4.5 Duality and Simplification: The Steady-State Filter and Adjoint

For linear models, the Lagrange multiplier method and the filter/smoother algorithms produce identical solutions. In both cases, the computation of the uncertainty remains an issue—in the former case because it is not part of the solution, and in the latter because it can overwhelm the computation. However, if the uncertainty is computed for the sequential estimator solutions,

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it must also represent the uncertainty derived from the Lagrange multiplier principle. In the interests of gaining insight into both methods, and of ultimately finding uncertainty estimates, consider again the covariance propagation equations for the Kalman filter:

$$\mathbf{P}(t,-) = \mathbf{A}(t-1)\mathbf{P}(t-1)\mathbf{A}(t-1)^T + \mathbf{\Gamma}(t-1)\mathbf{Q}(t-1)\mathbf{\Gamma}\,(t-1)^T\,, \tag{4.147}$$ {64001}

$$\begin{aligned}\mathbf{P}(t) &= \mathbf{P}(t,-)- \\ &\mathbf{P}(t,-)\mathbf{E}(t)^T\left[\mathbf{E}(t)\mathbf{P}(t,-)\mathbf{E}(t)^T + \mathbf{R}(t)\right]^{-1}\mathbf{E}(t)\mathbf{P}(t,-)\end{aligned} \tag{4.148}$$ {64002}

where $\mathbf{K}(t)$ has been written out. Make the substitutions shown in Table 4.1; the equations for evolution of the uncertainty of the Kalman filter are identical to those for the control matrix $\mathbf{S}(t)$, given in Equation (4.136); hence, the Kalman filter covariance *also satisfies a matrix Riccati equation.* To see that in Eq. (4.136)[129] put,

$$\mathbf{S}(t,-) \equiv \mathbf{S}(t+1) - \mathbf{S}(t+1)\mathbf{\Gamma}\left[\mathbf{\Gamma}^T\mathbf{S}(t+1)\mathbf{\Gamma} + \mathbf{Q}^{-1}\right]^{-1}\mathbf{\Gamma}^T\mathbf{S}(t+1) \tag{4.149}$$

and then,

$$\mathbf{S}(t) = \mathbf{A}^T\mathbf{S}(t,-)\mathbf{A} + \mathbf{R}(t)^{-1}, \tag{4.150}$$

which correspond to Eq. (4.147, 4.148). Time runs backward in the control formulation and forward in the estimation problem, but this difference is not fundamental. The significance of this result is that simplifications and insights obtained from one problem can be employed on the other (some software literally makes the substitutions of Table 4.1 to compute the Kalman filter solution from the algorithm for solving the control Riccati equation). {pagericcati}

This feature—that both problems produce a matrix Riccati equation—is referred to as the "duality" of estimation and control. It does *not* mean that they are the same problem; in particular, recall that the control problem is equivalent not to filtering, but to smoothing.

Covariances usually dominate the Kalman filter (and smoother) calculations and sometimes lead to the conclusion that the procedures are impractical. But as with all linear least-square—like estimation problems, the state vector uncertainty does not depend upon the actual data values, only upon the prior error covariances. Thus, the filter and smoother uncertainties (and the filter and smoother gains) can be computed in advance of the actual application to data, and stored. The computation can be done e.g., by stepping through the recursion in (4.147)–(4.148) starting from $t = 0$.

Furthermore, it was pointed out that in Kalman filter problems, the covariances and Kalman gain can approach a steady state, in which $\mathbf{P}(t)$, $\mathbf{P}(t,-)$, $\mathbf{K}(t)$ eventually do not depend upon

Table 4.1: Correspondences between the variables of the control formulation and that of the Kalman filter, which lead to the Riccati equation. Note that time runs backward for control cases and forward for the filter. {table1}

Adjoint/Control	Kalman Filter
$\mathbf{A}$	$\mathbf{A}^T$
$\mathbf{S}(t,-)$	$\mathbf{P}(t+1)$
$\mathbf{S}(t+1)$	$\mathbf{P}(t+1,-)$
$\mathbf{R}^{-1}$	$\Gamma Q \Gamma^T$
Γ	$\mathbf{E}^T$
$\mathbf{Q}^{-1}$	$\mathbf{R}$

time. Physically, the growth in error from the propagation Equation (4.147) is then just balanced by the reduction in uncertainty from the incoming data stream (4.148). This simple description supposes the data come in at every time step; often the data appear only intermittently, but periodically, and the steady-state solution is periodic—errors displaying a characteristic sawtooth structure between observation times.[130]

If these steady-state values can be found, then the necessity to update the covariances and gain matrix disappears, and the computational load is much reduced, potentially by many orders of magnitude (see also Chapter 5). The equivalent steady state for the control problem is best interpreted in terms of the feedback gain control matrix $\mathbf{K}_c$, which can also become time independent, meaning that the value of the control to be applied depends only upon the state observed at time t and need not be recomputed at each time step.

{steadystate} The great importance of steady-state estimation and control has led to a large number of methods for obtaining the solution of the various steady-state Riccati equations requiring one of, $\big(\mathbf{S}(t) = \mathbf{S}(t-1)$, $\mathbf{S}(t,-) = \mathbf{S}(t-1,-)$, $\mathbf{P}(t) = \mathbf{P}(t-1)$, or $\mathbf{P}(t,-) = \mathbf{P}(t-1,-)\big)$.[131] The steady-state equation is often known as the "algebraic Riccati equation."[132]

A steady-state solution to the Riccati equation corresponds not only to a determination of the steady-state filter and smoother covariances but also to the steady-state solution of the Lagrange multiplier normal equations—a so-called steady-state control. Generalizations to the steady-state problem exist; an important one is the possibility of a periodic steady state.[133]

Before seeking a steady-state solution, one must determine whether one exists. That no such solution will exist in general is readily seen by considering a physical system in which certain components (elements of the flow) are not readily observed. If these components are initialized with partially erroneous values, then if they are unstable, they will grow without bound, and there will be no limiting asymptotic value for the uncertainty, which will also have to grow without bound. Alternatively, suppose there are elements of the state vector whose values cannot be modified by the available control variables. Then no observations of the state vector produce information about the control variables; if the control vector uncertainty is described by $\mathbf{Q}$, then this uncertainty will accumulate from one time step to another, growing without bound with the number of time steps.