4.1 Background

The discussion so far has treated models and data that most naturally represent a static world. Much data, however, describe systems that are changing in time to some degree. Many familiar differential equations represent phenomena that are intrinsically time-dependent; a good example is the wave equation,

$$\frac{1}{c^2}\frac{\partial^2 x(t)}{\partial t^2} - \frac{\partial^2 x(t)}{\partial r^2} = 0. \tag{4.1}$$ {wave1}

One may well wonder if the methods described in Chapter 2 have any use with data thought to be described by (4.1). An approach to answering the question is to recognize that t is simply another coordinate, and can be regarded e.g., as the counterpart of one of the space coordinates encountered in the previous discussion of two dimensional partial differential equations. From this point of view, time dependent systems are nothing but versions of the systems already developed. (The statement is even more obvious for the simpler equation,

$$\frac{d^2 x(t)}{dt^2} = q(t). \tag{4.2}$$

That the coordinate is labelled t is a detail.)

On the other hand, time often has a somewhat different flavor to it than does a spatial coordinate because it has an associated direction. The most obvious example occurs when one has data up to and including some particular time t, and one asks for a *forecast* of some elements of the system at some future time $t' > t$. Even this role of time is not unique: one could imagine a completely equivalent spatial forecast problem, in which e.g., one required extrapolation of the map of an ore body beyond some area in which measurements exist. In state estimation, time does not introduce truly novel problems. The main issue is really a computational one: problems in two or more spatial dimensions, when time-dependent, typically generate system dimensions which are too large for conventionally available computer systems. To deal with the computational load, one seeks state estimation algorithms that are computationally more efficient than what can be achieved with the methods used so far. Consider as an example,

{tracer1}
$$\frac{\partial C}{\partial t} = \kappa \nabla^2 C, \tag{4.3}$$

a two dimensional generalization of the Laplace equation (a diffusion equation). Using a one-sided time difference, and the discrete form of the Laplacian in Eq. (1.13), one has

$$\frac{C_{ij}\left(\left(n+1\right)\Delta t\right) - C_{ij}\left(n\Delta t\right)}{\Delta t} = \tag{4.4}$$
$$\kappa \left\{ C_{i+1,j}\left(n\Delta t\right) - 2C_{i,j}\left(n\Delta t\right) + C_{i-1,j}\left(n\Delta t\right) + C_{i,j+1}\left(n\Delta t\right) - 2C_{i,j}\left(n\Delta t\right) + C_{i,j-1}\left(n\Delta t\right) \right\}$$

If there are N^2 elements defining C_{ij} at each time $n\Delta t$, then the number of elements over the entire time span of T time steps, would be TN^2 and which grows rapidly as the number of time steps increases. Typically the relevant observation numbers also grow rapidly through time. On the other hand, the operation,

$$\mathbf{x} = \mathrm{vec}\left(C_{ij}\left(n\Delta t\right)\right), \tag{4.5}$$

renders Eq. (4.4) in the familiar form

$$\mathbf{A}_1\mathbf{x} = \mathbf{0}, \tag{4.6}$$

and with some boundary conditions, some initial conditions and/or observations, and a big enough computer, one could use without change any of the methods of Chapter 2. There are however, many times when T, N become so large, that even the largest available computer is inadquate. Methods are sought that can take advantage of special structures built into time evolving equations to reduce the computational load. (Note however, that $\mathbf{A}_1$ is very sparse.)

This chapter is in no sense exhaustive; many entire books are devoted to the material and its extensions, which are important for understanding and practical use. The intention is to lay out the fundamental ideas, which are primarily algorithmic rearrangements of methods already described in Chapters 2 and 3 with the hope that they will permit the reader to penetrate the wider literature. Several very useful textbooks are available for readers who are not deterred by discussions in contexts differing from their own applications.[101] Most of the methods now being used in fields involving large-scale fluid dynamics, such as oceanography and meteorology, have been known for years under the general headings of control theory and control engineering. The experience in these latter areas is very helpful; the main issues in applications to fluid problems concern the size of the models and data sets encountered: they are typically many orders of magnitude larger than anything contemplated by engineers. In meteorology, specialized techniques used for forecasting are commonly called "data assimilation."[102] The reader may find it helpful to keep in mind, through the details that follow, that almost all methods in actual use are, beneath the mathematical disguises, nothing but versions of least-squares fitting of models to data, but reorganized, so as to increase the efficiency of solution, or to minimize storage requirements, or to accomodate continuing data streams.

Several notation systems are in wide use. The one chosen here is taken directly from the conrol theory literature; it is simple and adequate.[103]