2.11 Appendix 2. Differential Operators and Green Functions

Adjoints appear prominently in the theory of differential operators and are usually discussed independently of any optimization problem. Many of the concepts are those used in defining Green functions.

Consider an example. Suppose we want to solve an ordinary differential equation,

$$\frac{du(\xi)}{d\xi} + \frac{d^2u(\xi)}{d\xi^2} = \rho(\xi), \tag{2.451}$$ {diffeq1}

subject to boundary conditions on $u(\xi)$ at $\xi = 0, L$. To proceed, seek first a solution to,

$$\alpha\frac{\partial v(\xi,\xi_0)}{\partial\xi} + \frac{\partial^2 v(\xi,\xi_0)}{\partial\xi^2} = \delta(\xi_0 - \xi), \tag{2.452}$$ {diffeq2}

where α is arbitrary for the time being. Multiply (2.451) by v, and (2.452) by u and subtract:

$$\begin{aligned} & v(\xi,\xi_0)\frac{du(\xi)}{d\xi} + v(\xi,\xi_0)\frac{d^2u(\xi)}{d\xi^2} - u(\xi)\alpha\frac{\partial v(\xi,\xi_0)}{\partial\xi} - u(\xi)\frac{\partial^2 v(\xi,\xi_0)}{\partial\xi^2} \\ = \; & v(\xi,\xi_0)\rho(\xi) - u(\xi)v(\xi,\xi_0). \end{aligned} \tag{2.453}$$

Integrate this last equation over the domain,

$$\int_0^L \left\{ v(\xi,\xi_0)\frac{du(\xi)}{d\xi} + \right.$$

$$\left. v(\xi,\xi_0)\frac{d^2u(\xi)}{d\xi^2} - u(\xi)\alpha\frac{\partial v(\xi,\xi_0)}{\partial\xi} - u(\xi)\frac{\partial^2 v(\xi,\xi_0)}{\partial\xi^2} \right\} d\xi \tag{2.454}$$

$$= \int_0^L \left\{ v(\xi,\xi_0)\rho(\xi) - u(\xi)\delta(\xi_0-\xi) \right\} d\xi, \tag{2.455}$$

or,

$$\begin{aligned} & \int_0^L \frac{d}{d\xi}\left\{ v\frac{du}{d\xi} - \alpha u\frac{dv}{d\xi} \right\} d\xi + \int_0^L \left\{ u\frac{d^2v(\xi,\xi_0)}{d\xi^2} - u\frac{d^2v(\xi,\xi_0)}{d\xi^2} \right\} d\xi \\ = \; & \int_0^L v(\xi,\xi_0)\rho(\xi)\,d\xi - u(\xi_0). \end{aligned} \tag{2.456}$$

Choose $\alpha = -1$; then the first term on the left hand-side is integrable, as,

$$\int_0^L \frac{d}{d\xi}\{uv\}\,d\xi = uv|_0^L, \tag{2.457}$$

as is the second term on the left,

$$\int_0^L \frac{d}{d\xi}\left\{ u\frac{dv}{d\xi} - v\frac{du}{d\xi} \right\} d\xi = \left[u\frac{dv}{d\xi} - v\frac{du}{d\xi} \right]_0^L \tag{2.458}$$

and thus,

$$u\left(\xi_0\right) = \int_0^L v\left(\xi, \xi_0\right) \rho\left(\xi\right) d\xi + uv|_0^L + \left[u\frac{dv}{d\xi} - v\frac{du}{d\xi}\right]_0^L \tag{2.459}$$

Because the boundary conditions on v were not specified, we are free to choose them such that $v = 0$ on $\xi = 0, L$ such that e.g., the boundary terms reduce simply to $[udv/d\xi]_0^L$, which is then known.

Here, v is the adjoint solution to Eq. (2.452), with $\alpha = -1$, defining the adjoint equation to (2.451); it was found by requiring that the terms on the left-hand-side of Eq. (4.32) should be exactly integrable. v is also the problem Green function (although the Green function is sometimes defined so as to satisfy the forward operator, rather than the adjoint one). Textbooks show that for a general differential operator, $\mathcal{L}$, the requirement that v should render the analogous terms integrable is that,

{adjoint11}

$$u^T \mathcal{L} v = v^T \mathcal{L}^T u \tag{2.460}$$

where here the superscript T denotes the adjoint. Eq. (2.460) defines the adjoint operator (compare to (2.374a)).