3.1 The General Eigenvector/Eigenvalue Problem

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To understand some recent work on so-called pseudospectra and some surprising recent results on fluid instability, it helps to review the more general eigenvector/eigenvalue problem for arbitrary, *square,* matrices. Consider,

$$\mathbf{E}\mathbf{g}_i = \lambda_i \mathbf{g}_i, \quad 1 \leq i \leq N \tag{3.1}$$ {A1}

If there are no repeated eigenvalues $\mathbf{g}_i$, then it is possible to show that there are always N independent $\mathbf{g}_i$ which are a spanning set, but which are *not usually orthogonal.* Because in most problems dealt with in this book we can often make small perturbations to the elements of $\mathbf{E}$ without creating any physical damage, it suffices here to assume that such perturbations can always assure that there are N distinct λ_i. (Failure of the hypothesis leads to the Jordan form which requires a somewhat tedious discussion.) A matrix $\mathbf{E}$ is "normal" if it has an orthonormal spanning set. Otherwise it is "non-normal." (Any matrix of form $\mathbf{A}\mathbf{A}^T$ etc., is necessarily normal.)

Denote $\mathbf{G} = \{g_i\}, \mathbf{\Lambda} = diag(\lambda_i)$. It follows immediately that $\mathbf{E}$ can be diagonalized:

$$\mathbf{G}^{-1}\mathbf{E}\mathbf{G} = \mathbf{\Lambda} \tag{3.2}$$

but for a non-normal matrix, $\mathbf{G}^{=1} \neq \mathbf{G}^T$.

Matrix $\mathbf{E}^T$ has a different set of spanning eigenvectors, but the same eigenvalues:

$$\mathbf{E}^T\mathbf{f}_i = \mathbf{f}_i \tag{3.3}$$ {A3}

which can be written

{A4} $$\mathbf{f}_i^T\mathbf{E} = \lambda_i\mathbf{f}_i^T. \tag{3.4}$$

The $\mathbf{g}_i$ are hence known as the "right eigenvectors," and the $\mathbf{f}_i$ as the "left eigenvectors." Multiplying (3.1) on the left by $\mathbf{f}_j^T$, and (3.4) on the right by $\mathbf{g}_i$ and subtracting shows,

{A5} $$0 = (\lambda_i - \lambda_j)\,\mathbf{f}_j^T\mathbf{g}_i \tag{3.5}$$

or

{A6} $$\mathbf{f}_j^T\mathbf{g}_i = 0, i \neq j. \tag{3.6}$$

That is to say, the left and right eigenvectors are orthgonal for different eigenvalues, but $\mathbf{f}_j^T\mathbf{g}_j \neq 0$. (In general the eigenvectors and eigenvalues are complex even for purely real $\mathbf{E}$. *The reader is warned* that some software automatically conjugates a transposed vector or matrix, and the derivation of (3.6) shows that it applies only to the *non-conjugated* variables.)

Consider now a "model,"

{A7} $$\mathbf{A}\mathbf{x} = \mathbf{b}. \tag{3.7}$$

The norm of $\mathbf{b}$ is supposed bounded, $\|\mathbf{b}\| \leq b$, and the norm of $\mathbf{x}$ will be,

$$\|\mathbf{x}\| = \left\|\mathbf{A}^{-1}\mathbf{b}\right\| \tag{3.8}$$

What is the relationship of $\|\mathbf{x}\|$ to $\|\mathbf{b}\|$?

Let $\mathbf{g}_i$ be the right eigenvectors of $\mathbf{A}$. Write

$$\mathbf{b} = \sum_{i=1}^{N}\beta_i\mathbf{g}_i \tag{3.9}$$

$$\mathbf{x} = \sum_{i=1}^{N}\alpha_i\mathbf{g}_i \tag{3.10}$$

If the $\mathbf{g}_i$ were orthogonal, $|\beta_i| \leq \|\mathbf{b}\|$. But as they are not orthonormal, the β_i will need to be found through a system of simultaneous equations (ecall the discussion in Chapter 2 of the expansion of an arbitrary vector in non-orthogonal vectors) (2.3) and no simple bound on the β_i is then possible; some may be very large. Substituting into (3.7),

$$\sum_{i=1}^{N}\boldsymbol{\alpha}_i\lambda_i\mathbf{g}_i = \sum_{i=1}^{N}\beta_i\mathbf{g}_i. \tag{3.11}$$

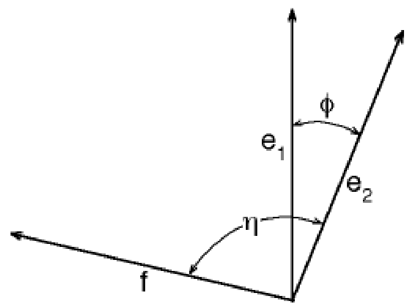

Figure 3.1: Two dimensional example of the expansion of a vector **f** in two non-orthogonal vectors. Although $\mathbf{e}_1$and $\mathbf{e}_2$ are a spanning set, the expansion coefficients become very large of **f** is nearly orthogonal to both. (There is "nearly a nullspace" as $\phi \to 0$.) {ocip3.1new.ep

A term–by-term solution is evidently no longer possible because of the lack of orthogonality. But multiplying on the left by $\mathbf{f}_j^T$, and invoking (3.6) produces,

$$\alpha_j \lambda_j \mathbf{f}_j^T \mathbf{g}_j = \beta_j \mathbf{f}_j^T \mathbf{g}_j \tag{3.12}$$

or,

$$\alpha_j = \beta_j / \lambda_j, \ \ \lambda_j \neq 0. \tag{3.13}$$

Even if the λ_j are all of the same order, the possibility that some of the β_j are very large implies that eigenstructures in the solution may be much larger than b. This possibility becomes very interesting when we turn to time-dependent systems. At the moment, note that partial differential equations that are self-adjoint produce discretizations which have coefficient matrices $\mathbf{A}$, such that $\mathbf{A}^T = \mathbf{A}$. Thus self-adjoint systems have normal matrices, and the eigencomponents of the solution are all immediately bounded by $\|\mathbf{b}\| / \lambda_i$. Non-self-adjoint systems produce non-normal coefficient matrices and so can therefore unexpectedly generate very large eigenvector contributions.