4. Energy equation for surface gravity waves

Equations of Motion

$$\rho \frac{d\vec{u}}{dt} = -\underline{\nabla} p - g\rho\hat{k} \quad (1) \qquad \rho = \text{constant}$$

$$\underline{\nabla} \bullet \vec{u} = o \quad (2) \qquad D = \text{constant}$$

Multiply (1) by $\vec{u}$

$$(\frac{1}{2}\rho\vec{u}\bullet\vec{u})_t + \vec{u}\bullet\underline{\nabla}p + g\rho w = o$$

In the linearized case, at every level z $\quad w = \frac{\partial z}{\partial t}$ and

$$[\frac{1}{2}\rho\vec{u}\bullet\vec{u} + g\rho z]_t + \underline{\nabla}\bullet(p\vec{u}) = 0$$

or rate of change (kinetic + potential energy) + divergence (energy flux) = 0

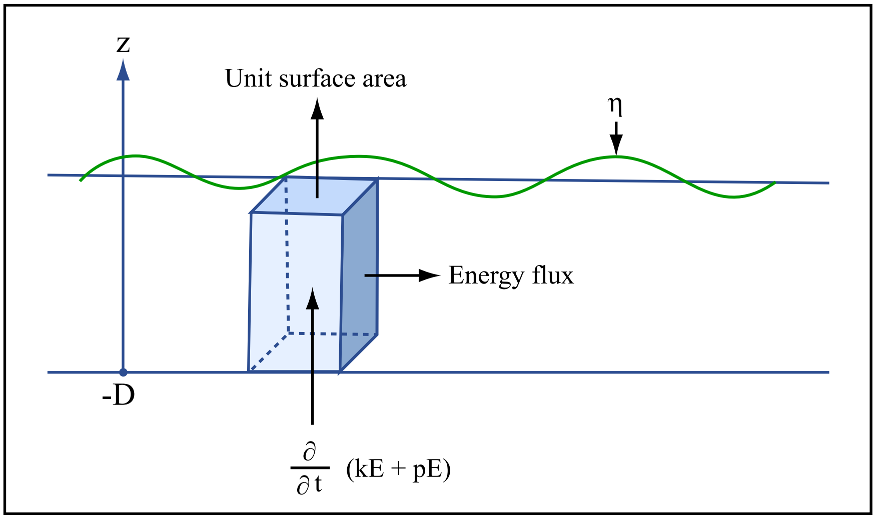

Figure by MIT OpenCourseWare.

Figure 1.

If we integrate from z = -D to z = η, we obtain the kinetic and potential energy and energy flux per unit horizontal area:

$$\frac{\partial}{\partial t}\left[\int_{-D}^{\eta}\frac{1}{2}\rho\vec{u}\bullet\vec{u}dz + \frac{1}{2}\rho g\eta^2\right] + \underline{\nabla}_H \bullet \int_{-D}^{\eta}(\vec{u}_H p)dz = 0$$

as p(η) = 0

and w = 0 at z = -D

$$\underline{\nabla}_H = \hat{i}\frac{\partial}{\partial x} + J\frac{\partial}{\partial y} \; ; \quad \int_{-D}^{\eta} g\rho z dz = \frac{1}{2}g\rho\frac{z^2}{2}\Big|_{-D}^{\eta} = \frac{1}{2}g\rho(\eta^2 - D^2)$$

$$\text{or} \quad \frac{\partial}{\partial t}[\overline{KE} + \overline{PE}] + \underline{\nabla}_H \bullet \overline{E}_{flux} = 0$$

Rate of change = horizontal divergence of wave energy flux

Bar denotes the quantities per unit horizontal area

Notice:

1) In the expression for the integrated potential density:

$\frac{1}{2}\rho g(\eta^2 - D^2)$ we have neglected the term proportional to D^2 as an irrelevant constant and $\frac{\partial D^2}{\partial t} = 0$.

2) In the integral for the kinetic energy we can integrate only to z = 0. In fact we are calculating energy to second order in the wave amplitude. To do this, for PE, we must integrate to η to obtain $\eta^2 (\equiv a^2)$. In the KE, the integral to η would include a correction of $0(u^2\eta) \equiv o(a^3)$, hence negligible. Let us now consider specifically the surface gravity wave field in one horizontal dimension (x,z,t):

$$\eta = a \ \cos(kx - \omega t) \qquad \omega^2 = gk \ \tanh(kD)$$

$$\phi = \frac{aw}{k \ \sinh(kD)} \cosh \ k(z + D) \cos(kx - \omega t)$$

$$p = -\rho g z + \frac{\rho \omega^2 a}{k \ \sinh(kD)} \cosh \ k(z + D) \cos(kx - \omega t)$$

$$u = \frac{a\omega}{\sinh(kD)} \cosh \ k(z + D) \cos(kx - \omega t)$$

$$w = \frac{a\omega}{\sinh(kD)} \sinh \ k(z + D) \sin(kx - \omega t)$$

$$PE = \frac{1}{2}\rho g a^2 \cos^2(kx - \omega t)$$

$$KE = + \int_{-D}^{o} \frac{\rho(u^2 + w^2)}{2} dz = \int_{-D}^{o} \frac{\rho a^2 \omega^2}{2} \left[\cos^2(kx - \omega t) \frac{\cosh^2 k(z+D)}{\sinh^2(kD)} + \sin^2(kx - \omega t) \frac{\sinh^2 k(z+D)}{\sinh^2(kD)} \right] dz$$

Let us now average both quantities over a wave period, indicated by < >

$$< PE > = \frac{1}{4}\rho g \; a^2$$

$$< kE > = \rho a^2 \omega^2 \int_{-D}^{o} \frac{1}{4} \frac{\cosh 2k(z+D)}{\sinh^2(kD)} dz = \qquad \text{as } \omega^2 = gk\tanh(kD)$$

$$= \rho \; a^2 \omega^2 \frac{1}{8} \frac{\sinh(2kD)}{k \sinh^2(kD)} =$$

$$= \rho a^2 g \tanh(kD) \frac{\sinh(kD)\cosh(kD)}{4 \sin^2(kD)} = \frac{1}{4}\rho g a^2$$

Averaged over a wave period

<PE>$\equiv$<kE> Equipartition of wave energy between potential and kinetic like in the oscillator problem. η is a linear oscillator!

And $< E_{total} > = < KE > + < PE > = \frac{\rho g a^2}{2}$

If we now calculate the energy flux vector and average it over one wave period we get:

$$< E_{flux} > = < \int_{-D}^{o} (up) dz > =$$

$$= \frac{1}{2}\rho g a^2 \left(\frac{\omega^2}{gK} \coth(kD) \;\; c \left[\frac{1}{2} + \frac{kD}{\sinh(2kD)}\right]\right.$$

$$\text{But} \quad c_g = \frac{\partial\omega}{\partial k} = c\left[\frac{1}{2} + \frac{kD}{\sinh(kD)}\right]$$

Thus the period average of the energy equation is:

$$\frac{\partial}{\partial t} < E > + \underline{\nabla}_H \bullet [\vec{c}_g < E >] = 0$$

Thus we have the important result that the energy in the wave propagates with the group velocity. If the medium is homogeneous, $\vec{c}_g = \frac{\partial\omega}{\partial\vec{k}}(|\vec{k}|)$ only and we can write

$$\frac{\partial}{\partial t} < E > + \vec{c}_g \bullet \nabla_H < E >= 0$$

For an observer moving horizontally with the group velocity the energy averaged over one phase of the wave is constant.

<u>Dispersion relationship for waves moving on a current</u>

Suppose I have a wave encountering a current $\vec{U}(x,y)$, the dispersion relationship is modified by the Doppler shift becoming

$$\sigma = \vec{k}(x,y) \bullet \vec{U}(x,y) + \omega \qquad \text{where} \quad \omega = \sqrt{gk\tanh(1k))D} \quad \text{is the intrinsic frequency}$$

Consider in fact the 1-D example

$U = U(x)$ only. Then $\sigma = kU + \omega$.