Part I

MATHEMATICAL PRELIMINARIES

1

Introduction to Matrix Algebra

1–1. DEFINITION OF A MATRIX

An ordered set of quantities may be a one-dimensional array, such as

$$a_1, a_2, \ldots, a_n$$

or a two-dimensional array, such as

$$\begin{matrix} a_{11}, a_{12}, \ldots, a_{1n} \\ a_{21}, a_{22}, \ldots, a_{2n} \\ \vdots \quad \vdots \qquad \vdots \\ a_{m1}, a_{m2}, \ldots, a_{mn} \end{matrix}$$

In a two-dimensional array, the first subscript defines the row location of an element and the second subscript its column location.

A two-dimensional array having m rows and n columns is called a matrix of order m by n if certain arithmetic operations (addition, subtraction, multiplication) associated with it are defined. The array is usually enclosed in square brackets and written as*

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = [a_{ij}] = \mathbf{a} \qquad (1\text{–}1)$$

Note that the first term in the order pertains to the number of rows and the second term to the number of columns. For convenience, we refer to the order of a matrix as simply $m \times n$ rather than of order m by n.

* In print, a matrix is represented by a boldfaced letter.

A matrix having only one row is called a row matrix. Similarly, a matrix having only one column is called a column matrix or column vector.* Braces instead of brackets are commonly used to denote a column matrix and the column subscript is eliminated. Also, the elements are arranged horizontally instead of vertically, to save space. The various column-matrix notations are:

$$\begin{bmatrix} c_{11} \\ c_{21} \\ \vdots \\ c_{n1} \end{bmatrix} = \begin{Bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{Bmatrix} = \{c_1, c_2, \ldots, c_n\} = \{c_i\} = \mathbf{c} \tag{1-2}$$

If the number of rows and the number of columns are equal, the matrix is said to be *square*. (Special types of square matrices are discussed in a later section.) Finally, if all the elements are zero, the matrix is called a *null* matrix, and is represented by **0** (boldface, as in the previous case).

Example 1-1

3 × 4 Matrix

$$\begin{bmatrix} 4 & 2 & -1 & 2 \\ 3 & -7 & 1 & -8 \\ 2 & 4 & -3 & 1 \end{bmatrix}$$

1 × 3 Row Matrix

$$[3 \quad 4 \quad 2]$$

3 × 1 Column Matrix

$$\begin{Bmatrix} 3 \\ 4 \\ 2 \end{Bmatrix} \text{ or } \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix} \text{ or } \{3, 4, 2\}$$

2 × 2 Square Matrix

$$\begin{bmatrix} 8 & 5 \\ 2 & 7 \end{bmatrix}$$

2 × 2 Null Matrix

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

* This is the mathematical definition of a vector. In mechanics, a vector is defined as a quantity having both magnitude and direction. We will denote a mechanics vector quantity, such as force or moment, by means of an italic letter topped by an arrow, e.g., $\vec{F}$. A knowledge of vector algebra is assumed in this text. For a review, see Ref. 2 (at end of chapter, preceding Problems).

1-2. EQUALITY, ADDITION, AND SUBTRACTION OF MATRICES

Two matrices, **a** and **b**, are equal if they are of the same order and if corresponding elements are equal:

$$\mathbf{a} = \mathbf{b} \quad \text{when} \quad a_{ij} = b_{ij} \tag{1-3}$$

If **a** is of order $m \times n$, the matrix equation

$$\mathbf{a} = \mathbf{b}$$

corresponds to mn equations:

$$a_{ij} = b_{ij} \qquad \begin{matrix} i = 1, 2, \ldots, m \\ j = 1, 2, \ldots, n \end{matrix}$$

Addition and subtraction operations are defined only for matrices of the same order. The sum of two $m \times n$ matrices, **a** and **b**, is defined to be the $m \times n$ matrix $[a_{ij} + b_{ij}]$:

$$[a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}] \tag{1-4}$$

Similarly,

$$[a_{ij}] - [b_{ij}] = [a_{ij} - b_{ij}] \tag{1-5}$$

For example, if

$$\mathbf{a} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 0 & -1 & -1 \\ 3 & 1 & 0 \end{bmatrix}$$

then

$$\mathbf{a} + \mathbf{b} = \begin{bmatrix} 1 & 1 & 0 \\ 4 & 1 & -1 \end{bmatrix}$$

and

$$\mathbf{a} - \mathbf{b} = \begin{bmatrix} 1 & 3 & 2 \\ -2 & -1 & -1 \end{bmatrix}$$

It is obvious from the example that addition is commutative and associative:

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a} \tag{1-6}$$

$$\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c} \tag{1-7}$$

1-3. MATRIX MULTIPLICATION

The product of a scalar k and a matrix **a** is defined to be the matrix $[ka_{ij}]$, in which each element of **a** is multiplied by k. For example, if

$$k = 5 \quad \text{and} \quad \mathbf{a} = \begin{bmatrix} -2 & 7 \\ 2 & 1 \end{bmatrix}$$

then

$$k\mathbf{a} = \begin{bmatrix} -10 & +35 \\ 10 & 5 \end{bmatrix}$$

Scalar multiplication is commutative. That is,

$$k\mathbf{a} = \mathbf{a}k = [ka_{ij}] \tag{1-8}$$

To establish the definition of a matrix multiplied by a column matrix, we consider a system of m linear algebraic equations in n unknowns, $x_1, x_2, \ldots, x_n$:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= c_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= c_2 \\ \vdots \qquad \vdots \qquad\quad \vdots \quad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= c_m \end{aligned} \tag{1-9}$$

This set can be written as

$$\sum_{k=1}^{n} a_{ik}x_k = c_i \qquad i = 1, 2, \ldots, m \tag{a}$$

where k is a dummy index. Using column matrix notation, (1–9) takes the form

$$\left\{\sum_{k=1}^{n} a_{ik}x_k\right\} = \{c_i\} \qquad i = 1, 2, \ldots, m \tag{1-10}$$

Now, we write (1–9) as a matrix product:

$$[a_{ij}]\,\{x_j\} = \{c_i\} \qquad \begin{matrix} i = 1, 2, \ldots, m \\ j = 1, 2, \ldots, n \end{matrix} \tag{1-11}$$

Since (1–10) and (1–11) must be equivalent, it follows that the definition equation for a matrix multiplied by a column matrix is

$$\mathbf{ax} = [a_{ij}]\,\{x_j\} = \left\{\sum_{k=1}^{n} a_{ik}x_k\right\} \qquad i = 1, 2, \ldots, m \tag{1-12}$$

This product is defined *only* when the column order of **a** is equal to the row order of **x**. The result is a column matrix, the row order of which is equal to that of **a**. In general, if **a** is of order $r \times s$, and **x** of order $s \times 1$, the product **ax** is of order $r \times 1$.

Example 1–2

$$\mathbf{a} = \begin{bmatrix} 1 & -1 \\ 8 & -4 \\ 0 & 3 \end{bmatrix} \qquad \mathbf{x} = \begin{Bmatrix} 2 \\ 3 \end{Bmatrix}$$

$$\mathbf{ax} = \begin{Bmatrix} (1)(2) + (-1)(3) \\ (8)(2) + (-4)(3) \\ (0)(2) + (3)(3) \end{Bmatrix} = \begin{Bmatrix} -1 \\ 4 \\ 9 \end{Bmatrix}$$

We consider next the product of two matrices. This product is associated with a linear transformation of variables. Suppose that the n original variables $x_1, x_2, \ldots, x_n$ in (1–9) are expressed as a linear combination of s new variables $y_1, y_2, \ldots, y_s$:

$$x_k = \sum_{j=1}^{s} b_{kj}y_j \qquad k = 1, 2, \ldots, n \tag{1-13}$$

Substituting for x_k in (1–10),

$$\left\{\sum_{k=1}^{n} a_{ij}\left(\sum_{j=1}^{s} b_{kj}y_j\right)\right\} = \{c_i\} \qquad i = 1, 2, \ldots, m \tag{a}$$

Interchanging the order of summation, and letting

$$p_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj} \qquad \begin{matrix} i = 1, 2, \ldots, m \\ j = 1, 2, \ldots, s \end{matrix} \tag{1-14}$$

the transformed equations take the form

$$\left\{\sum_{j=1}^{s} p_{ij}y_j\right\} = \{c_i\} \qquad i = 1, 2, \ldots, m \tag{1-15}$$

Noting (1–12), we can write (1–15) as

$$\mathbf{py} = \mathbf{c} \tag{1-16}$$

where **p** is $m \times s$ and **y** is $s \times 1$. Now, we also express (1–13), which defines the transformation of variables, in matrix form,

$$\mathbf{x} = \mathbf{by} \tag{1-17}$$

where **b** is $n \times s$. Substituting for **x** in (1–11),

$$\mathbf{aby} = \mathbf{c} \tag{1-18}$$

and requiring (1–16) and (1–18) to be equivalent, results in the following definition equation for the product, **ab**:

$$\mathbf{ab} = [a_{ik}]\,[b_{kj}] = [p_{ij}] \qquad \begin{matrix} i = 1, 2, \ldots, m \\ k = 1, 2, \ldots, n \\ j = 1, 2, \ldots, s \end{matrix} \tag{1-19}$$

$$p_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$$

This product is defined *only* when the column order of **a** is equal to the row order of **b**. In general, if **a** is of order $r \times n$, and **b** of order $n \times q$, the product **ab** is of order $r \times q$. The element at the ith row and jth column of the product is obtained by multiplying corresponding elements in the ith row of the *first* matrix and the jth column of the *second* matrix.

Example 1–3

$$\mathbf{a} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & 2 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & -1 & 3 \end{bmatrix}$$

$$\mathbf{ab} = \left[\begin{array}{c|c|c|c} (1)(1)+(0)(0) & (1)(1)+(0)(1) & (1)(0)+(0)(-1) & (1)(-1)+(0)(3) \\ (-1)(1)+(1)(0) & (-1)(1)+(1)(1) & (-1)(0)+(1)(-1) & (-1)(-1)+(1)(3) \\ (0)(1)+(2)(0) & (0)(1)+(2)(1) & (0)(0)+(2)(-1) & (0)(-1)+(2)(3) \end{array}\right]$$

$$\mathbf{ab} = \begin{bmatrix} +1 & +1 & 0 & -1 \\ -1 & 0 & -1 & +4 \\ 0 & +2 & -2 & +6 \end{bmatrix}$$

If the product **ab** is defined, **a** and **b** are said to be *conformable* in the order stated. One should note that **a** and **b** will be conformable in either order only when **a** is $m \times n$ and **b** is $n \times m$. In the previous example, **a** and **b** are conformable but **b** and **a** are not since the product **ba** is not defined.

When the relevant products are defined, multiplication of matrices is associative,

$$\mathbf{a}(\mathbf{bc}) = (\mathbf{ab})\mathbf{c} \tag{1–20}$$

and distributive,

$$\begin{aligned} \mathbf{a}(\mathbf{b} + \mathbf{c}) &= \mathbf{ab} + \mathbf{ac} \\ (\mathbf{b} + \mathbf{c})\mathbf{a} &= \mathbf{ba} + \mathbf{ca} \end{aligned} \tag{1–21}$$

but, in general, *not* commutative,

$$\mathbf{ab} \neq \mathbf{ba} \tag{1–22}$$

Therefore, in multiplying **b** by **a**, one should distinguish *premultiplication*, **ab**, from *postmultiplication* **ba**. For example, if **a** and **b** are square matrices of order 2, the products are

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \left[\begin{array}{c|c} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{array}\right]$$

$$\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \left[\begin{array}{c|c} b_{11}a_{11} + b_{12}a_{21} & b_{11}a_{12} + b_{12}a_{22} \\ b_{21}a_{11} + b_{22}a_{21} & b_{21}a_{12} + b_{22}a_{22} \end{array}\right]$$

When **ab** = **ba**, the matrices are said to commute or to be permutable.

1–4. TRANSPOSE OF A MATRIX

The transpose of $\mathbf{a} = [a_{ij}]$ is defined as the matrix obtained from **a** by interchanging rows and columns. We shall indicate the transpose of **a** by

$\mathbf{a}^T = [a_{ij}^T]$:

$$\underset{(m\times n)}{\mathbf{a} = [a_{ij}]} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix} \tag{1–23}$$

$$\underset{(n\times m)}{\mathbf{a}^T = [a_{ij}^T]} = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$$

The element, a_{ij}^T, at the ith row and jth column of $\mathbf{a}^T$, where now i varies from 1 to n and j from 1 to m, is given by

$$a_{ij}^T = a_{ji} \tag{1–24}$$

where a_{ji} is the element at the jth row and ith column of **a**. For example,

$$\mathbf{a} = \begin{bmatrix} 3 & 2 \\ 7 & 1 \\ 5 & 4 \end{bmatrix} \qquad \mathbf{a}^T = \begin{bmatrix} 3 & 7 & 5 \\ 2 & 1 & 4 \end{bmatrix}$$

Since the transpose of a column matrix is a row matrix, an alternate notation for a row matrix is

$$[a_1, a_2, \ldots, a_n] = \{a_i\}^T \tag{1–25}$$

We consider next the transpose matrix associated with the product of two matrices. Let

$$\mathbf{p} = \mathbf{ab} \tag{a}$$

where **a** is $m \times n$ and **b** is $n \times s$. The product, **p**, is $m \times s$ and the element, p_{ij}, is

$$p_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj} \qquad \begin{aligned} i &= 1, 2, \ldots, m \\ j &= 1, 2, \ldots, s \end{aligned} \tag{b}$$

The transpose of **p** will be of order $s \times m$ and the typical element is

$$p_{ij}^T = p_{ji} \tag{c}$$

where now $i = 1, 2, \ldots, s$ and $j = 1, 2, \ldots, m$. Using (1–24) and (b), we can write (c) as

$$p_{ij}^T = \sum_{k=1}^{n} a_{jk}b_{ki} = \sum_{k=1}^{n} b_{ik}^T a_{kj}^T \qquad \begin{aligned} i &= 1, 2, \ldots, s \\ j &= 1, 2, \ldots, m \end{aligned} \tag{d}$$

It follows from (d) that

$$\mathbf{p}^T = (\mathbf{ab})^T = \mathbf{b}^T\mathbf{a}^T \tag{1–26}$$

Equation (1–26) states that the transpose of a product is the product of the transposed matrices in *reversed* order. This rule is also applicable to multiple products. For example, the transpose of **abc** is

$$(\mathbf{abc})^T = \mathbf{c}^T(\mathbf{ab})^T = \mathbf{c}^T\mathbf{b}^T\mathbf{a}^T \tag{1-27}$$

Example 1–4

$$\mathbf{a} = \begin{bmatrix} 3 & 2 \\ 7 & 1 \\ 5 & 4 \end{bmatrix} \qquad \mathbf{b} = \begin{Bmatrix} 2 \\ -1 \end{Bmatrix}$$

$$\mathbf{ab} = \begin{Bmatrix} 4 \\ 13 \\ 6 \end{Bmatrix} \qquad (\mathbf{ab})^T = [4 \quad 13 \quad 6]$$

Alternatively,

$$\mathbf{a}^T = \begin{bmatrix} 3 & 7 & 5 \\ 2 & 1 & 4 \end{bmatrix} \qquad \mathbf{b}^T = [2 \quad -1]$$

$$(\mathbf{ab})^T = \mathbf{b}^T\mathbf{a}^T = [2 \quad -1]\begin{bmatrix} 3 & 7 & 5 \\ 2 & 1 & 4 \end{bmatrix} = [4 \quad 13 \quad 6]$$

1–5. SPECIAL SQUARE MATRICES

If the numbers of rows and of columns are equal, the matrix is said to be *square* and *of* order n, where n is the number of rows. The elements a_{ii} ($i = 1, 2, \ldots, n$) lie on the principal diagonal. If all the elements except the principal-diagonal elements are zero, the matrix is called a *diagonal matrix*. We will use **d** for diagonal matrices. If the elements of a diagonal matrix are all unity, the diagonal matrix is referred to as a *unit matrix*. A unit matrix is usually indicated by $\mathbf{I}_n$, where n is the order of the matrix.

Example 1–5

Square Matrix, Order 2

$$\begin{bmatrix} 1 & 7 \\ 3 & 2 \end{bmatrix}$$

Diagonal Matrix, Order 3

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Unit Matrix, Order 2

$$\mathbf{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We introduce the Kronecker delta notation:

$$\begin{aligned} \delta_{ij} &= 0 \qquad i \neq j \\ \delta_{ij} &= +1 \qquad i = j \end{aligned} \tag{1-28}$$

With this notation, the unit matrix can be written as

$$\mathbf{I}_n = [\delta_{ij}] \qquad i, j = 1, 2, \ldots, n \tag{1-29}$$

Also, the diagonal matrix, **d**, takes the form

$$\mathbf{d} = [d_i\delta_{ij}] \tag{1-30}$$

where $d_1, d_2, \ldots, d_n$ are the principal elements. If the principal diagonal elements are all equal to k, the matrix reduces to

$$[k\delta_{ij}] = k[\delta_{ij}] = k\mathbf{I}_n \tag{1-31}$$

and is called a *scalar matrix*.

Let **a** be of order $m \times n$. One can easily show that multiplication of **a** by a conformable unit matrix does not change **a**:

$$\begin{aligned} \mathbf{aI}_n &= \mathbf{a} \\ \mathbf{I}_m\mathbf{a} &= \mathbf{a} \end{aligned} \tag{1-32}$$

A unit matrix is commutative with any square matrix of the same order. Similarly, two diagonal matrices of order n are commutative and the product is a diagonal matrix of order n. Premultiplication of **a** by a conformable diagonal matrix **d** multiplies the ith row of **a** by d_i and postmultiplication multiplies the jth column by d_j.

Example 1–6

$$\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & -5 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 2 & 7 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ -2 & -7 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 \\ 2 & 7 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 6 & -1 \\ 4 & -7 \end{bmatrix}$$

A square matrix **a** for which $a_{ij} = a_{ji}$ is called *symmetrical* and has the property that $\mathbf{a} = \mathbf{a}^T$. If $a_{ij} = -a_{ji}$ ($i \neq j$) and the principal diagonal elements all equal zero, the matrix is said to be skew-symmetrical. In this case, $\mathbf{a}^T = -\mathbf{a}$. Any square matrix can be reduced to the sum of a symmetrical matrix and a skew-symmetrical matrix:

$$\mathbf{a} = \mathbf{b} + \mathbf{c}$$

$$\begin{aligned} b_{ij} &= \tfrac{1}{2}(a_{ij} + a_{ji}) \\ c_{ij} &= \tfrac{1}{2}(a_{ij} - a_{ji}) \end{aligned} \tag{1-33}$$

The product of two symmetrical matrices is symmetrical only when the matrices are commutative.* Finally, one can easily show that products of the type

$$(\mathbf{a}^T\mathbf{a}) \qquad (\mathbf{a}\mathbf{a}^T) \qquad (\mathbf{a}^T\mathbf{b}\mathbf{a})$$

where **a** is an arbitrary matrix and **b** a symmetrical matrix, result in symmetrical matrices.

A square matrix having zero elements to the left (right) of the principal diagonal is called an upper (lower) triangular matrix. Examples are:

Upper Triangular Matrix

$$\begin{bmatrix} 3 & 5 & 2 \\ 0 & 7 & 1 \\ 0 & 0 & 4 \end{bmatrix}$$

Lower Triangular Matrix

$$\begin{bmatrix} 3 & 0 & 0 \\ 5 & 7 & 0 \\ 2 & 1 & 4 \end{bmatrix}$$

Triangular matrices are encountered in many of the computational procedures developed for linear systems. Some important properties of triangular matrices are:

1. The transpose of an upper triangular matrix is a lower triangular matrix and vice versa.
2. The product of two triangular matrices of like structure is a triangular matrix of the same structure.

$$\begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} b_{11} & 0 \\ b_{21} & b_{22} \end{bmatrix} = \left[\begin{array}{c|c} a_{11}b_{11} & 0 \\ \hline a_{21}b_{11} + a_{22}b_{21} & a_{22}b_{22} \end{array}\right]$$

1–6. OPERATIONS ON PARTITIONED MATRICES

Operations on a matrix of high order can be simplified by considering the matrix to be divided into smaller matrices, called *submatrices* or cells. The partitioning is usually indicated by dashed lines. A matrix can be partitioned in a number of ways. For example,

$$a = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \left[\begin{array}{cc|c} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \hline a_{31} & a_{32} & a_{33} \end{array}\right] = \left[\begin{array}{cc|c} a_{11} & a_{12} & a_{13} \\ \hline a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array}\right]$$

Note that the partition lines are always straight and extend across the entire matrix. To reduce the amount of writing, the submatrices are represented by

* See Prob. 1–7.

a single symbol. We will use upper case letters to denote the submatrices whenever possible and omit the partition lines.

Example 1–7

We represent

$$\mathbf{a} = \left[\begin{array}{cc|c} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \hline a_{31} & a_{32} & a_{33} \end{array}\right]$$

as

$$\mathbf{a} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \quad \text{or} \quad \mathbf{a} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}$$

where

$$\mathbf{A}_{11} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad \mathbf{A}_{12} = \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix}$$

$$\mathbf{A}_{21} = [a_{31} \quad a_{32}] \qquad \mathbf{A}_{22} = [a_{33}]$$

If two matrices of the *same* order are *identically* partitioned, the rules of matrix addition are applicable to the submatrices. Let

$$\mathbf{a} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{bmatrix} \tag{1–34}$$

where $\mathbf{B}_{ij}$ and $\mathbf{A}_{ij}$ are of the *same* order. The sum is

$$\mathbf{a} + \mathbf{b} = \left[\begin{array}{c|c} \mathbf{A}_{11} + \mathbf{B}_{11} & \mathbf{A}_{12} + \mathbf{B}_{12} \\ \hline \mathbf{A}_{21} + \mathbf{B}_{21} & \mathbf{A}_{22} + \mathbf{B}_{22} \end{array}\right] \tag{1–35}$$

The rules of matrix multiplication are applicable to partitioned matrices provided that the partitioned matrices are conformable for multiplication. In general, two partitioned matrices are conformable for multiplication if the partitioning of the *rows* of the *second* matrix is identical to the partitioning of the *columns* of the first matrix. This restriction allows us to treat the various submatrices as single elements provided that we preserve the order of multiplication. Let **a** and **b** be two partitioned matrices:

$$\begin{aligned} \mathbf{a} &= [\mathbf{A}_{ij}] \qquad & i &= 1, 2, \ldots, N \\ & & j &= 1, 2, \ldots, M \\ \mathbf{b} &= [\mathbf{B}_{jk}] \qquad & j &= 1, 2, \ldots, M \\ & & k &= 1, 2, \ldots, S \end{aligned} \tag{1–36}$$

We can write the product as

$$\mathbf{c} = \mathbf{ab} = [\mathbf{C}_{ik}]$$

$$\mathbf{C}_{ik} = \sum_{j=1}^{M} \mathbf{A}_{ij}\mathbf{B}_{jk} \qquad \begin{aligned} i &= 1, 2, \ldots, N \\ k &= 1, 2, \ldots, S \end{aligned} \tag{1–37}$$

when the row partitions of **b** are consistent with the column partitions of **a**.

As an illustration, we consider the product

$$\mathbf{ab} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix}$$

Suppose we partition $\mathbf{a}$ with a vertical partition between the second and third columns.

$$\mathbf{a} = \left[\begin{array}{cc|c} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array}\right] = [\mathbf{A}_{11}\mathbf{A}_{12}]$$

For the rules of matrix multiplication to be applicable to the submatrices of $\mathbf{a}$, we must partition $\mathbf{b}$ with a horizontal partition between the second and third rows. Taking

$$\mathbf{b} = \left\{\begin{array}{c} b_1 \\ b_2 \\ \hline b_3 \end{array}\right\} = \begin{Bmatrix} \mathbf{B}_{11} \\ \mathbf{B}_{21} \end{Bmatrix}$$

the product has the form

$$\mathbf{ab} = [\mathbf{A}_{11}\mathbf{A}_{12}] \begin{Bmatrix} \mathbf{B}_{11} \\ \mathbf{B}_{21} \end{Bmatrix} = \mathbf{A}_{11}\mathbf{B}_{11} + \mathbf{A}_{12}\mathbf{B}_{21}$$

The conformability of two partitioned matrices does not depend on the horizontal partitioning of the first matrix or the vertical partitioning of the second matrix. To show this, we consider the product

$$\mathbf{ab} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$$

Suppose we partition $\mathbf{a}$ with a horizontal partition between the second and third rows:

$$\mathbf{a} = \left[\begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \hline a_{31} & a_{32} & a_{33} \end{array}\right] = \begin{bmatrix} \mathbf{A}_{11} \\ \mathbf{A}_{21} \end{bmatrix}$$

Since the column order of $\mathbf{A}_{11}$ and $\mathbf{A}_{21}$ is equal to the row order of $\mathbf{b}$, *no* partitioning of $\mathbf{b}$ is required. The product is

$$\mathbf{ab} = \begin{bmatrix} \mathbf{A}_{11} \\ \mathbf{A}_{21} \end{bmatrix} \mathbf{b} = \begin{bmatrix} \mathbf{A}_{11}\mathbf{b} \\ \mathbf{A}_{21}\mathbf{b} \end{bmatrix}$$

As an alternative, we partition $\mathbf{b}$ with a vertical partition.

$$\mathbf{b} = \left[\begin{array}{c|c} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{array}\right] = [\mathbf{B}_{11}\mathbf{B}_{12}]$$

In this case, since the row order of $\mathbf{B}_{11}$ and $\mathbf{B}_{12}$ is the same as the column

order of $\mathbf{a}$, no partitioning of $\mathbf{a}$ is necessary and the product has the form

$$\mathbf{ab} = \mathbf{a}[\mathbf{B}_{11}\mathbf{B}_{12}] = [\mathbf{aB}_{11} \mid \mathbf{aB}_{12}]$$

To transpose a partitioned matrix, one first interchanges the off-diagonal submatrices and then transposes each submatrix. If

$$\mathbf{a} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \cdots & \mathbf{A}_{1n} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \cdots & \mathbf{A}_{2n} \\ \vdots & \vdots & & \vdots \\ \mathbf{A}_{m1} & \mathbf{A}_{m2} & \cdots & \mathbf{A}_{mn} \end{bmatrix}$$

then

$$\mathbf{a}^T = \begin{bmatrix} \mathbf{A}_{11}^T & \mathbf{A}_{21}^T & \cdots & \mathbf{A}_{m1}^T \\ \mathbf{A}_{12}^T & \mathbf{A}_{22}^T & \cdots & \mathbf{A}_{m2}^T \\ \vdots & \vdots & & \vdots \\ \mathbf{A}_{1n}^T & \mathbf{A}_{2n}^T & \cdots & \mathbf{A}_{mn}^T \end{bmatrix} \tag{1–38}$$

A particular type of matrix encountered frequently is the *quasi-diagonal* matrix. This is a partitioned matrix whose diagonal submatrices are *square* of various orders, and whose off-diagonal submatrices are *null* matrices. An example is

$$\mathbf{a} = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix}$$

which can be written in partitioned form as

$$\mathbf{a} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_2 \end{bmatrix} = [\mathbf{A}_i\delta_{ij}]$$

where

$$\mathbf{A}_1 = [a_{11}] \qquad \mathbf{A}_2 = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$$

and $\mathbf{0}$ denotes a null matrix. The product of two quasi-diagonal matrices of like structure (corresponding diagonal submatrices are of the same order) is a quasi-diagonal matrix of the same structure.

$$\begin{bmatrix} \mathbf{A}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{A}_n \end{bmatrix} \begin{bmatrix} \mathbf{B}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{B}_n \end{bmatrix} = \begin{bmatrix} \mathbf{A}_1\mathbf{B}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_2\mathbf{B}_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{A}_n\mathbf{B}_n \end{bmatrix} \tag{1–39}$$

where $\mathbf{A}_i$ and $\mathbf{B}_i$ are of the same order.

We use the term *quasi* to distinguish between partitioned and unpartitioned matrices having the *same form*. For example, we call

$$\begin{bmatrix} \mathbf{A}_{11} & \mathbf{0} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \tag{1–40}$$

a lower quasi-triangular matrix.

1–7. DEFINITION AND PROPERTIES OF A DETERMINANT

The concept of a determinant was originally developed in connection with the solution of square systems of linear algebraic equations. To illustrate how this concept evolved, we consider the simple case of two equations:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= c_1 \\ a_{21}x_1 + a_{22}x_2 &= c_2 \end{aligned} \tag{a}$$

Solving (a) for x_1 and x_2, we obtain

$$\begin{aligned} (a_{11}a_{22} - a_{12}a_{21})x_1 &= c_1a_{22} - c_2a_{12} \\ (a_{11}a_{22} - a_{12}a_{21})x_2 &= -c_1a_{21} + c_2a_{11} \end{aligned} \tag{b}$$

The scalar quantity, $a_{11}a_{22} - a_{21}a_{21}$, is defined as the determinant of the second-order square array a_{ij} $(i, j = 1, 2)$. The determinant of an array (or matrix) is usually indicated by enclosing the array (or matrix) with vertical lines:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = |\mathbf{a}| = a_{11}a_{22} - a_{12}a_{21} \tag{1–41}$$

We use the terms *array* and *matrix* interchangeably, since they are synonymous. Also, we refer to the determinant of an nth-order array as an nth-order determinant. It should be noted that determinants are associated only with square arrays, that is, with *square* matrices.

The determinant of a third-order array is defined as

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{matrix} +a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} \\ -a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} \\ +a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} \end{matrix} \tag{1–42}$$

This number is the coefficient of x_1, x_2, and x_3, obtained when the third-order system $\mathbf{ax} = \mathbf{c}$ is solved successively for x_1, x_2, and x_3. Comparing (1–41) and (1–42), we see that both expansions involve products which have the following properties:

1. Each product contains only one element from any row or column and no element occurs twice in the same product. The products differ only in the column subscripts.
2. The sign of a product depends on the order of the column subscripts, e.g., $+a_{11}a_{22}a_{33}$ and $-a_{11}a_{23}a_{32}$.

These properties are associated with the arrangement of the column subscripts and can be conveniently described using the concept of a permutation, which is discussed below.

A set of distinct integers is considered to be in *natural order* if each integer is followed only by larger integers. A rearrangement of the natural order is called a *permutation* of the set. For example, (1, 3, 5) is in natural order and (1, 5, 3) is a permutation of (1, 3, 5). If an integer is followed by a smaller integer, the pair is said to form an *inversion*. The number of inversions for a set is defined as the sum of the inversions for each integer. As an illustration, we consider the set (3, 1, 4, 2). Working from left to right, the integer inversions are:

Integer	Inversions	Total
3	(3, 1)(3, 2)	2
1	None	0
4	(4, 2)	1
2	None	0
		3

This set has three inversions. A permutation is classified as even (odd) if the total number of inversions for the set is an even (odd) integer. According to this convention, (1, 2, 3) and (3, 1, 2) are even permutations and (1, 3, 2) is an odd permutation. Instead of counting the inversions, we can determine the number of integer interchanges required to rearrange the set in its natural order since an even (odd) number of interchanges corresponds to an even (odd) number of inversions. For example, (3, 2, 1) has three inversions and requires one interchange. Working with interchanges rather than inversions is practical only when the set is small.

Referring back to (1–41) and (1–42), we see that each product is a permutation of the set of column subscripts and the sign is negative when the permutation is odd. The number of products is equal to the number of possible permutations of the column subscripts that can be formed. One can easily show that there are n-factorial* possible permutations for a set of n distinct integers.

We let $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ be a permutation of the set $(1, 2, \ldots, n)$ and define $e_{\alpha_1\alpha_2\cdots\alpha_n}$ as

$$\begin{aligned} e_{\alpha_1\alpha_2\cdots\alpha_n} &= +1 \quad \text{when } (\alpha_1, \alpha_2, \ldots, \alpha_n) \text{ is an even permutation} \\ e_{\alpha_1\alpha_2\cdots\alpha_n} &= -1 \quad \text{when } (\alpha_1, \alpha_2, \ldots, \alpha_n) \text{ is an odd permutation} \end{aligned} \tag{1–43}$$

Using (1–43), the definition equation for an nth-order determinant can be written as

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \sum e_{\alpha_1\alpha_2\cdots\alpha_n} a_{1\alpha_1}a_{2\alpha_2}\cdots a_{n\alpha_n} \tag{1–44}$$

where the summation is taken over all possible permutations of $(1, 2, \ldots, n)$.

* Factorial $n = n! = n(n-1)(n-2)\cdots(2)(1)$.

Example 1–8

The permutations for $n = 3$ are

$$\begin{array}{llll} \alpha_1 = 1 & \alpha_2 = 2 & \alpha_3 = 3 & e_{123} = +1 \\ \alpha_1 = 1 & \alpha_2 = 3 & \alpha_3 = 2 & e_{132} = -1 \\ \alpha_1 = 2 & \alpha_2 = 1 & \alpha_3 = 3 & e_{213} = -1 \\ \alpha_1 = 2 & \alpha_2 = 3 & \alpha_3 = 1 & e_{231} = +1 \\ \alpha_1 = 3 & \alpha_2 = 1 & \alpha_3 = 2 & e_{312} = +1 \\ \alpha_1 = 3 & \alpha_2 = 2 & \alpha_3 = 1 & e_{321} = -1 \end{array}$$

Using (1–44), we obtain

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{array}{l} a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} \\ -a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} \\ +a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} \end{array}$$

This result coincides with (1–42).

The following properties of determinants can be established* from (1–44):

1. If all elements of any row (or column) are zero, the determinant is zero.
2. The value of the determinant is unchanged if the rows and columns are interchanged; that is, $|\mathbf{a}^T| = |\mathbf{a}|$.
3. If two successive rows (or two successive columns) are interchanged, the sign of the determinant is changed.
4. If all elements of one row (or one column) are multiplied by a number k, the determinant is multiplied by k.
5. If corresponding elements of two rows (or two columns) are equal or in a constant ratio, then the determinant is zero.
6. If each element in one row (or one column) is expressed as the sum of two terms, then the determinant is equal to the sum of two determinants, in each of which one of the two terms is deleted in each element of that row (or column).
7. If to the elements of any row (column) are added k times the corresponding elements of any other row (column), the determinant is unchanged.

We demonstrate these properties for the case of a second-order matrix. Let

$$\mathbf{a} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

The determinant is

$$|\mathbf{a}| = a_{11}a_{22} - a_{12}a_{21}$$

Properties 1 and 2 are obvious. It follows from property 2 that $|\mathbf{a}^T| = |\mathbf{a}|$. We

* See Probs. 1–17, 1–18, 1–19.

illustrate the third by interchanging the rows of $\mathbf{a}$:

$$\mathbf{a}' = \begin{bmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{bmatrix}$$

$$|\mathbf{a}'| = a_{21}a_{12} - a_{11}a_{22} = -|\mathbf{a}|$$

Property 4 is also obvious from (b). To demonstrate the fifth, we take

$$a_{21} = ka_{11} \qquad a_{22} = ka_{12}$$

Then

$$|\mathbf{a}| = a_{11}(ka_{12}) - a_{12}(ka_{11}) = 0$$

Next, let

$$a_{11} = b_{11} + c_{11} \qquad a_{12} = b_{12} + c_{12}$$

According to property 6,

$$|\mathbf{a}| = |\mathbf{b}| + |\mathbf{c}|$$

where

$$|\mathbf{b}| = \begin{vmatrix} b_{11} & b_{12} \\ a_{21} & a_{22} \end{vmatrix} \qquad |\mathbf{c}| = \begin{vmatrix} c_{11} & c_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

This result can be obtained by substituting for a_{11} and a_{12} in (b). Finally, to illustrate property 7, we take

$$\begin{aligned} b_{11} &= a_{11} + ka_{21} \\ b_{12} &= a_{12} + ka_{22} \\ b_{21} &= a_{21} \\ b_{22} &= a_{22} \end{aligned}$$

Then,

$$|\mathbf{b}| = (a_{11} + ka_{21})a_{22} - (a_{12} + ka_{22})a_{21} = |\mathbf{a}|$$

1–8. COFACTOR EXPANSION FORMULA

If the row and column containing an element, a_{ij}, in the square matrix, $\mathbf{a}$, are deleted, the determinant of the remaining square array is called the *minor* of a_{ij}, and is denoted by M_{ij}. The *cofactor* of a_{ij}, denoted by A_{ij}, is related to the minor of M_{ij} by

$$A_{ij} = (-1)^{i+j}M_{ij} \tag{1–45}$$

As an illustration, we take

$$\mathbf{a} = \begin{bmatrix} 3 & 2 & 8 \\ 1 & 7 & 4 \\ 5 & 3 & 1 \end{bmatrix}$$

The values of M_{ij} and A_{ij} associated with a_{23} and a_{22} are

$$M_{23} = \begin{vmatrix} 3 & 2 \\ 5 & 3 \end{vmatrix} = -1 \qquad A_{23} = (-1)^5 M_{23} = +1$$

$$M_{22} = \begin{vmatrix} 3 & 8 \\ 5 & 1 \end{vmatrix} = -37 \qquad A_{22} = (-1)^4 M_{22} = -37$$

Cofactors occur naturally when (1–44) is expanded* in terms of the elements of a row or column. This leads to the following expansion formula, called Laplace's expansion by cofactors or simply Laplace's expansion:†

$$|\mathbf{a}| = \sum_{k=1}^{n} a_{ik}A_{ik} = \sum_{k=1}^{n} a_{kj}A_{kj} \tag{1–46}$$

Equation (1–46) states that the determinant is equal to the sum of the products of the elements of any single row or column by their cofactors.

Since the determinant is zero if two rows or columns are identical, if follows that

$$\begin{aligned} \sum_{k=1}^{n} a_{rk}A_{ik} &= 0 \qquad r \neq i \\ \sum_{k=1}^{n} a_{ks}A_{kj} &= 0 \qquad s \neq j \end{aligned} \tag{1–47}$$

The above identities are used to establish Cramer's rule in the following section.

Example 1–9

(1) We apply (1–46) to a third-order array and expand with respect to the first row:

$$\begin{aligned} &\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ &= a_{11}(-1)^2\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}(-1)^3\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}(-1)^4\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) + a_{12}(-a_{21}a_{33} + a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \end{aligned}$$

To illustrate (1–47), we take the cofactors for the first row and the elements of the second row:

$$\begin{aligned} &\sum_{k=1}^{3} a_{2k}A_{1k} \\ &= a_{21}(a_{22}a_{33} - a_{23}a_{32}) + a_{22}(-a_{21}a_{33} + a_{23}a_{31}) + a_{23}(a_{21}a_{32} - a_{22}a_{31}) = 0 \end{aligned}$$

(2) Suppose the array is triangular in form, for example, lower triangular. Expanding with respect to the first row, we have

$$\begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & 0 \\ a_{32} & a_{33} \end{vmatrix} = (a_{11})(a_{22}a_{33}) = a_{11}a_{22}a_{33}$$

Generalizing this result, we find that the determinant of a triangular matrix is equal to the product of the diagonal elements. This result is quite useful.

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* See Probs. 1–20, 1–21.

† See Ref. 4, sect. 3–15, for a discussion of the general Laplace expansion method. The expansion in terms of cofactors for a row or a column is a special case of the general method.

The evaluation of a determinant, using the definition equation (1–44) or the cofactor expansion formula (1–46) is quite tedious, particularly when the array is large. A number of alternate and more efficient numerical procedures for evaluating determinants have been developed. These procedures are described in References 9–13.

Suppose a square matrix, say **c**, is expressed as the product of two square matrices,

$$\mathbf{c} = \mathbf{ab} \tag{a}$$

and we want $|\mathbf{c}|$. It can be shown* that the determinant of the product of two square matrices is equal to the product of the determinants:

$$|\mathbf{c}| = |\mathbf{a}|\,|\mathbf{b}| \tag{1–48}$$

Whether we use (1–48) or first multiply **a** and **b** and then determine $|\mathbf{ab}|$ depends on the form and order of **a** and **b**. If they are diagonal or triangular, (1–48) is quite efficient.†

Example 1–10

(1)

$$\mathbf{a} = \begin{bmatrix} 1 & 3 \\ 3 & 5 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$$

$$|\mathbf{a}| = -4 \qquad |\mathbf{b}| = 5 \qquad |\mathbf{c}| = -20$$

Alternatively,

$$\mathbf{c} = \begin{bmatrix} 5 & 15 \\ 11 & 29 \end{bmatrix} \quad \text{and} \quad |\mathbf{c}| = -20$$

(2)

$$\mathbf{a} = \begin{bmatrix} 1 & 3 \\ 0 & 5 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 2 & 0 \\ 1 & 4 \end{bmatrix}$$

$$|\mathbf{a}| = 5 \qquad |\mathbf{b}| = 8 \qquad |\mathbf{c}| = +40$$

Determining **c** first, we obtain

$$\mathbf{c} = \begin{bmatrix} 5 & 12 \\ 5 & 20 \end{bmatrix} \quad \text{and} \quad |\mathbf{c}| = +40$$

1–9. CRAMER'S RULE

We consider next a set of n equations in n unknowns:

$$\sum_{k=1}^{n} a_{jk}x_k = c_j \qquad j = 1, 2, \ldots, n \tag{a}$$

* See Ref. 4, section 3–16.

† See Prob. 1–25 for an important theoretical application of Eq. 1–48.

Multiplying both sides of (a) by A_{jr}, where r is an arbitrary integer from 1 to n, and summing with respect to j, we obtain (after interchanging the order of summation)

$$\sum_{k=1}^{n}\left(\sum_{j=1}^{n} a_{jk}A_{jr}\right)x_k = \sum_{j=1}^{n} A_{jr}c_j \tag{b}$$

Now, the inner sum vanishes when $r \neq k$ and equals $|\mathbf{a}|$ when $r = k$. This follows from (1–47). Then, (b) reduces to

$$|\mathbf{a}|x_r = \sum_{j=1}^{n} A_{jr}c_j \tag{c}$$

The expansion on the right side of (c) differs from the expansion

$$|\mathbf{a}| = \sum_{j=1}^{n} a_{jr}A_{jr} \tag{d}$$

only in that the rth column of $\mathbf{a}$ is replaced by $\mathbf{c}$. Equation (c) leads to Cramer's rule, which can be stated as follows:

> A set of n linear algebraic equations in n unknowns, $\mathbf{ax} = \mathbf{c}$, has a *unique* solution when $|\mathbf{a}| \neq 0$. The expression for x_r ($r = 1, 2, \ldots, n$) is the ratio of two determinants; the denominator is $|\mathbf{a}|$ and the numerator is the determinant of the matrix obtained from $\mathbf{a}$ by replacing the rth column by $\mathbf{c}$.

If $|\mathbf{a}| = 0$, $\mathbf{a}$ is said to be *singular*. Whether a solution exists in this case will depend on $\mathbf{c}$. All we can conclude from Cramer's rule is that the solution, if it exists, will not be unique. Singular matrices and the question of solvability are discussed in Sec. 1–13.

1–10. ADJOINT AND INVERSE MATRICES

We have shown in the previous section that the solution to a system of n equations in n unknowns,

$$[a_{ij}]\{x_j\} = \{c_i\} \qquad i, j = 1, 2, \ldots, n \tag{a}$$

can be expressed as

$$x_i = \frac{1}{|\mathbf{a}|}\sum_{j=1}^{n} A_{ji}c_j \qquad i = 1, 2, \ldots, n \tag{b}$$

(note that we have taken $r = i$ in Eq. c of Sec. 1–9). Using matrix notation, (b) takes the form

$$x_i = \frac{1}{|\mathbf{a}|}[A_{ij}]^T\{c_j\} \tag{c}$$

Equation (c) leads naturally to the definition of adjoint and inverse matrices.

We define the *adjoint* and *inverse matrices* for the square matrix $\mathbf{a}$ of order n as

$$\text{adjoint } \mathbf{a} = \text{Adj } \mathbf{a} = [A_{ij}]^T \tag{1–49}$$

$$\text{inverse } \mathbf{a} = \mathbf{a}^{-1} = \frac{1}{|\mathbf{a}|}\text{Adj } \mathbf{a} \tag{1–50}$$

Note that the inverse matrix is defined only for a nonsingular square matrix.

Example 1–11

We determine the adjoint and inverse matrices for

$$\mathbf{a} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 4 & 1 & 2 \end{bmatrix}$$

The matrix of cofactors is

$$[A_{ij}] = \begin{bmatrix} 5 & 0 & -10 \\ -1 & -10 & +7 \\ -7 & +5 & -1 \end{bmatrix}$$

Also, $|\mathbf{a}| = -25$. Then

$$\text{Adj } \mathbf{a} = [A_{ij}]^T = \begin{bmatrix} 5 & -1 & -7 \\ 0 & -10 & +5 \\ -10 & +7 & -1 \end{bmatrix}$$

$$\mathbf{a}^{-1} = \frac{1}{|\mathbf{a}|}\text{Adj } \mathbf{a} = \begin{bmatrix} -1/5 & +1/25 & +7/25 \\ 0 & +2/5 & -1/5 \\ +2/5 & -7/25 & +1/25 \end{bmatrix}$$

Using the inverse-matrix notation, we can write the solution of (a) as

$$\mathbf{x} = \mathbf{a}^{-1}\mathbf{c} \tag{d}$$

Substituting for $\mathbf{x}$ in (a) and $\mathbf{c}$ in (d), we see that $\mathbf{a}^{-1}$ has the property that

$$\mathbf{a}^{-1}\mathbf{a} = \mathbf{a}\mathbf{a}^{-1} = \mathbf{I}_n \tag{1–51}$$

Equation (1–51) is frequently taken as the definition of the inverse matrix instead of (1–50). Applying (1–48) to (1–51), we obtain

$$|\mathbf{a}^{-1}|\,|\mathbf{a}| = \mathbf{I}$$

It follows that (1–51) is valid only when $|\mathbf{a}| \neq 0$. Multiplication by the inverse matrix is analogous to division in ordinary algebra.

If $\mathbf{a}$ is symmetrical, then $\mathbf{a}^{-1}$ is also symmetrical. To show this, we take the transpose of (1–51), and use the fact that $\mathbf{a} = \mathbf{a}^T$:

$$(\mathbf{a}^{-1}\mathbf{a})^T = \mathbf{a}\mathbf{a}^{-1,T} = \mathbf{I}_n$$

Premultiplication by $\mathbf{a}^{-1}$ results in

$$\mathbf{a}^{-1,T} = \mathbf{a}^{-1}$$

and therefore $\mathbf{a}^{-1}$ is also symmetrical. One can also show* that, for any nonsingular square matrix, the inverse and transpose operations can be interchanged:

$$\mathbf{b}^{T,-1} = \mathbf{b}^{-1,T} \tag{1-52}$$

We consider next the inverse matrix associated with the product of two square matrices. Let

$$\mathbf{c} = \mathbf{ab}$$

where $\mathbf{a}$ and $\mathbf{b}$ are both of order $n \times n$ and nonsingular. Premultiplication by $\mathbf{a}^{-1}$ and then $\mathbf{b}^{-1}$ results in

$$\mathbf{a}^{-1}\mathbf{c} = \mathbf{b}$$
$$(\mathbf{b}^{-1}\mathbf{a}^{-1})\mathbf{c} = \mathbf{I}_n$$

It follows from the definition of the inverse matrix that

$$(\mathbf{ab})^{-1} = \mathbf{b}^{-1}\mathbf{a}^{-1} \tag{1-53}$$

In general, the inverse of a multiple matrix product is equal to the product of the inverse matrices in reverse order. For example,

$$(\mathbf{abcd})^{-1} = \mathbf{d}^{-1}\mathbf{c}^{-1}\mathbf{b}^{-1}\mathbf{a}^{-1}$$

The determination of the inverse matrix using the definition equation (1–50) is too laborious when the order is large. A number of inversion procedures based on (1–51) have been developed. These methods are described in Ref. 9–13.

1–11. ELEMENTARY OPERATIONS ON A MATRIX

The elementary operations on a matrix are:

1. The interchange of two rows or of two columns.
2. The multiplication of the elements of a row or a column by a number other than zero.
3. The addition, to the elements of a row or column, of k times the corresponding element of another row or column.

These operations can be effected by premultiplying (for row operation) or postmultiplying (for column operation) the matrix by an appropriate matrix, called an elementary operation matrix.

We consider a matrix $\mathbf{a}$ of order $m \times n$. Suppose that we want to interchange rows j and k. Then, we *premultiply* $\mathbf{a}$ by an $m \times m$ matrix obtained by modifying the mth-order *unit* matrix, $\mathbf{I}_m$, in the following way:

1. Interchange δ_{jj} and δ_{jk}.
2. Interchange δ_{kk} and δ_{kj}.

* See Prob. 1–28.

For example, if $\mathbf{a}$ is 3×4, premultiplication by

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

interchanges rows 1 and 3 and postmultiplication by

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

interchanges columns 2 and 4. This simple example shows that to interchange rows, we first interchange the rows of the conformable unit matrix and *premultiply*. Similarly, to interchange columns, we interchange columns of the conformable unit matrix and postmultiply.

The elementary operation matrices for operations (2) and (3) are also obtained by operating on the *corresponding* conformable unit matrix. The matrix which multiplies row j by α is an mth order diagonal matrix having $d_i = 1$ for $i \neq j$ and $d_j = \alpha$. Similarly, postmultiplication by an nth order diagonal matrix having $d_i = 1$ for $i \neq j$ and $d_j = \alpha$ will multiply the jth column by α. Suppose that we want to add α times row j to row k. Then, we insert α in the kth row and jth column of $\mathbf{I}_m$ and premultiply. To add α times column j to column k, we put α in the jth row and kth column of $\mathbf{I}_n$ and postmultiply.

We let $\mathbf{e}$ denote an elementary operation matrix. Then, $\mathbf{ea}$ represents the result of applying a set of elementary operations to the rows of $\mathbf{a}$. Similarly, $\mathbf{ae}$ represents the result of applying a set of elementary operations to the columns of $\mathbf{a}$. In general, we obtain $\mathbf{e}$ by applying the *same* operations to the conformable unit matrix. Since we start with a unit matrix and since the elementary operations, at most, change the value of the determinant by a nonzero scalar factor,* it follows that $\mathbf{e}$ will always be nonsingular.

Example 1–12

We illustrate these operations on a third matrix:

$$\mathbf{a} = \begin{bmatrix} 1 & 1/2 & 1/5 \\ 3 & 7 & 2 \\ -2 & 1 & 5 \end{bmatrix}$$

We first:

1. Add (−3) times the first row to the second row.
2. Add (2) times the first row to the third row.

* See properties of determinants (page 18).

These operations are carried out by premultiplying by

$$\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

and the result is

$$\begin{bmatrix} 1 & 1/2 & 1/5 \\ 0 & 11/2 & 7/5 \\ 0 & 2 & 27/5 \end{bmatrix}$$

Continuing, we multiply the second row by 2/11:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2/11 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1/2 & 1/5 \\ 0 & 11/2 & 7/5 \\ 0 & 2 & 27/5 \end{bmatrix} = \begin{bmatrix} 1 & 1/2 & 1/5 \\ 0 & 1 & 14/55 \\ 0 & 2 & 27/5 \end{bmatrix}.$$

Next, we add (−2) times the second row to the third row:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1/2 & 1/5 \\ 0 & 1 & 14/55 \\ 0 & 2 & 27/5 \end{bmatrix} = \begin{bmatrix} 1 & 1/2 & 1/5 \\ 0 & 1 & 14/55 \\ 0 & 0 & 269/55 \end{bmatrix}$$

Finally, we multiply the third row by 55/269. The complete set of operations is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 55/269 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2/11 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1/2 & 1/5 \\ 3 & 7 & 2 \\ -2 & 1 & 5 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 1/2 & 1/5 \\ 0 & 1 & 14/55 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{b}$$

This example illustrates the reduction of a square matrix to a *triangular* matrix using elementary operations on rows, and is the basis for the Gauss elimination solution scheme (Refs. 9, 11, 13). We write the result as

$$\mathbf{ea} = \mathbf{b}$$

where **e** is the product of the four operation matrices listed above:

$$\mathbf{e} = \begin{bmatrix} 1 & 0 & 0 \\ -6/11 & 2/11 & 0 \\ +1870/2959 & -220/2959 & 55/269 \end{bmatrix}$$

We obtain **e** by applying successive operations, starting with a unit matrix. This is more convenient than listing and then multiplying the operation matrices for the various steps. The form of **e** *after* each step is listed below:

Initial, Step 1, Step 2:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ -6/11 & 2/11 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

Step 3, Step 4:

$$\begin{bmatrix} 1 & 0 & 0 \\ -6/11 & 2/11 & 0 \\ +34/11 & -4/11 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ -6/11 & 2/11 & 0 \\ +1870/2959 & -220/2959 & 55/269 \end{bmatrix}$$

Two matrices are said to be equivalent if one can be derived from the other by any finite number of elementary operations. Referring to Example 1–12, the matrices

$$\begin{bmatrix} 1 & 1/2 & 1/5 \\ 3 & 7 & 2 \\ -2 & 1 & 5 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 1/2 & 1/5 \\ 0 & 1 & 14/55 \\ 0 & 0 & 1 \end{bmatrix}$$

are equivalent. In general, **a** and **b** are equivalent if

$$\mathbf{b} = \mathbf{paq} \tag{1–54}$$

where **p** and **q** are *nonsingular*. This follows from the fact that the elementary operation matrices are nonsingular.

1–12. RANK OF A MATRIX

The rank, r, of a matrix is defined as the order of the largest square array, formed by deleting certain rows and columns, which has a nonvanishing determinant. The concept of rank is quite important since, as we shall see in the next section, the solvability of a set of linear algebraic equations is dependent on the rank of certain matrices associated with the set.

Let **a** be of order $m \times n$. Suppose the rank of **a** is r. Then **a** has r rows which are linearly independent, that is, which contain a nonvanishing determinant of order r, and the remaining $m - r$ rows are linear combinations of these r rows. Also, it has $n - r$ columns which are linear combinations of r linearly independent columns.

To establish this result, we suppose the determinant associated with the first r rows and columns does not vanish. If **a** is of rank r, one can always rearrange the rows and columns such that this condition is satisfied. We consider the $(r + 1)$th-order determinant associated with the first r rows and columns, row p, and column q where $r < p \leqslant m$, $r < q \leqslant n$.

$$A_{r+1} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1r} & a_{1q} \\ a_{21} & a_{22} & \cdots & a_{2r} & a_{2q} \\ \vdots & \vdots & & \vdots & \vdots \\ a_{r1} & a_{r2} & \cdots & a_{rr} & a_{rq} \\ a_{p1} & a_{p2} & \cdots & a_{pr} & a_{pq} \end{vmatrix} \tag{1–55}$$

We multiply the elements in row j by λ_{pj} $(j = 1, 2, \ldots, r)$ and subtract the result from the last row. This operation will not change the magnitude of A_{r+1} (see Sec. 1–7). In particular, we determine the constants such that the first r elements in the last row vanish:

$$\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{r1} \\ a_{12} & a_{22} & \cdots & a_{r2} \\ \vdots & \vdots & & \vdots \\ a_{1r} & a_{2r} & \cdots & a_{rr} \end{bmatrix} \begin{Bmatrix} \lambda_{p1} \\ \lambda_{p2} \\ \vdots \\ \lambda_{pr} \end{Bmatrix} = \begin{Bmatrix} a_{p1} \\ a_{p2} \\ \vdots \\ a_{pr} \end{Bmatrix} \tag{1-56}$$

Equation (1–56) has a unique solution since the coefficient matrix is non-singular. Then (1–55) reduces to

$$A_{r+1} = \left| \begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1r} & a_{1q} \\ a_{21} & a_{22} & \cdots & a_{2r} & a_{2q} \\ \vdots & \vdots & & \vdots & \vdots \\ a_{r1} & a_{r2} & \cdots & a_{rr} & a_{rq} \\ \hline 0 & 0 & \cdots & 0 & a_{pq}^{(1)} \end{array} \right| \tag{1-57}$$

where

$$a_{pq}^{(1)} = a_{pq} - [a_{1q}, a_{2q}, \ldots, a_{rq}] \begin{Bmatrix} \lambda_{p1} \\ \lambda_{p2} \\ \vdots \\ \lambda_{pr} \end{Bmatrix} \tag{1-58}$$

Applying Laplace's expansion formula to (1–57), we see that A_{r+1} vanishes when $a_{pq}^{(1)} = 0$.

Now, if **a** is of rank r, A_{r+1} vanishes for all combinations of p and q. It follows that

$$a_{pq} = [a_{1q}, a_{2q}, \ldots, a_{rq}] \begin{Bmatrix} \lambda_{p1} \\ \lambda_{p2} \\ \vdots \\ \lambda_{pr} \end{Bmatrix} \qquad \begin{array}{l} q = r+1, \ldots, n \\ p = r+1, r+2, \ldots, m \end{array} \tag{1-59}$$

Combining (1–56) and (1–59), we have

$$\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{r1} \\ a_{12} & a_{22} & \cdots & a_{r2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{rn} \end{bmatrix} \begin{Bmatrix} \lambda_{p1} \\ \lambda_{p2} \\ \vdots \\ \lambda_{pr} \end{Bmatrix} = \begin{Bmatrix} a_{p1} \\ a_{p2} \\ \vdots \\ a_{pn} \end{Bmatrix} \qquad p = r+1, r+2, \ldots, m \tag{1-60}$$

Equation (1–60) states that the last $m - r$ rows of **a** are linear combinations of the first r rows. One can also show* that the last $n - r$ columns of **a** are linear combinations of the first r columns.

Example 1–13

Consider the 3×4 matrix

$$\mathbf{a} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 2 \\ 5 & 7 & 12 & 14 \end{bmatrix}$$

* See Prob. 1–39.

We see that **a** is at least of rank 2 since the determinant associated with the first two rows and columns is finite. Then, the first two rows are linearly independent. We consider the determinant of the third-order array consisting of columns 1, 2, and q:

$$A_3 = \begin{vmatrix} 1 & 2 & a_{1q} \\ 2 & 1 & a_{2q} \\ 5 & 7 & a_{3q} \end{vmatrix}$$

Solving the system,

$$\begin{aligned} \lambda_1 + 2\lambda_2 &= 5 \\ 2\lambda_1 + \lambda_2 &= 7 \end{aligned}$$

we obtain

$$\lambda_1 = 3 \qquad \lambda_2 = 1$$

If **a** is of rank 2, A_3 must vanish. This requires

$$\begin{aligned} a_{3q} &= \lambda_1 a_{1q} + \lambda_2 a_{2q} = 3a_{1q} + a_{2q} \\ q &= 3, 4 \end{aligned}$$

Since a_{33} and a_{34} satisfy this requirement, we conclude that **a** is of rank 2. The rows are related by

(third row) = +3 (first row) + (second row)

One can show* that the elementary operations do not change the rank of a matrix. This fact can be used to determine the rank of a matrix. Suppose **b** defined by (1–61) is obtained by applying elementary operations to **a**. We know that **b** and **a** have the same rank. It follows that **a** is of rank p. A matrix having the form of **b** is called an echelon matrix. When **a** is large, it is more efficient to reduce it to an echelon matrix rather than try to find the largest nonvanishing determinant:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \Rightarrow \mathbf{b} = \left[\begin{array}{cccc|c} \multicolumn{4}{c|}{(p \times p)} & (p \times (n-p)) \\ 1 & b_{12} & \cdots & b_{1p} & \\ 0 & 1 & \cdots & b_{2p} & \mathbf{B}_{12} \\ \vdots & \vdots & & \vdots & \\ 0 & 0 & \cdots & 1 & \\ \hline \multicolumn{4}{c|}{\mathbf{0}} & \mathbf{0} \\ \multicolumn{4}{c|}{((m-p) \times p)} & ((m-p) \times (n-p)) \end{array} \right] \tag{1-61}$$

Example 1–14

$$\mathbf{a} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 2 \\ 5 & 7 & 12 & 12 \end{bmatrix}$$

First, we eliminate a_{21} and a_{31}, using the first row:

$$\mathbf{a}^{(1)} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -3 & -6 \\ 0 & -3 & -3 & -8 \end{bmatrix}$$

* See Prob. 1–40.

Next, we eliminate $a_{32}^{(1)}$, using the second row:

$$\mathbf{a}^{(2)} = \begin{bmatrix} -1 & 2 & 3 & 4 \\ 0 & -3 & -3 & -6 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

At this point, we see that $r = 3$. To obtain $\mathbf{b}$, we multiply the second row by $-1/3$, the third row by $-1/2$, and interchange the third and fourth columns:

$$\mathbf{b} = \begin{bmatrix} 1 & 2 & 4 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Suppose $\mathbf{a}$ is expressed as the product of two rectangular matrices:

$$\underset{(m \times s)}{\mathbf{a}} = \underset{(m \times n)}{\mathbf{b}} \; \underset{(n \times s)}{\mathbf{c}} \tag{1-62}$$

One can show* that the rank of $\mathbf{a}$ cannot be greater than the minimum value of r associated with $\mathbf{b}$ and $\mathbf{c}$:

$$r(\mathbf{a}) \leqslant \min\,[r(\mathbf{b}), r(\mathbf{c})] \tag{1-63}$$

As an illustration, consider the product

$$\mathbf{a} = \begin{bmatrix} -1/2 & +1/2 & 0 \\ -1/2 & +1/2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Since each matrix is of rank 2, the rank of $\mathbf{a}$ will be $\leqslant 2$. Evaluating the product, we obtain

$$\mathbf{a} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

It follows that $\mathbf{a}$ is of rank 1.

1–13. SOLVABILITY OF LINEAR ALGEBRAIC EQUATIONS

We consider first a system of two equations in three unknowns:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} \tag{1-64}$$

$$\mathbf{ax} = \mathbf{c}$$

Suppose $\mathbf{a}$ is of rank 2 and

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0 \tag{1-65}$$

* See Prob. 1–44.

If $\mathbf{a}$ is of rank 2, we can always renumber the rows and columns such that (1–65) is satisfied. We partition $\mathbf{a}$ and $\mathbf{x}$,

$$\mathbf{a} = \left[\begin{array}{cc|c} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{array}\right] = [\mathbf{A}_1 \quad \mathbf{A}_2] \tag{1-66}$$

$$\mathbf{x} = \left\{\begin{array}{c} x_1 \\ x_2 \\ \hline x_3 \end{array}\right\} = \begin{Bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{Bmatrix}$$

and write (1–64) as $\mathbf{A}_1\mathbf{X}_1 + \mathbf{A}_2\mathbf{X}_2 = \mathbf{c}$. Next, we transfer the term involving $\mathbf{X}_2$ to the right-hand side:

$$\mathbf{A}_1\mathbf{X}_1 = \mathbf{c} - \mathbf{A}_2\mathbf{X}_2 \tag{1-67}$$

Since $|\mathbf{A}_1| \neq 0$, it follows from Cramer's rule that (1–67) has a unique solution for $\mathbf{X}_1$. Finally, we can write the solution as

$$\mathbf{X}_1 = \mathbf{A}_1^{-1}(\mathbf{c} - \mathbf{A}_2\mathbf{X}_2) \tag{1-68}$$

Since $\mathbf{X}_2$ is arbitrary, the system does not have a unique solution for a given $\mathbf{c}$. The order of $\mathbf{X}_2$ is generally called the *defect of the system*. The defect for this system is 1.

If $\mathbf{a}$ is of rank 1, the second row is a scalar multiple, say λ, of the first row. Multiplying the second equation in (1–64) by $1/\lambda$, we have

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= c_1 \\ a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= c_2/\lambda \end{aligned} \tag{1-69}$$

If $c_2 \neq \lambda c_1$, the equations are inconsistent and no solution exists. Then, when $\mathbf{a}$ is of rank 1, (1–64) has a solution only if the rows of $\mathbf{c}$ are related in the same manner as the rows of $\mathbf{a}$. If this condition is satisfied, the two equations in (1–69) are identical and one can be disregarded. Assuming that $a_{11} \neq 0$, the solution is

$$x_1 = (1/a_{11})(c_1 - a_{12}x_2 - a_{13}x_3) \tag{1-70}$$

The defect of this system is 2.

The procedure followed for the simple case of 2 equations in 3 unknowns is also applicable to the general case of m equations in n unknowns:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} = \begin{Bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{Bmatrix} \tag{1-71}$$

If $\mathbf{a}$ is of rank m, there exists an mth order array which has a nonvanishing determinant. We rearrange the columns such that the first m columns are linearly independent. Partitioning **a** and **x**,

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} & | & a_{1,m+1} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2m} & | & a_{2,m+1} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & | & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} & | & a_{m,m+1} & \cdots & a_{mn} \end{bmatrix} = [\underset{(m\times m)}{\mathbf{A}_1} \quad \underset{(m\times(n-m))}{\mathbf{A}_2}] \tag{1-72}$$

$$\{x_1 \quad x_2 \quad \cdots \quad x_m \mid x_{m+1} \quad \cdots \quad x_n\} = \{\underset{(m\times 1)}{\mathbf{X}_1} \quad \underset{((n-m)\times 1)}{\mathbf{X}_2}\}$$

we write (1–71) as

$$\mathbf{A}_1\mathbf{X}_1 = \mathbf{c} - \mathbf{A}_2\mathbf{X}_2 \tag{1-73}$$

Since $|\mathbf{A}_1| \neq 0$, (1–73) can be solved for $\mathbf{X}_1$ in terms of **c** and $\mathbf{X}_2$. The defect of the set is $n - m$, that is, the solution involves $n - m$ arbitrary constants represented by $\mathbf{X}_2$.

Suppose **a** is of rank r where $r < m$. Then, **a** has r rows which contain an rth-order array having a nonvanishing determinant. The remaining $m - r$ rows are linear combinations of these r rows. For (1–71) to be consistent, that is, have a solution, the relations between the rows of **c** must be the *same* as those for **a**. The defect for this case is $n - r$.

Example 1–15

As an illustration, consider the third-order system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= c_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= c_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= c_3 \end{aligned} \tag{a}$$

Suppose that $r = 2$ and the rows of **a** are related by

$$(\text{third row}) = \lambda_1\,(\text{first row}) + \lambda_2\,(\text{second row}) \tag{b}$$

For (a) to be consistent, the elements of **c** must satisfy the requirement,

$$c_3 = \lambda_1 c_1 + \lambda_2 c_2 \tag{c}$$

To show this, we multiply the first equation in (a) by $-\lambda_1$, the second by $-\lambda_2$, and add to these equations the third equation. Using (b), we obtain

$$0 = c_3 - \lambda_1 c_1 - \lambda_2 c_2 \tag{d}$$

Unless the right-hand side vanishes, the equations are contradictory or inconsistent and no solution exists. When $\mathbf{c} = \mathbf{0}$, (c) is identically satisfied and we see that (a) has a nontrivial solution ($\mathbf{x} \neq 0$) only when $r < 3$. The general case is handled in the same manner.*

* See Prob. 1–45.

In general, (1–71) can be solved when $r < m$ if the relations between the rows of **a** and **c** are *identical*. We define the augmented matrix, $\boldsymbol{\alpha}$, for (1–71) as

$$\boldsymbol{\alpha} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & | & c_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & | & c_2 \\ \vdots & \vdots & & \vdots & | & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & | & c_m \end{bmatrix} = [\mathbf{a} \quad \mathbf{c}] \tag{1-74}$$

When the rows of **a** and **c** are related in the *same* way, the rank of $\boldsymbol{\alpha}$ is equal to the rank of **a**. It follows that (1–71) has a solution only if the rank of the augmented matrix is *equal* to the rank of the coefficient matrix:

$$r(\boldsymbol{\alpha}) = r(\mathbf{a}) \tag{1-75}$$

Note that (1–75) is always satisfied when $r(\mathbf{a}) = m$ for *arbitrary* **c**.

We can determine $r(\boldsymbol{\alpha})$ and $r(\mathbf{a})$ simultaneously using elementary operations on $\boldsymbol{\alpha}$ provided that we do not interchange the elements in the last column. The reduction can be represented as

$$\boldsymbol{\alpha} = [\mathbf{a} \quad \mathbf{c}] \Rightarrow \left[\begin{array}{c|c} \mathbf{A}_1^{(1)} & \mathbf{C}_1^{(1)} \\ \hline \mathbf{0} & \mathbf{C}_2^{(1)} \end{array}\right] \tag{1-76}$$

where $\mathbf{A}_1^{(1)}$ is of rank $r(\mathbf{a})$. If $\mathbf{C}_2^{(1)}$ has a nonvanishing element, $r(\boldsymbol{\alpha}) > r(\mathbf{a})$ and no solution exists.

When $r(\boldsymbol{\alpha}) = r(\mathbf{a})$, (1–71) contains r independent equations involving n unknowns. The remaining $m - r$ equations are linear combinations of these r equations and can be disregarded. Thus, the problem reduces to first finding $r(\boldsymbol{\alpha})$ and then solving a set of r independent equations in n unknowns. The complete problem can be efficiently handled by using the Gauss elimination procedure (Refs. 9, 11, 13).

REFERENCES

1. FRAZER, R. A., W. J. DUNCAN and A. R. COLLAR: *Elementary Matrices*, Cambridge University Press, London, 1963.
2. THOMAS, G. B., JR.: *Calculus and Analytical Geometry*, Addison-Wesley Publishing Co., Reading, Mass., 1953.
3. BODEWIG, E.: *Matrix Calculus*, Interscience Publishers, New York, 1956.
4. HOHN, F. E.: *Elementary Matrix Algebra*, Macmillan Co., New York, 1958.
5. HADLEY, G.: *Linear Algebra*, Addison-Wesley Publishing Co., Reading, Mass., 1961.
6. HOUSEHOLDER, A. S.: *The Theory of Matrices in Numerical Analysis*, Blaisdell, Waltham, Mass., 1964.
7. NOBLE, B.: *Applied Linear Algebra*, Prentice-Hall, New York, 1969.
8. HIL DEBRAND, F. B.: *Methods of Applied Mathematics*, Prentice-Hall, New York, 1952.
9. Faddeeva, V. N.: *Computational Methods of Linear Algebra*, Dover Publications, New York, 1959.

10. RALSTON, A. and H. S. WILF: *Mathematical Methods for Digital Computers*, Vol. 1, Wiley, New York, 1960.
11. RALSTON, A. and H. S. WILF: *Mathematical Methods for Digital Computers*, Vol. 2, Wiley, New York, 1967.
12. BEREZIN, I. S. and N. P. ZHIDKOV: *Computing Methods*, Vols. 1 and 2, Addison-Wesley Publishing Co., Reading, Mass., 1965.
13. FORSYTHE, G. E., and C. B. MALER: *Computer Solution of Linear Algebraic Systems*, Prentice-Hall, New York, 1967.
14. VARGA, R. S.: *Matrix Iterative Analysis*, Prentice-Hall, New York, 1962.
15. CONTE, S. D.: *Elementary Numerical Analysis*, McGraw-Hill, New York, 1965.

PROBLEMS

1–1. Carry out the indicated operations:

(a)

$$\begin{bmatrix} 1 & 4 & 0 \\ 3 & 2 & 1 \\ 5 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 3 & 5 \\ 7 & 1 & 3 \\ 0 & 5 & 6 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 2 & 7 & 3 \\ 5 & 1 & 6 \end{bmatrix} - \begin{bmatrix} -4 & 1 & 5 \\ 3 & -1 & 2 \end{bmatrix}$$

(c)

$$3\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + 2\begin{bmatrix} -1 & 1 \\ 0 & 3 \end{bmatrix} - 4\begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}$$

(d)

$$\begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix} \begin{Bmatrix} 2 \\ 5 \end{Bmatrix}$$

(e)

$$\begin{bmatrix} -1 & 1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$$

(f)

$$\begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 2 & -3 \end{bmatrix}$$

1–2. Expand the following products:

(a)

$$[a_1, a_2, \ldots, a_n] \{b_1, b_2, \ldots, b_n\}$$

(b)

$$\{a_1, a_2, \ldots, a_n\} [b_1, b_2, \ldots, b_n]$$

(c)

$$\begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

(d)

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix}$$

1–3. Show that the product of

$$S_1 = a_1 + a_2 + a_3 = \sum_{k=1}^{3} a_k$$

$$S_2 = b_1 + b_2 + b_3 = \sum_{k=1}^{3} b_k$$

can be written as

$$S_1 S_2 = \sum_{k=1}^{3} \sum_{j=1}^{3} a_k b_j$$

Generalize this result for the sum of n elements.

1–4. Suppose the elements of **a** and **b** are functions of y. Let

$$\frac{d\mathbf{a}}{dy} = \left[\frac{da_{ik}}{dy}\right] \qquad \frac{d\mathbf{b}}{dy} = \left[\frac{db_{ik}}{dy}\right]$$

Using (1–19), show that if

$$\mathbf{c} = \mathbf{ab}$$

then

$$\frac{d\mathbf{c}}{dy} = \mathbf{a}\frac{d\mathbf{b}}{dy} + \frac{d\mathbf{a}}{dy}\mathbf{b}$$

1–5. Consider the triple product, **abc**. When is this product defined? Let

$$\mathbf{p} = \mathbf{abc}$$

Determine an expression for p_{ij}. What is the order of **p**? Determine p_{ij} for the case where $\mathbf{c} = \mathbf{a}^T$.

1–6. Evaluate the following products:

(a)

$$\begin{bmatrix} 3 & 4 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ 4 & 1 \end{bmatrix}$$

(b)

$(\mathbf{a} + \mathbf{b})(\mathbf{a} + \mathbf{b})$ where **a** is a square matrix.

1–7. Show that the product of two symmetrical matrices is symmetrical only when they are commutative.

1–8. Show that the following products are symmetrical:

(a)

$\mathbf{a}^T\mathbf{a}$

(b)

$\mathbf{a}^T\mathbf{ba}$ where **b** is symmetrical

(c)

$\mathbf{b}^T\mathbf{a}^T\mathbf{cab}$ where **c** is symmetrical

1–9. Evaluate the following matrix product, using the indicated submatrices:

$$\left[\begin{array}{cc|c} 1 & 3 & 2 \\ 2 & 1 & 3 \\ \hline 4 & 1 & 2 \end{array}\right]\left[\begin{array}{cc} 2 & 1 \\ 1 & 3 \\ \hline 5 & 1 \end{array}\right]$$

1–10. Let $\mathbf{c} = \mathbf{ab}$. Show that the horizontal partitions of $\mathbf{c}$ correspond to those of $\mathbf{a}$ and the vertical partitions of $\mathbf{c}$ correspond to those of $\mathbf{b}$. *Hint*: See Eq. (1–37).

1–11. A matrix is said to be symmetrically partitioned if the locations of the row and column partitions coincide. For example,

$$\left[\begin{array}{cc|c} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \hline a_{31} & a_{32} & a_{33} \end{array}\right]$$

is symmetrically partitioned and

$$\left[\begin{array}{c|c|c} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \hline a_{31} & a_{32} & a_{33} \end{array}\right]$$

is unsymmetrically partitioned. Suppose we partition a *square* matrix with $N - 1$ *symmetrical* partitions.

$$\mathbf{a} = [\mathbf{A}_{ij}] \qquad i, j = 1, 2, \ldots, N$$

(a) Deduce that the diagonal submatrices are square and $\mathbf{A}_{rs}$, $\mathbf{A}_{sr}^T$ have the same order.

(b) If $\mathbf{a} = \mathbf{a}^T$, deduce that

$$\mathbf{A}_{rs} = \mathbf{A}_{sr}^T$$

1–12. Consider the product of two square nth order matrices.

$$\mathbf{c} = \mathbf{ab}$$

(a) If $\mathbf{a}$ and $\mathbf{b}$ are symmetrically partitioned, show that $\mathbf{C}_{jk}$, $\mathbf{A}_{jk}$, $\mathbf{B}_{jk}$ are of the same order. Illustrate for the case of one partition, e.g.,

$$\mathbf{a} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}$$

(b) Suppose we symmetrically partition $\mathbf{c}$. What restrictions are placed on the partitions of $\mathbf{a}$ and $\mathbf{b}$? Does it follow that we must also partition $\mathbf{a}$ and $\mathbf{b}$ symmetrically? *Hint*: See Prob. 1–10.

1–13. Consider the triple product,

$$\mathbf{c} = \mathbf{a}^T\mathbf{ba}$$

where $\mathbf{b}$ is a symmetric rth-order square matrix and $\mathbf{a}$ is of order $r \times n$. Suppose we symmetrically partition $\mathbf{c}$. The order of the partitioned matrices are indicated in parentheses.

$$\underset{(n\times n)}{\mathbf{c}} = \begin{bmatrix} \overset{(p\times p)}{\mathbf{C}_{11}} & \overset{(p\times q)}{\mathbf{C}_{12}} \\ \underset{(q\times p)}{\mathbf{C}_{21}} & \underset{(q\times q)}{\mathbf{C}_{22}} \end{bmatrix}$$

(a) Show that the following partitioning of $\mathbf{a}$ is consistent with that of $\mathbf{c}$.

$$\underset{(r\times n)}{\mathbf{a}} = [\underset{(r\times p)}{\mathbf{A}_1} \quad \underset{(r\times q)}{\mathbf{A}_2}]$$

(b) Express $\mathbf{C}_{jk}$ $(j, k = 1, 2)$ in terms of $\mathbf{A}_1$, $\mathbf{A}_2$, and $\mathbf{b}$.

1–14. Let $\mathbf{d} = [\mathbf{D}_j]$ be a quasi-diagonal matrix. Show that

$$\mathbf{da} = [\mathbf{D}_j\mathbf{A}_{jk}]$$
$$\mathbf{bd} = [\mathbf{B}_{jk}\mathbf{D}_k]$$

when the matrices are conformably partitioned.

1–15. Determine the number of inversions and interchanges for the following sets.

(a) $(4, 3, 1, 2)$

(b) $(3, 4, 2, 1)$

1–16. How many permutations does $(1, 2, 3, 4, 5)$ have?

1–17. Consider the terms

$$e_{\alpha_1\alpha_2\alpha_3} a_{1\alpha_1} a_{2\alpha_2} a_{3\alpha_3} \qquad \text{(a)}$$

and

$$e_{\beta_1\beta_2\beta_3} a_{\beta_1 1} a_{\beta_2 2} a_{\beta_3 3} \qquad \text{(b)}$$

The first subscripts in (a) are in natural order. We obtain (b) by rearranging (a) such that the second subscripts are in natural order. For example, rearranging

$$e_{231} a_{12} a_{23} a_{31} \qquad \alpha = (2, 3, 1)$$

we obtain

$$e_{312} a_{31} a_{12} a_{23} \qquad \beta = (3, 1, 2)$$

Show that if $(\alpha_1, \alpha_2, \alpha_3)$ is an even permutation, $(\beta_1, \beta_2, \beta_3)$ is also an even permutation. Using this result, show that

$$\sum_{\alpha} e_{\alpha_1\alpha_2\alpha_3} a_{1\alpha_1} a_{2\alpha_2} a_{3\alpha_3} = \sum_{\beta} e_{\beta_1\beta_2\beta_3} a_{\beta_1 1} a_{\beta_2 2} a_{\beta_3 3}$$

and, in general,

$$|\mathbf{a}| = |\mathbf{a}^T|$$

1–18. Consider the terms

$$e_{\alpha_1\alpha_2\alpha_3} a_{1\alpha_1} a_{2\alpha_2} a_{3\alpha_3} \qquad \text{(a)}$$
$$e_{\alpha_1\alpha_2\alpha_3} b_{1\alpha_1} b_{2\alpha_2} b_{3\alpha_3} \qquad \text{(b)}$$

Suppose that

$$b_{1\alpha_1} = a_{2\alpha_1} \qquad b_{2\alpha_2} = a_{1\alpha_2} \qquad b_{3\alpha_3} = a_{3\alpha_3}$$

Then, (b) takes the form

$$e_{\alpha_1\alpha_2\alpha_3} a_{2\alpha_1} a_{1\alpha_2} a_{3\alpha_3} \qquad \text{(c)}$$

Show that

$$(c) = -(a)$$

Generalize this result and establish that the sign of a determinant is reversed when two rows are interchanged.

1–19. Consider the third-order determinant

$$|\mathbf{a}| = \sum e_{\alpha_1\alpha_2\alpha_3} a_{1\alpha_1} a_{2\alpha_2} a_{3\alpha_3}$$

Suppose the second row is a multiple of the first row:

$$\alpha_{2j} = k\alpha_{1j}$$

Show that $|\ | = 0$. (*Hint*: $e_{\alpha_2\alpha_1\alpha_3} = -e_{\alpha_1\alpha_2\alpha_3}$). Generalize this result and establish properties 5 and 7 of Sec. 1–7.

1–20. Suppose all the integers of a set are in natural order except for one integer, say n, which is located at position p. We can put the set in natural order by successively interchanging adjacent integers. For example,

$$\begin{matrix} 3 & 1 & 2 & \rightarrow & 1 & 3 & 2 & \rightarrow & 1 & 2 & 3 \\ 2 & 3 & 1 & \rightarrow & 2 & 1 & 3 & \rightarrow & 1 & 2 & 3 \end{matrix}$$

Show that $|n - p|$ adjacent interchanges (called transpositions) are required. It follows that the sign of the resulting set is changed by

$$(-1)^{|n-p|}$$

1–21. We can write the expansion for the third-order determinant as

$$\sum_{i=1}^{3} a_{1i}\left(\sum_j \sum_k e_{ijk} a_{2j} a_{3k}\right) \qquad \begin{matrix} i \neq j \neq k \\ i, j, k = 1, 2, 3 \end{matrix}$$

Using the result of the previous problem,

$$e_{ijk} = (-1)^{|i-1|} e_{jk} = (-1)^{i+1} e_{jk} \tag{b}$$

and (a) reduces to

$$\sum_{i=1}^{3} a_{1i}(-1)^{i+1} M_{1i} = \sum_{i=1}^{3} a_{1i} A_{1i} \tag{c}$$

Following this approach, establish Laplace's cofactor expansion formula for an nth-order determinant.

1–22. Use Laplace's expansion formula to show that

$$\left|\begin{array}{cccc|cccc} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ \hline b_{11} & b_{12} & \cdots & b_{1p} & a_{11} & a_{12} & \cdots & a_{1n} \\ b_{21} & b_{22} & \cdots & b_{2p} & a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} & a_{n1} & a_{n2} & \cdots & a_{nn} \end{array}\right| = \left|\begin{array}{c|c} \underset{(p\times p)}{\mathbf{I}} & \underset{(p\times n)}{\mathbf{0}} \\ \hline \underset{(n\times p)}{\mathbf{b}} & \underset{(n\times n)}{\mathbf{a}} \end{array}\right| = |\mathbf{a}|$$

1–23. Consider the quasi-diagonal matrix,

$$\mathbf{d} = \begin{bmatrix} \underset{(p\times p)}{\mathbf{D}_1} & \underset{(p\times q)}{\mathbf{0}} \\ \underset{(q\times p)}{\mathbf{0}} & \underset{(q\times q)}{\mathbf{D}_2} \end{bmatrix}$$

By expressing $\mathbf{d}$ as

$$\mathbf{d} = \begin{bmatrix} \mathbf{D}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_q \end{bmatrix} \begin{bmatrix} \mathbf{I}_p & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_2 \end{bmatrix}$$

show that $|\mathbf{d}| = |\mathbf{D}_1|\,|\mathbf{D}_2|$. Verify this result for

$$\mathbf{d} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 5 & 3 \end{bmatrix}$$

Generalize for

$$\mathbf{d} = [\mathbf{A}_i\,\delta_{ij}]$$

1–24. Let

$$\mathbf{g} = \begin{bmatrix} \underset{(p\times p)}{\mathbf{G}_{11}} & \underset{(p\times q)}{\mathbf{0}} \\ \underset{(q\times p)}{\mathbf{G}_{21}} & \underset{(q\times q)}{\mathbf{G}_{22}} \end{bmatrix}$$

Show that

$$|\mathbf{g}| = |\mathbf{G}_{11}|\,|\mathbf{G}_{22}|$$

Generalize for a quasi-triangular matrix whose diagonal submatrices are square, of various orders.

1–25. Suppose we express $\mathbf{a}$ as the product of a lower triangular matrix, $\mathbf{g}$, and an upper triangular matrix, $\mathbf{b}$.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} g_{11} & 0 & \cdots & 0 \\ g_{21} & g_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ g_{n1} & g_{n2} & \cdots & g_{nn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ 0 & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & b_{nn} \end{bmatrix}$$

We introduce symmetrical partitions after row (and column) p and write the product as

$$\begin{bmatrix} \underset{(p\times p)}{\mathbf{A}_{11}} & \underset{(p\times q)}{\mathbf{A}_{12}} \\ \underset{(q\times p)}{\mathbf{A}_{21}} & \underset{(q\times q)}{\mathbf{A}_{22}} \end{bmatrix} = \begin{bmatrix} \underset{(p\times p)}{\mathbf{G}_{11}} & \underset{(p\times q)}{\mathbf{0}} \\ \underset{(q\times p)}{\mathbf{G}_{21}} & \underset{(q\times q)}{\mathbf{G}_{22}} \end{bmatrix} \begin{bmatrix} \underset{(p\times p)}{\mathbf{B}_{11}} & \underset{(p\times q)}{\mathbf{B}_{12}} \\ \underset{(q\times p)}{\mathbf{0}} & \underset{(q\times q)}{\mathbf{B}_{22}} \end{bmatrix}$$

Note that the diagonal submatrices of $\mathbf{g}$ and $\mathbf{b}$ are triangular in form.

(a) Show that

$$\begin{aligned} \mathbf{G}_{11}\mathbf{B}_{11} &= \mathbf{A}_{11} \\ \mathbf{G}_{11}\mathbf{B}_{12} &= \mathbf{A}_{12} \\ \mathbf{G}_{21}\mathbf{B}_{11} &= \mathbf{A}_{21} \\ \mathbf{G}_{21}\mathbf{B}_{12} + \mathbf{G}_{22}\mathbf{B}_{22} &= \mathbf{A}_{22} \end{aligned}$$

(b) Show that

$$|\mathbf{A}_{11}| = |\mathbf{G}_{11}|\,|\mathbf{B}_{11}|$$

and

$$|\mathbf{a}| = |\mathbf{G}_{11}|\,|\mathbf{G}_{22}|\,|\mathbf{B}_{11}|\,|\mathbf{B}_{22}|$$

(c) Suppose we require that

$$|\mathbf{g}| \neq 0 \qquad |\mathbf{b}| \neq 0$$

By taking $p = 1, 2, \ldots, n - 1$, deduce that this requirement leads to the following n conditions on the elements of $\mathbf{a}$:

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1j} \\ a_{21} & a_{22} & \cdots & a_{2j} \\ \vdots & \vdots & & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jj} \end{vmatrix} \neq 0 \qquad j = 1, 2, \ldots n$$

The determinant of the array contained in the first j rows and columns is called the jth-order discriminant.

1–26. Does the following set of equations have a unique solution?

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 3 & 7 & 11 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 3 \\ 5 \end{Bmatrix}$$

1–27. Determine the adjoint matrix for

$$\mathbf{a} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 3 & 7 & 11 \end{bmatrix}$$

Does $\mathbf{a}^{-1}$ exist?

1–28. Show that $\mathbf{b}^{-1,T} = \mathbf{b}^{T,-1}$

1–29. Find the inverse of

(a)
$$\begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix}$$

(b)
$$\begin{bmatrix} 2 & 4 \\ 1 & 5 \end{bmatrix}$$

(c)
$$\begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 1 & 5 \end{bmatrix}$$

(d)
$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 1 & 5 \end{bmatrix}$$

Let

$$\mathbf{a} = \left[\begin{array}{cc|c} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \hline a_{31} & a_{32} & a_{33} \end{array}\right] = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}$$

and

$$\mathbf{a}^{-1} = \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{bmatrix}$$

where the order of $\mathbf{B}_{jk}$ is the same as $\mathbf{A}_{jk}$. Starting with the condition

$$\mathbf{a}\mathbf{a}^{-1} = \mathbf{I}_3$$

determine the four matrix equations relating $\mathbf{B}_{jk}$ and $\mathbf{A}_{jk}$ $(j, k = 1, 2)$. Use this result to find the inverse of

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

1–31. Find the inverse of

$$\begin{bmatrix} \mathbf{I}_p & \mathbf{A} \\ \mathbf{0} & \mathbf{I}_q \end{bmatrix}$$

Note that $\mathbf{A}$ is $(p \times q)$.

1–32. Find the inverse of

$$\mathbf{d} = \begin{bmatrix} \mathbf{D}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_2 \end{bmatrix}$$

1–33. Use the results of Probs. 1–31 and 1–32 to find the inverse of

$$\mathbf{b} = \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{0} & \mathbf{B}_{22} \end{bmatrix}$$

where $\mathbf{B}_{11}$, $\mathbf{B}_{22}$ are square and nonsingular. *Hint*: write $\mathbf{b}$ as

$$\mathbf{b} = \begin{bmatrix} \mathbf{B}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{B}_{11}^{-1}\mathbf{B}_{12}\mathbf{B}_{22}^{-1} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_{22} \end{bmatrix}$$

1–34. Consider the 3×4 matrix

$$\mathbf{a} = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 3 & 2 & 2 \\ 1 & 2 & 1 & 1 \end{bmatrix}$$

Determine the elementary row operation matrix which results in $a_{21} = a_{31} = a_{32} = 0$ and $a_{11} = a_{22} = a_{33} = +1$.

1–35. Let

$$\underset{(n \times n)}{\mathbf{a}} = \begin{bmatrix} \underset{(p \times p)}{\mathbf{A}_{11}} & \underset{(p \times q)}{\mathbf{A}_{12}} \\ \underset{(q \times p)}{\mathbf{A}_{21}} & \underset{(q \times q)}{\mathbf{A}_{22}} \end{bmatrix}$$

where $|\mathbf{A}_{11}| \neq 0$ and $|\mathbf{a}| \neq 0$. Show that the following elementary operations on the partitioned rows of $\mathbf{a}$ reduce $\mathbf{a}$ to a triangular matrix. Determine $\mathbf{A}_{12}^{(1)}$.

$$\begin{bmatrix} \mathbf{I}_p & \mathbf{0} \\ \mathbf{0} & \mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{I}_p & \mathbf{0} \\ -\mathbf{A}_{21} & \mathbf{I}_q \end{bmatrix} \begin{bmatrix} \mathbf{A}_{11}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_q \end{bmatrix} \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{I}_p & \mathbf{A}_{12}^{(1)} \\ \mathbf{0} & \mathbf{I}_q \end{bmatrix}$$

$$\mathbf{C} = (\mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12})^{-1}$$

1–36. Suppose we want to rearrange the columns of $\mathbf{a}$ in the following way:

$$\mathbf{a} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 4 & 5 \end{bmatrix} \qquad \begin{array}{l} \text{col } 1 \rightarrow \text{col } 3 \\ \text{col } 2 \rightarrow \text{col } 1 \\ \text{col } 3 \rightarrow \text{col } 2 \end{array}$$

(a) Show that *postmultiplication* by $\mathbf{\Pi}$ (which is called a permutation matrix) results in the desired column rearrangement:

$$\mathbf{\Pi} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Note that we just rearrange the corresponding columns of $\mathbf{I}_3$.

(b) Show that *premultiplication* by $\mathbf{\Pi}^T$ rearranges the rows of $\mathbf{a}$ in the *same* way.

(c) Show that $\mathbf{\Pi}^T\mathbf{\Pi} = \mathbf{I}_3$.

(d) Generalize for the case where $\mathbf{a}$ is $n \times n$.

(e) Show that

$$|\mathbf{\Pi}^T\mathbf{a}\,\mathbf{\Pi}| = |\mathbf{a}|$$

1–37. Let $\mathbf{a}$ be of order $2 \times n$, where $n \geqslant 2$. Show that $\mathbf{a}$ is of rank 1 when the second row is a multiple of the first row. Also, show that when $r = 1$, the second, third, ..., nth columns are multiples of the first column.

1–38. Determine the rank of

(a)

$$\begin{bmatrix} 1 & 3 & 7 \\ 5 & 2 & 4 \\ 3 & -4 & -10 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & 4 & 6 & -2 \\ -1 & -2 & -3 & 1 \end{bmatrix}$$

1–39. Let $\mathbf{a}$ be of order $m \times n$ and rank r. Show that $\mathbf{a}$ has $n - r$ columns which are linear combinations of r linear independent columns. Verify for

$$\mathbf{a} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 2 \\ 5 & 7 & 12 & 14 \end{bmatrix}$$

1–40. Using properties 3, 4, and 7 of determinants (see Sec. 1–7), deduce that the elementary operations do not change the rank of a matrix. For convenience, consider the first r rows and columns to be linearly independent.

1–41. Find the rank of $\mathbf{a}$ by reducing it to an echelon matrix.

$$\mathbf{a} = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 1 & 3 & 2 \\ 3 & 4 & 2 & 1 \\ 7 & 7 & 9 & 7 \end{bmatrix}$$

1–42. Show that $\mathbf{c}$ is at most of rank 1.

$$\mathbf{c} = \begin{Bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{Bmatrix} [b_1 b_2 \cdots b_n]$$

When will $r(\mathbf{c}) = 0$?

1–43. Consider the product,

$$\mathbf{c} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ \vdots & \vdots \\ a_{m1} & a_{m2} \end{bmatrix} \begin{bmatrix} b_{11} b_{12} \cdots b_{1n} \\ b_{21} b_{22} \cdots b_{2n} \end{bmatrix}$$

(a) Suppose $\mathbf{b}$ is of rank 1 and $b_{11} \neq 0$. Then, we can write

$$\begin{Bmatrix} b_{1k} \\ b_{2k} \end{Bmatrix} = \alpha_k \begin{Bmatrix} b_{11} \\ b_{21} \end{Bmatrix} \qquad k = 2, 3, \ldots, n$$

Show that the second, third, ..., nth columns of $\mathbf{c}$ are multiples of the first column and therefore $r(\mathbf{c}) \leqslant 1$. When will $r(\mathbf{c}) = 0$?

(b) Suppose $r(\mathbf{a}) = 1$ and $a_{11} \neq 0$. In this case, we can write

$$[a_{j1} a_{j2}] = \lambda_j [a_{11} a_{12}] \qquad j = 2, 3, \ldots, m$$

Show that the second, third, ..., mth rows of $\mathbf{c}$ are multiples of the first row and therefore $r(\mathbf{c}) \leqslant 1$. When will $r(\mathbf{c}) = 0$?

1–44. Consider the product

$$\mathbf{c} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1s} \\ a_{21} & a_{22} & \cdots & a_{2s} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{ms} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{s1} & b_{s2} & \cdots & b_{sn} \end{bmatrix} \qquad \text{(a)}$$

Let

$$\mathbf{A}_j = [a_{j1} a_{j2} \cdots a_{js}]$$
$$\mathbf{B}_k = \{b_{1k} b_{2k} \cdots b_{sk}\} \qquad \text{(b)}$$

Using (b), we can write (a) as

$$\mathbf{c} = \begin{Bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ \vdots \\ \mathbf{A}_m \end{Bmatrix} [\mathbf{B}_1 \mathbf{B}_2 \cdots \mathbf{B}_n] \qquad \text{(c)}$$

Suppose $r(\mathbf{a}) = r_a$, $r(\mathbf{b}) = r_b$. For convenience, we assume the first r_a rows of $\mathbf{a}$ and the first r_b columns of $\mathbf{b}$ are linearly independent. Then,

$$\mathbf{A}_j = \sum_{p=1}^{r_a} \lambda_{jp} \mathbf{A}_p \qquad j = r_a + 1, r_a + 2, \ldots, m$$

$$\mathbf{B}_k = \sum_{q=1}^{r_b} \alpha_{kq}\mathbf{B}_q \qquad k = r_b + 1, r_b + 2, \ldots, n$$

(a) Show that rows $r_a + 1, r_a + 2, \ldots, m$ of **c** are linear combinations of the first r_a rows.
(b) Show that columns $r_b + 1, r_b + 2, \ldots, n$ of **c** are linear combinations of the first r_b columns.
(c) From (a) and (b), what can you conclude about an upper bound on $r(\mathbf{c})$?
(d) To determine the actual rank of **c**, we must find the rank of

$$\begin{Bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ \vdots \\ \mathbf{A}_{r_a} \end{Bmatrix} [\mathbf{B}_1\mathbf{B}_2 \cdots \mathbf{B}_{r_b}] = \begin{bmatrix} \mathbf{A}_1\mathbf{B}_1 & \mathbf{A}_1\mathbf{B}_2 & \cdots & \mathbf{A}_1\mathbf{B}_{r_b} \\ \mathbf{A}_2\mathbf{B}_1 & \mathbf{A}_2\mathbf{B}_2 & \cdots & \mathbf{A}_2\mathbf{B}_{r_b} \\ \vdots & \vdots & & \vdots \\ \mathbf{A}_{r_a}\mathbf{B}_1 & \mathbf{A}_{r_a}\mathbf{B}_2 & \cdots & \mathbf{A}_{r_a}\mathbf{B}_{r_b} \end{bmatrix}$$

Suppose $r_a \leqslant r_b$. What can you conclude about $r(\mathbf{c})$ if $\mathbf{A}_1^T$ is orthogonal to $\mathbf{B}_1, \mathbf{B}_2, \ldots, \mathbf{B}_{r_b}$?
(e) Utilize these results to find the rank of

$$\begin{bmatrix} -1/2 & 1/2 & 0 \\ -1/2 & 1/2 & 1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

(f) Suppose $r_a = r_b = s$. Show that $r(\mathbf{c}) = s$. Verify for

$$\begin{bmatrix} 1 & 3 \\ 2 & -2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & 4 \end{bmatrix}$$

1–45. Consider the $m \times n$ system

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} = \begin{Bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{Bmatrix} \qquad \text{(a)}$$

Let

$$\mathbf{A}_j = [a_{j1}a_{j2} \cdots a_{jn}] \qquad j = 1, 2, \ldots, m \qquad \text{(b)}$$

Using (b), we write (a) as

$$\mathbf{A}_j\mathbf{x} = c_j \qquad j = 1, 2, \ldots, m \qquad \text{(c)}$$

Now, suppose **a** is of rank r and the first r rows are linearly independent. Then,

$$\mathbf{A}_k = \sum_{p=1}^{r} \lambda_{kp}\mathbf{A}_p \qquad k = r + 1, r + 2, \ldots, m$$

(a) Show that the system is consistent only if

$$c_k = \sum_{p=1}^{r} \lambda_{kp}c_p \qquad k = r + 1, r + 2, \ldots, m$$

Note that this requirement is *independent* of whether $m < n$ or $m > n$.

(b) If $m < n$ and $r = m$, the equations are consistent for an arbibrary **c**. Is this also true when $m > n$ and $r = n$? Illustrate for

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix}$$

and

$$\begin{bmatrix} 1 & 1 \\ -1 & 2 \\ 1 & 4 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix}$$

1–46. Consider the following system of equations:

$$\begin{aligned} x_1 + x_2 + 2x_3 + 2x_4 &= 4 \\ 2x_1 + x_2 + 3x_3 + 2x_4 &= 6 \\ 3x_1 + 4x_2 + 2x_3 + x_4 &= 9 \\ 7x_1 + 7x_2 + 9x_3 + 7x_4 &= 23 \end{aligned}$$

(a) Determine whether the above system is consistent using elementary operations on the augmented matrix.
(b) Find the solution in terms of x_4.