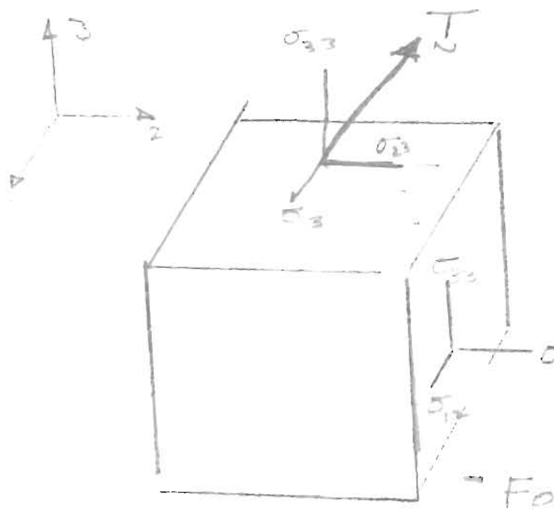


Stress

Traction $\underset{\sim}{T}$
mag. depend... on

Body f... $\underset{\sim}{g}$
depen...
reaches ... the d

- Forces inside bod... are a reaction to trac... on
- Homogeneous ... of T and s...

Then tr... on each face of

... to be ... static ...
(+2 face) = (−2 face) and ...

tra... /a

⇒ $\underset{\sim}{\sigma} \cdot \underset{\sim}{A}$ ... stress ... is
area normal ... ⇒ t

$\sigma \cdot A$

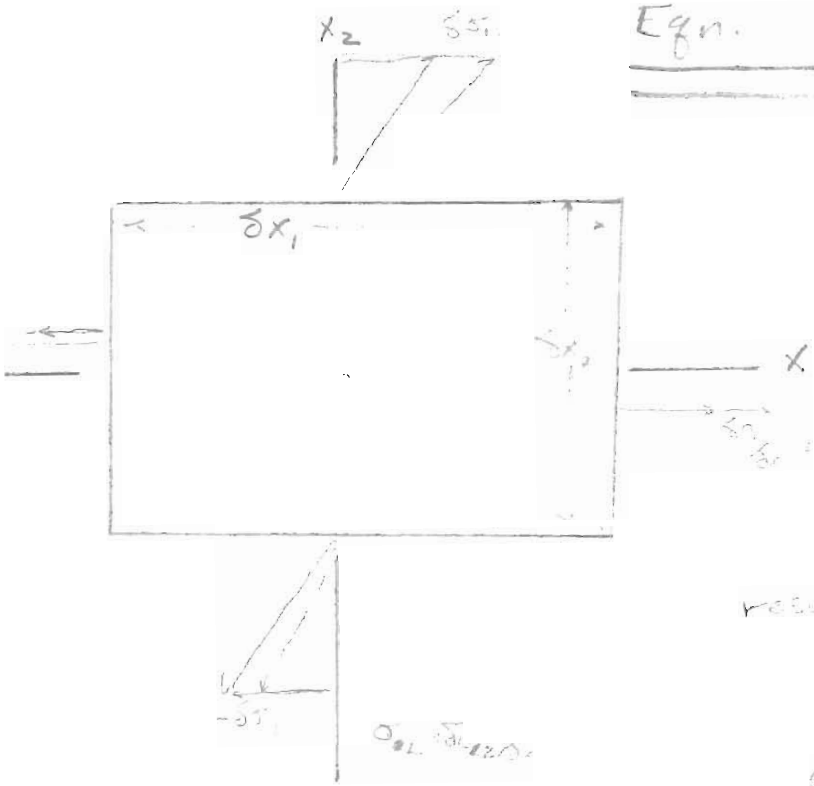
σ_{ij}

+ if ...

σ_{11}, σ_{12}

| if

Eqn. Equilibrium



x direct

$+ \sigma_{11} \delta x_2 \delta x_3 + \frac{\partial \sigma_{11}}{\partial x_1} \frac{\delta x_1}{2} \partial x_2 \partial x_3$

$-\left[\sigma_{11} \partial x_2 \partial x_3 - \frac{\partial \sigma_{11}}{\partial x_1} \frac{\partial x_1}{2} \partial x_2 \partial x_3 \right]$

result $\frac{\delta \sigma_{11}}{\partial x_1} \left(\partial x_1 \partial x_2 \partial x_3 \right)$

On x_2 ... direction

$\frac{\partial \sigma_{12}}{\partial x_2}$

B... sum of fo...

$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} + \rho g_1 = \rho \ddot{x}_1$

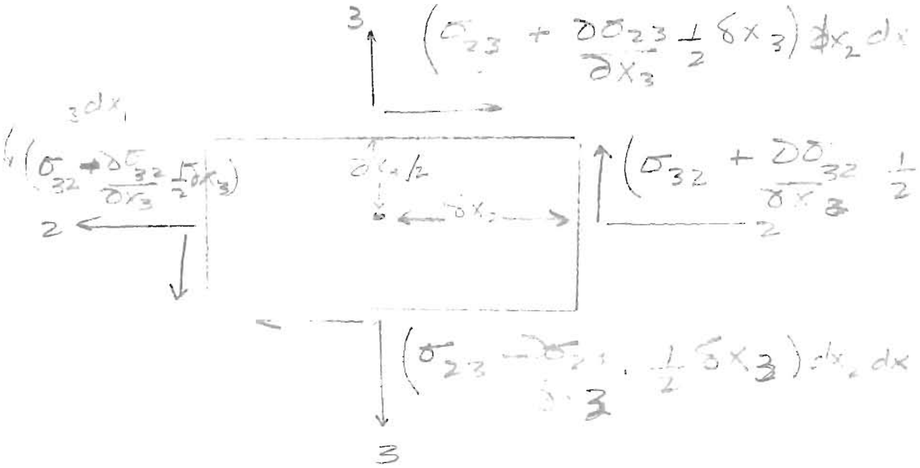
in general,

$$\boxed{\frac{\partial \sigma_{ij}}{\partial x_j} + \rho g_i = \rho \ddot{x}_i}$$

If $\ddot{x} = 0$, i.e. static eq

$$\boxed{\frac{\partial \sigma_{ij}}{\partial x_j} + \rho g_i = 0}$$

Shear moments and symmetry



moment arm for 2 face is $\frac{1}{2}\delta x_2$

moment axis

moment arm for 3 face is $\frac{1}{2}\delta x_3$
moments are cloc

$$2\sigma_{32}\frac{dx_3}{2}(dx_2\, dx_1) - 2\sigma_{23}\frac{dx_2}{2}\, dx_3\, dx_1 + G_1\, dx_1\, dx_2\, dx_3 = I_1\,\theta_1$$

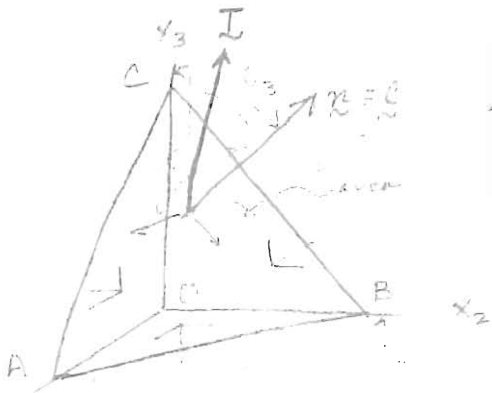
but assume no body torques $G_1 = 0$
and note I_1 order of mag $\delta x^5 \to 0$ ...

then $(\sigma_{32} - \sigma_{23})\, dx_1\, dx_2\, dx_3 = 0$

$\Rightarrow \sigma_{32} = \sigma_{23}$

or $\sigma_{ij} = \sigma_{ji}$

$\Rightarrow$ Stress tensor symmetric



Proof that stre[illegible] a tensor

Cauchy teh[illegible]

Total fo[illegible] face [illegible] ABC

$\underset{\sim}{T} \overset{\Delta}{=} \underset{\sim}{p} \cdot (ABC)$

$p_1 (ABC) = \sigma_{11} (BOC) + \sigma_{1} A$

$p_1 = \sigma$

where l [illegible] $\angle x_i \, \underset{\sim}{e}$

Sim. $p_2 = \sigma_{21} l_1 \quad \sigma_{22} l_2 + \sigma_{23} l_3$

$p_3 = \sigma_{31} l \quad \sigma_{32} l_2 + \sigma_{33} l_3$

or $p_i = \sigma_{ij} l_j$

i.e tensor transformation rule

⇒ Stress is a 2nd rank symmetric tensor

⇒ Principal stresses Quadric

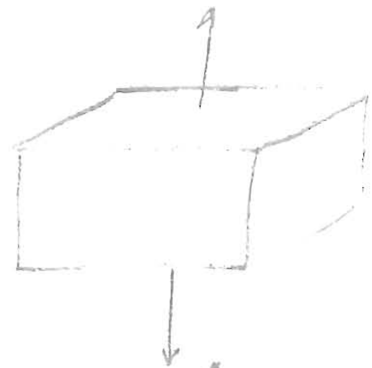
$\sigma_{11} \; \sigma_{12} \; \sigma_{13}$
$\sigma_{21} \; \sigma_{22} \; \sigma_{23}$
$\sigma_{31} \; \sigma_{32} \; \sigma_{33}$

$\sigma_1 \; 0 \; 0$
$0 \; \sigma_2 \; 0$
$0 \; 0 \; \sigma_3$

Special Forms of Stress

i.) uniaxial stress

$$\begin{matrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \end{matrix}$$



ii) biaxial stress

$$\begin{matrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & 0 \end{matrix}$$

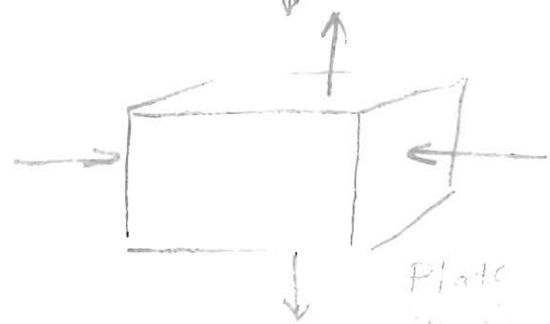


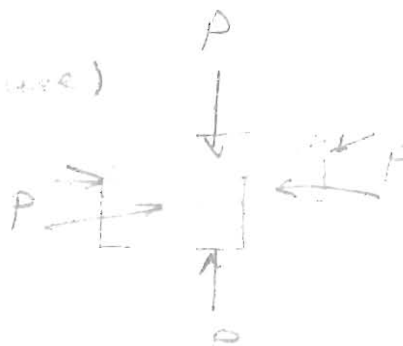
Plate [illegible] at [illegible] $\Delta x_1, \Delta x_3 \ll x_2$

iii) triaxial stress

$$\begin{matrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{matrix}$$

iv) hy[illegible] stress (pressure)

$$\begin{matrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{matrix} = -p\delta_{ij}$$



v.) Pure shear

$$\begin{bmatrix} -\sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

vi.) Simple shear

$$\begin{matrix} 0 & \sigma & 0 \\ \sigma & 0 & 0 \\ 0 & 0 & 0 \end{matrix}$$

Summary Stress Tensor

1.) Traction on a plane with direction cosines $\underset{\sim}{l}$

$$\underset{\sim}{T} = \underset{\approx}{\sigma}\;\underset{\sim}{l} \qquad T \quad \sigma_{ij}\, l_j$$

2.) Eqn of motion

$$\frac{\partial \sigma_{ij}}{\partial x_i} + \rho g_j = m\ddot{x}_j$$

3.) Stress is symmetric

$$\sigma_{ij} = \sigma_{ji}$$

4.) Principal stress directions
values

$$\begin{matrix} \sigma_{I} & 0 & 0 \\ 0 & \sigma_{II} & 0 \\ 0 & 0 & \sigma_{III} \end{matrix}$$

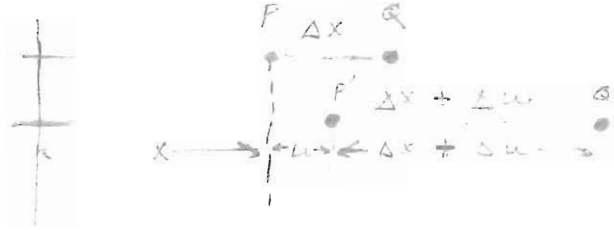
5.) Stress quadric construction

Stress, Strain Elasticity:

Nye Chap. 5 & 6, pp. 82–105

1. One dimensional Strain

Relative displacements important



inhom

x

P is

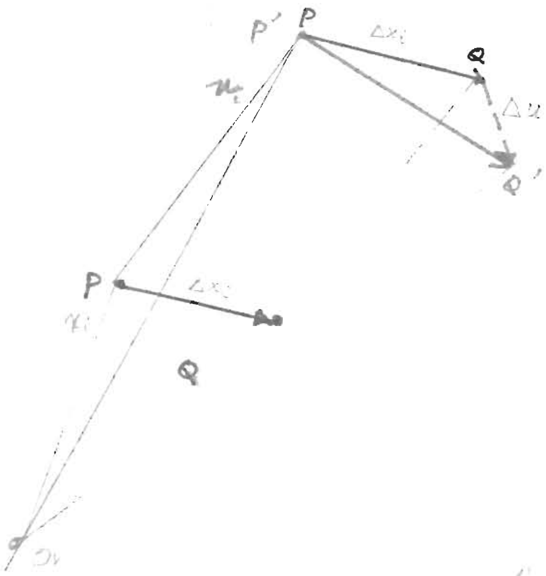
$= P'Q' \quad PQ \quad \Delta u$

above,

Goal: For a given deformation of body, Define a tensor that describes the change of direction and length of any vector in body.

Want to map vector in body (undef.) into vector in body (def.)

Infinitesimal Strain (2 of 8)



Point P can ... P'

$[OP] = x_i$

$[OP' - OP] = u_i$ (translatio...

Consider ... PQ

$[\overline{PQ}] = \Delta x_i$

Point ... Q

... not ...

$$\Delta PQ = \overline{QQ'} = P'Q' - PQ = \Delta u$$

but for any continuous diff. variable

$$\Delta u \quad \frac{\partial u}{\partial x_j} \Delta x_j$$

$$\Delta u \quad e_{ij} \Delta x_j$$

$$\boxed{\text{Define} \quad e_{ij} = \frac{\partial u_i}{\partial x_j} \quad \text{Displacement Gradient Tensor}}$$

Statements

1.) e_{ij} is a second rank tensor

$u_i' = a_{ik} u_k$

old

$$\Rightarrow \frac{\partial}{\partial x_j'} = \frac{\partial}{\partial x_l}\frac{\partial x_l}{\partial x_j'} = a_{jl}\frac{\partial}{\partial x_l}$$

then $$\frac{\partial u_i'}{\partial x_j'} = a_{ik}\, a_{jl}\, \frac{\partial u_k}{\partial x_l}$$

$$\Rightarrow e_{ij}' = a_{ik}\, a_{jl}\, e_{kl}$$

2.)

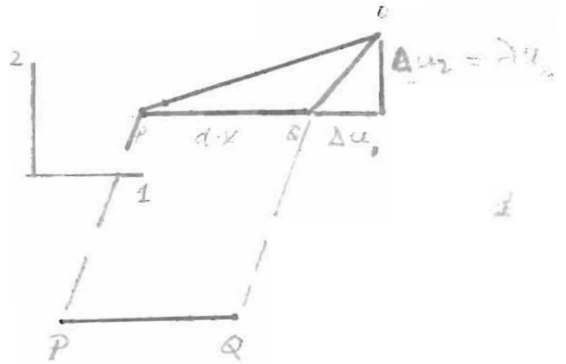
Strain

not

Specific Examples of Strain

Now consider a special case

$PQ = \hat{e}_1 \, dx$ where $\hat{e}_1$ unit vector along x_1 axis



$$\Delta u_i = \frac{\partial u_i}{\partial x_j} \Delta x_j$$

$$\Delta u_1 = \frac{\partial u_1}{\partial x_1} \Delta x_1 + \frac{\partial u_1}{\partial x_2} \Delta x_2 + \frac{\partial u_1}{\partial x_3} \Delta x_3$$

by definition $\Delta u_i \;\; e_{11} \Delta x_1 + e_{12} \Delta x_2 + e_{13} \Delta x_3$

e_{11} is change of length of $dx_1 \vec{i}$ in 1 direction

<u>normal strain</u> note: $\frac{l+\Delta l}{l}$ = stretch = $1+e_{11}$

e_{12} is change of length of $dx_1 \vec{i}$ in 2 direction

shear strain

note $\quad \frac{\partial u_2}{\partial x_1} dx_1 \Big/ dx_1 + \frac{\partial u_1}{\partial x} dx_1 = \tan\theta$

but $\theta \ll 1$ so $\tan\theta \simeq \theta$

and $\partial u_2 / \partial x_1 \ll 1$ so $dx \;\; \frac{\partial u_1}{\partial x} dx_1 \simeq dx_1$

$$\Rightarrow \frac{\partial u_2}{\partial x_1} = e_{21} \simeq \theta$$

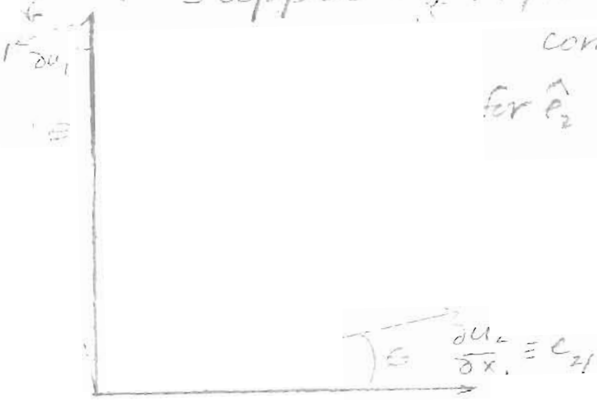
so e_{11} measure change of length in 1 direction of a line in 1 direction

e_{12} measures change of length in 1 dir of a line (originally) in 2 direction

Infinitesimal Strain

Example 2

Suppose rigid body rotation



consider rotation of vectors $\hat{e}$ and $\hat{e}$

for $\hat{e}_2$ $\quad \frac{\partial u_1}{\partial x_2} = -\theta \quad$ because $\partial u_1 < 0$

from definition

$$\omega_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right)$$

$$= \frac{1}{2}\left(-\theta - +\theta\right) = -\theta$$

but $\epsilon_{12} = \frac{1}{2}\left(e_{12} + e_{21}\right) = 0$

Particular Special Types of Strain

Homogenous strain:

All elements in body strained same

$$\epsilon_{ij} \neq \epsilon_{ij}(x_1, x_2, x_3)$$

st lines → st lines
p‖ line → p‖ lines
all st. lines ext. cont. by same ratio

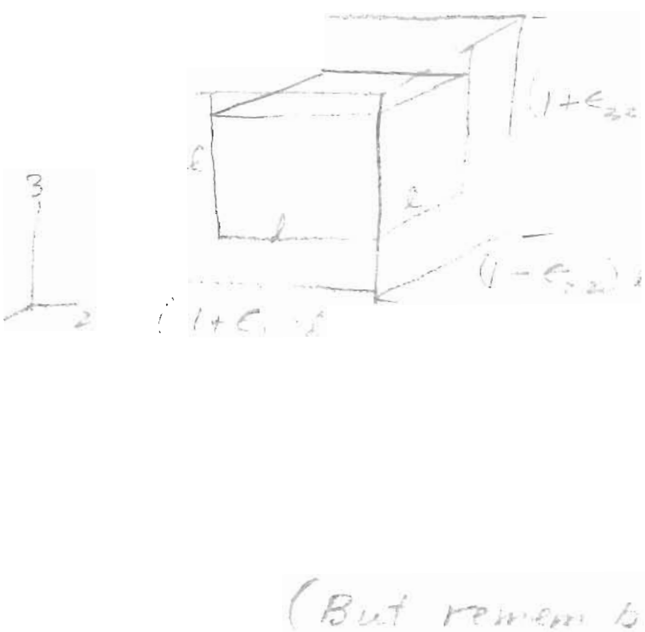
an ellipse becomes a diff. ellipse
circle " ellipse

Heterogenous Strain

$$\epsilon_{ij} = \epsilon_{ij}(x_k)$$

Plane Strain All elements in a plane in original remain in a plane in deformed body (actually impossible) but nearly achieved if z dim very large compared to x, y dimensions

Volumetric Strain



$\epsilon_{33})(1+\epsilon_{22}$ ℓ^3

$\ell^3(1+\epsilon_{11}+\epsilon_{22}+\epsilon_{33}+\dots)$

fractional change in vol

$$\frac{\Delta V}{V} = \frac{\ell^3(1+\epsilon_{11}+\epsilon_{22}+\epsilon_{33}) - \ell^3}{\ell^3}$$

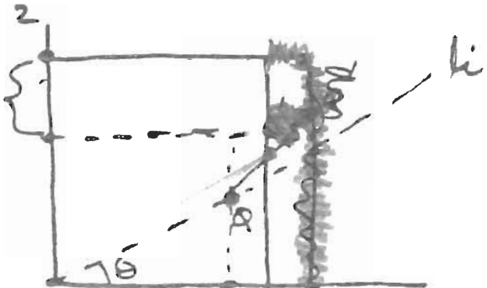
$$= \epsilon_{11}+\epsilon_{22}+\epsilon_{33}$$

(But remember trace of tensor is one of the

IB3c2

Example :

$\Delta l_2 = \epsilon_{22} l_2$



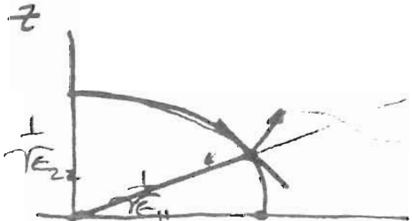
$\Delta l_1 = \epsilon_{11} l_1$

Body

No shear along axes
$\Rightarrow$ these must be principal axes

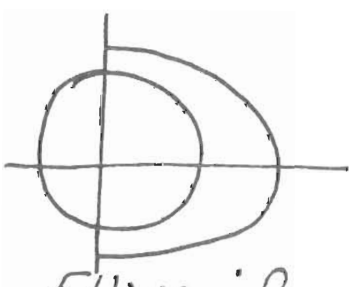
Assume
$\epsilon_{11} < \epsilon_{22}$

$\frac{1}{\sqrt{\epsilon_{11}}} > \frac{1}{\sqrt{\epsilon_{22}}}$



$\Delta l_i = \epsilon_{ij} l_j$

Quadric



Ellipsoid

Equation of circle $= x^2 + y^2 = 1$

Equation of deformed ellipse

$$x \rightarrow (1+\epsilon_1)x = x'$$

$$\frac{x'}{1+\epsilon_1} = x$$

$$\Rightarrow \frac{x'^2}{(1+\epsilon_1)^2} + \frac{y'^2}{(1+\epsilon_2)^2} + \frac{z'^2}{(1+\epsilon_3)^2} = 1$$

<u>Homework:</u>

Exercise 6.1 in Nye: A small def. is defined by the tensor

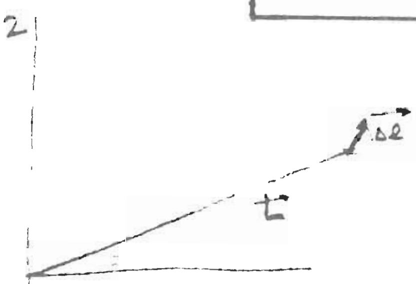
$$e_{ij} = \begin{bmatrix} 8 & -1 & -1 \\ 1 & 6 & 0 \\ -5 & 0 & 2 \end{bmatrix} \times 10^{-6}$$

Find magnitudes of principal

Summary: Strain

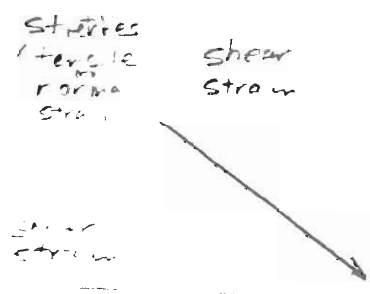
- Strain is the symmetric part of the deformation gradient tensor

$$\boxed{\begin{aligned}\epsilon_{ij} &\triangleq \tfrac{1}{2}\left[e_{ij} + e_{ji}\right] \\ &= \tfrac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right]\end{aligned}}$$



$$\underset{\sim}{\Delta \ell} = \underset{\approx}{\epsilon}\ \underset{\sim}{L}$$

$$\Delta L_i = \epsilon_{ij} L_j \quad \left(= \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right) \; dx_j$$



- Mohr's circle construction applies for 2 dimensional strain

  Also because symmetric 2nd rank tensor

  $\Rightarrow$ 3 principal directions

  3 principal values

- stretch in an arbitrary direction

$$\Delta L_i = \epsilon_{ij}\,(\vec{L} \cdot \vec{e}_j) = \epsilon_{ij}\, a_j\, |L|$$

$$\lambda = \frac{\Delta L_i}{|L|} = \epsilon_{ij}\, a_j \quad \text{where } a_j \text{ is } \cos(\angle \text{ between } \vec{e}_j \text{ and } \vec{L})$$

Reading: Creep of crystals, Poirier, Chap 1.

Work:

$$\underbrace{\sigma_{ij}\, d\epsilon_{ij}}_{\text{full stress tensor}} \quad \text{or} \quad \underbrace{P\,dV}_{\text{"isotropic" hydrostatic lithostatic}} \qquad \underbrace{\sigma_{ij} - P\delta_{ij}}_{\text{non-hydrostatic}}$$

non-hydrostatic

deviatoric stress

differential stress

$\sigma_{I} - \sigma_{III}$

(2.) In general, deformation occuring by crystal plastic mechanisms responds to differential stress or non-hydrostatic stress.

(1.) When material is not porous, then $\Delta V_{permanent} = 0$ when crystal plasticity (i.e. vacancies, dislocations twinning gb. sliding)
[Exception is phase change]

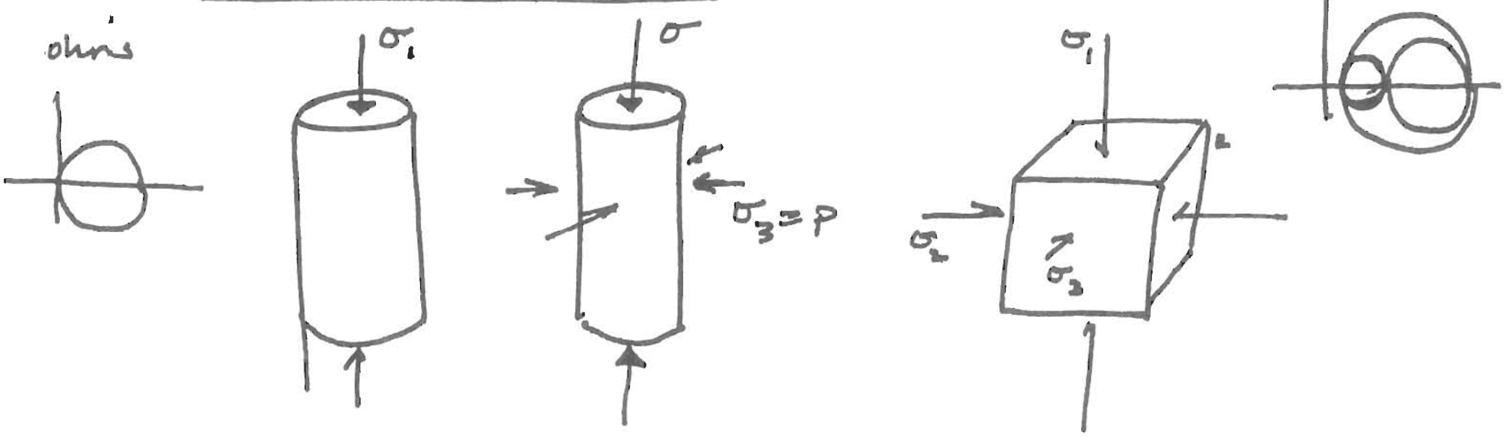
(6.) For elastic processes, all work converted potential energy of separation of bonds. Recoverable (Eqn. of State)

(3.) Effect of Intermediate Stress:

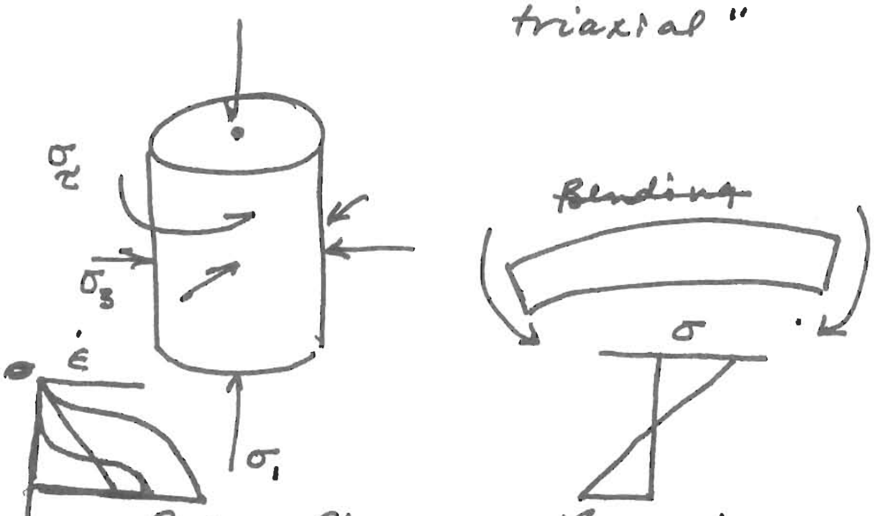
$$\sigma_{OCTAHEDRAL} = \sqrt{(\sigma_{I} - \sigma_{II})^2 + (\sigma_{II} - \sigma_{III})^2 + (\sigma_{III} - \sigma_{II})^2}$$

$$\sigma_{DIFFERENTIAL} = \sigma_{I} - \sigma_{III}$$

Mechanical Tests:



uniaxial — "conventional triaxial" — true triaxial



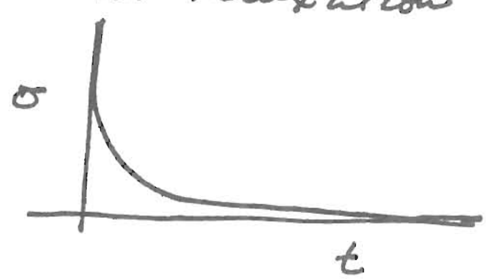
Rotary Shear — Bending

Uniaxial & Conv. triaxial (Boundary Conditions)

Creep tests : constant load (or const. σ)

Constant $\dot\epsilon$: constant displ. rate (or const. $\dot\epsilon$)

Relaxation : Load to yield; stop crosshead wait for relaxation



Observed Variables:

$T,\ P,\ l_s(t),\ (d_s(t)),\ \text{Load}(t)$

T → thermocouple
P → pressure transducer
l → displacement transducer DCDT or LVDT
L → load cell (displacement in elastic element)

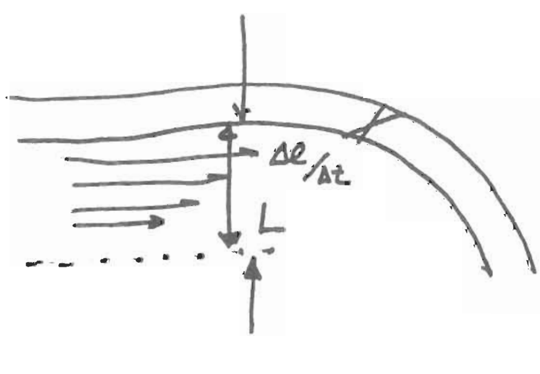
- convert displacement to load using Hooke's law
  convert Load to stress using calculated area or measured area

- Calculate stress
  Load / Area (t)
  Calculate area (1. measurement
  2. assume no volume change
  3. interpolate from initial to final

typical $\overline{T}$: 0 − 1600 K, $\overline{P}$: (0 − 1000 MPa) (0 − 50 GPa), $\dot{\epsilon}$: $10^{-6} - 10^{-8}$, $\overline{t}$: ⇒ 10^{6} secs

Natural Boundary Conditions

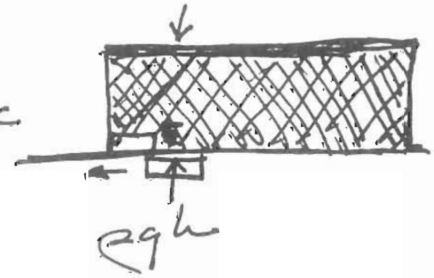
constant $\dot{\text{displacement}}$ rate − descending plate
constant load − mountain



$\frac{10^{-2}\ (1\,cm)}{3\cdot 10^{7}\ (1\,yr)} \cong 2\cdot 10^{-16}$ /sec

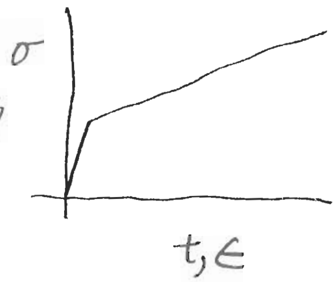
~~10 or 100·10^3~~
$1.5\cdot 10^{3}\cdot 10^{3}$ (whole mantle)

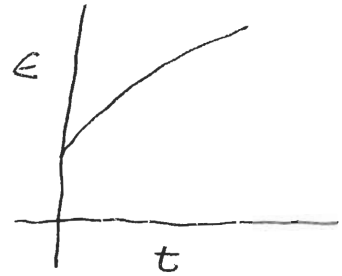
Olympus Mons



Macroscopic Mech. Behavior

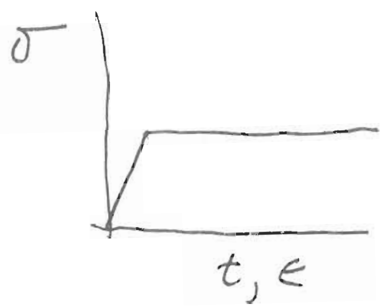
work hardening

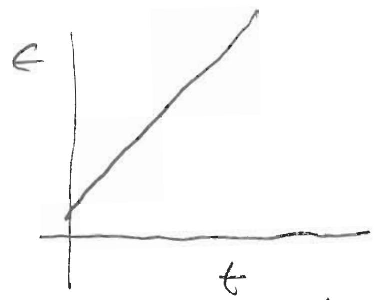




decelerating creep rate

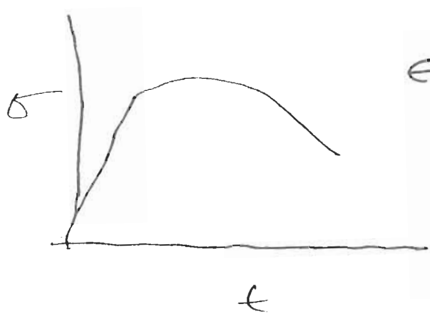
steady state

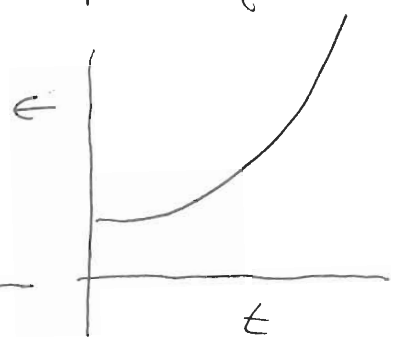




st. st

work softening





accel. creep

yield pt

sigmoidal creep curve

Stability of deformation:

instability – material props
loading conditions
geometric

$\Delta \epsilon \Rightarrow \Delta T_{bad} < 0$

Constitutive Law:

- Predict strength or deformation rate using observable quantities at a given instant in time. (i.e. thermodynamic state variables that do not depend on path)
  $\Rightarrow$ strain & time specifically excluded.

$$\dot{\epsilon} = f(\sigma, T, P, f_{ci}, \mathcal{Y}(\epsilon, t), \ldots)$$

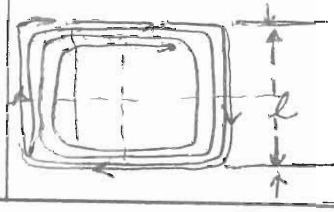
$\mathcal{Y}(\epsilon,t)$ - e.g. defect # (extrinsic chemistry)
geometry

$\mathcal{Y}(\epsilon,t)$ may not be a function of ϵ, t
should be determined by separate observation.

$\mathcal{Y}(\epsilon,t)$ internal state variable

- For fixed "external" state variables
  internal variable might change
  instability, weakening, or hardening

Lab & Natural Loading Conditions

	σ	P	T	$\dot{\varepsilon}$
lab	0 – 100 MPa 0 – 1000 MPa	0 – 600 MPa	0 – 1600 K	.01 in 1 day — 10^{-7}/s 1 wk — 10^{-8}/s 1 mnth — $5\cdot10^{-10}$/s 1 yr — $5\cdot10^{-11}$/s
nat	0 – 500 MPa 3	50 km ~ [illegible] MPa 100 km ~ 3200 MPa IC/OC ~$3.2\cdot10^{11}$ Pa $0.3\cdot10^{12}$ Pa	0 – 20°/km @ 95 km 1400°C 1600 K @ 95 km	$\frac{1\,cm}{yr}\left(\frac{1}{300\,km}\right) \approx 10^{-15}$/s  ℓ

$R_c = 3473$ km
$R_m = 6338$
$\Delta R_c = 0 - 50$ km

1 mb = 10^{11} Pa
1 kb = 10^{8} Pa

C. Stein in Ahrens, Handbook of Physical Constants.

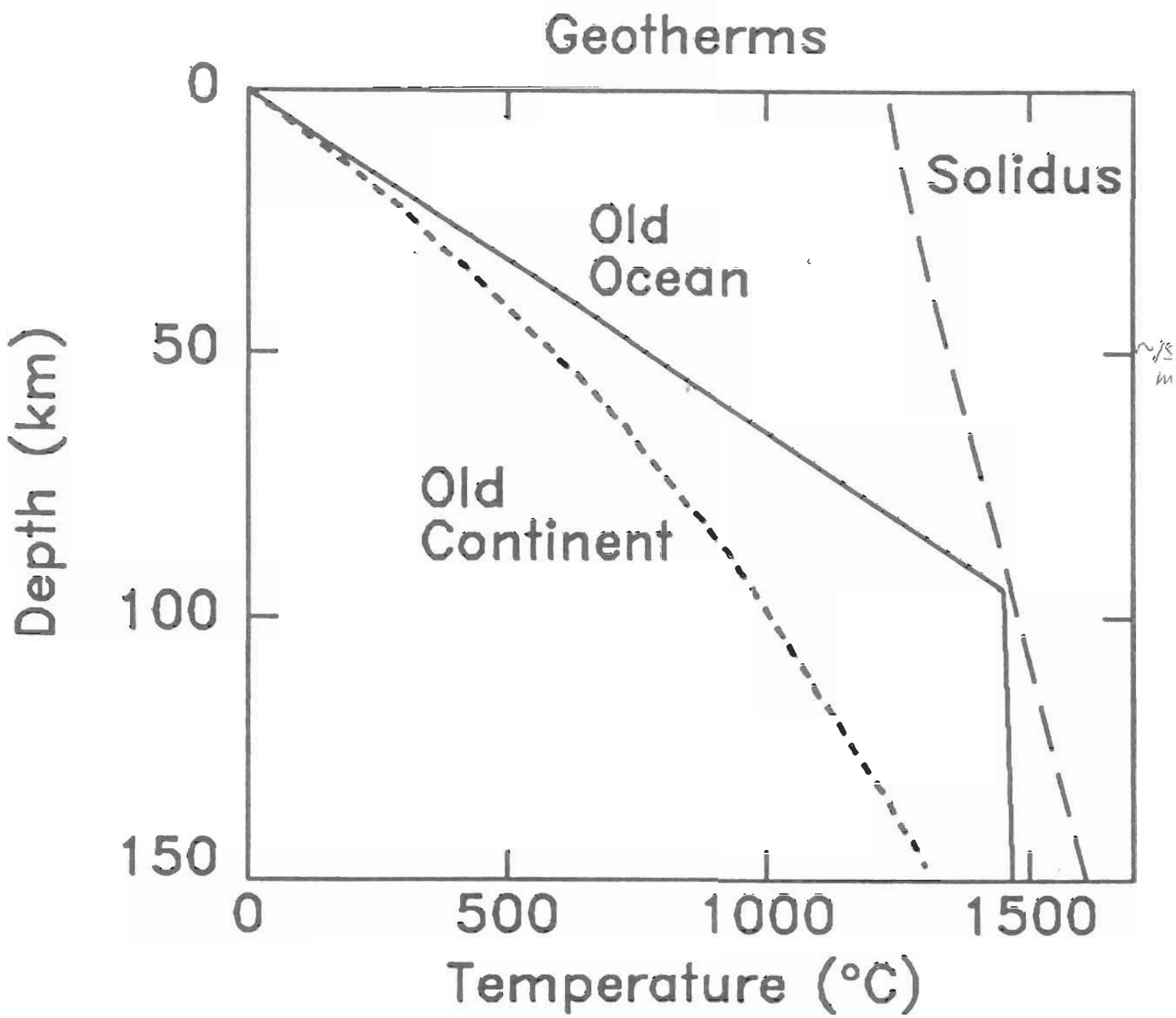


Fig. 5: Geotherms for old oceanic [105] and old con-