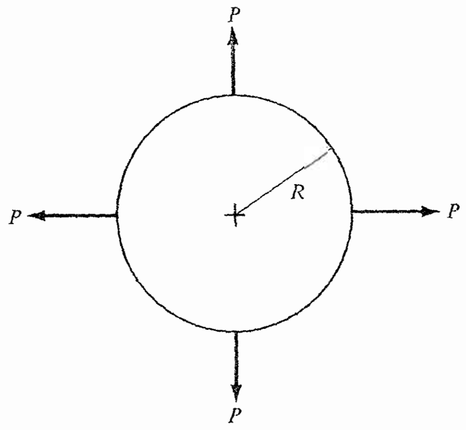



(a) Determine the fixed end forces and radial displacement at point $B$ with the force method. Consider only bending deformation and utilize symmetry at $B$.
(b) Generalize for an arbitrarily located radial force.



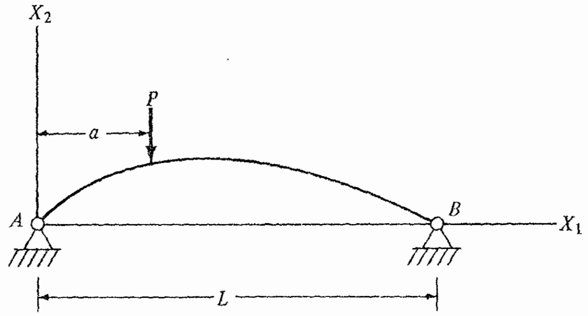
**14–15.** Refer to Example 14–7.
(a) Determine the radial displacement at $B$ defined in Fig. E14–7.
(b) Determine the force solution for the loading shown.

**Prob. 14–15**




**14–16.** The sketch defines a *thin* parabolic two-hinged arch.

**Prob. 14–16**

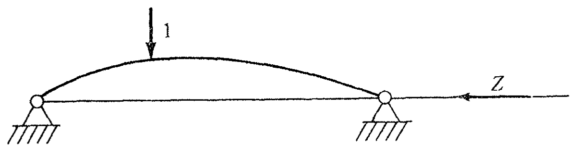



$$f = \frac{4h}{L}\left(x_1 - \frac{x_1^2}{L}\right)$$
$$I = I_o/\cos\theta$$

(a) Determine the horizontal reaction at $B$ due to the concentrated load. Consider the arch to be nonshallow.
(b) Utilize the results of (a) to obtain the solution for a distributed loading $p_2(x)$ per unit $x_1$.
(c) Determine the reactions resulting from a uniform temperature increase, $\Delta T$.
(d) Suppose the horizontal support at $B$ is replaced by a prismatic member extending from $A$ to $B$. Assume the connections are frictionless hinges. Repeat parts a and c.

**14–17.** Consider the arbitrary two-hinged arch shown. Discuss how you

Prob. 14–17




would generate the influence line for the horizontal reaction. Utilize the results contained in Examples 14–10 and 14–11.

# 15

# Engineering Theory of an Arbitrary Member

## 15–1. INTRODUCTION; GEOMETRICAL RELATIONS

In the first part of this chapter, we establish the governing equations for a member whose centroidal axis is an arbitrary space curve. The formulation is restricted to linear geometry and negligible warping and is referred to as the *engineering theory*. Examples illustrating the application of the displacement and force methods are presented. Next, we outline a restrained warping formulation and apply it to a planar circular member. Lastly, we cast the force method for the engineering theory in matrix form and develop the member force-displacement relations which are required for the analysis of a system of member elements.

The geometrical relations for a member are derived in Chapter 4. For convenience, we summarize the differentiation formulas here. Figure 15–1 shows the *natural* and *local* frames. They are related by

$$\begin{aligned}\vec{t}_1 &= \vec{t}\\ \vec{t}_2 &= \cos\phi\vec{n} + \sin\phi\vec{b}\\ \vec{t}_3 &= -\sin\phi\vec{n} + \cos\phi\vec{b}\end{aligned} \tag{a}$$

where $\phi = \phi(s)$. Differentiating (a) and using the Frenet equations (4–20), we obtain

$$\frac{d\mathbf{t}}{dS} = \mathbf{at}$$

$$\begin{Bmatrix}\dfrac{d\vec{t}_1}{dS}\\[2mm] \dfrac{d\vec{t}_2}{dS}\\[2mm] \dfrac{d\vec{t}_3}{dS}\end{Bmatrix} = \begin{bmatrix} 0 & K\cos\phi & -K\sin\phi\\ -K\cos\phi & 0 & \tau + \dfrac{d\phi}{dS}\\ K\sin\phi & -\left(\tau + \dfrac{d\phi}{dS}\right) & 0\end{bmatrix}\begin{Bmatrix}\vec{t}_1\\ \vec{t}_2\\ \vec{t}_3\end{Bmatrix} \tag{15–1}$$

Note that **a** is skew-symmetric.