4.8 Forward Models

The focus we have had on the solution of inverse problems has perhaps given the impression that there is some fundamental distinction between forward and inverse modeling. The point was made at the beginning of this book that inverse *methods* are important in solving forward as well as inverse *problems*. Almost all the inverse problems discussed here involved the use of an objective function, and such objective functions do not normally appear in forward modeling. The presence or absence of objective functions thus might be considered a fundamental difference between the problem types.

But numerical models do not produce universal, uniformly accurate solutions to the fluid equations. Any modeler makes a series of decisions about which aspects of the flow are most important for accurate depiction—the energy or vorticity flux, the large-scale velocities, the nonlinear cascades, etc.—and which cannot normally be achieved simultaneously with equal fidelity. It is rare that these goals are written explicitly, but they could be, and the modeler could choose the grid and differencing scheme, etc., to minimize a specific objective function. The use of such explicit objective functions would prove beneficial because it would quantify the purpose of the model.

One can also consider the solution of ill-posed forward problems. In view of the discussion throughout this book, the remedy is straightforward: One must introduce an explicit objective function of the now-familiar type, involving state vectors, observations, control, etc., and this approach is precisely that recommended. If a Lagrange multiplier method is adopted, then Eqs. (2.332, 2.333) show that an over- or under-specified forward model produces a complementary under- or overspecified adjoint model, and it is difficult to sustain a claim that modeling in the forward direction is fundamentally distinct from that in the inverse sense.

Example

Consider the ordinary differential equation

$$\frac{d^2x(t)}{dt^2} - k^2 x(t) = 0\,. \tag{4.183}$$ {69001a}

Formulated as an initial value problem, it is properly posed with Cauchy conditions $x(0) = x_0$, $x'(0) = x_0'$. *The solution is*

$$x(t) = A\exp(kt) + B\exp(-kt)\,, \tag{4.184}$$ {69001b}

with A, B *determined by the initial conditions. If we add another condition—for example, at the end of the interval of interest,* $x(t_f) = x_{t_f}$*—the problem is ill-posed because it is now overspecified. To analyze and solve such a problem using the methods of this book, discretize it as*

$$x(t+1) - (2+k^2)x(t) + x(t-1) = 0\,, \tag{4.185}$$ {69001c}

taking $\Delta t = 1$, *with corresponding redefinition of* k^2. *A canonical form is,*

$$\mathbf{x}(t) = \mathbf{A}\mathbf{x}(t-1), \quad \mathbf{x}(t) = \left[x(t), \quad x(t-1)\right]^T, \quad \mathbf{A} = \left\{\begin{matrix} 2+k^2 & -1 \\ 1 & 0 \end{matrix}\right\}.$$

A reduced form *of equations (4.166-4.170) are easily solved (the only "observations" are at the final time) by a backward sweep of the adjoint model (4.101) to obtain* $\boldsymbol{\mu}(1)$, *which through (??) produces* $\tilde{\mathbf{x}}(1)$ *in terms of* $\mathbf{x}(t_f) - \mathbf{x}_d(t_f)$. *A forward sweep of the model, to* t_f, *produces the numerical value of* $\tilde{\mathbf{x}}(t_f)$*; the backward sweep of the adjoint model gives the corresponding numerical value of* $\tilde{\mathbf{x}}(1)$*, and a final forward sweep of the model completes the solution. The subproblem forward and backward sweeps are always well-posed. This recipe was run for*

$$k^2 = 0.05, \quad \Delta t = 1, \quad \tilde{\mathbf{x}}(1) = [0.805,\ 1.0]^T, \quad \mathbf{P}(1) = 10^{-2}\mathbf{I}\,,$$
$$\tilde{x}(t_f) = 1.427\times 10^{-5}, \quad \mathbf{P}(t_f) = \mathrm{diag}\left\{\ 10^{-4}\ \ 10^{4}\ \right\}, \quad t_f = 50$$

with results in Figure 4.17. (The large subelement uncertainty in $\mathbf{P}(50)$*, corresponding to scalar element,* $x(49)$*, is present because we sought to specify only scalar element* $x(50)$*, in* $\mathbf{x}(50)$*.) The solution produces a new estimated value* $\tilde{\mathbf{x}}(0) = [0.800, 1.00]^T$ which is exactly the value used in Fig. 4.17 to generate the stable forward computation. *Notice that the original ill-posedness in both overspecification and instability of the initial value problem have been dealt with. The*

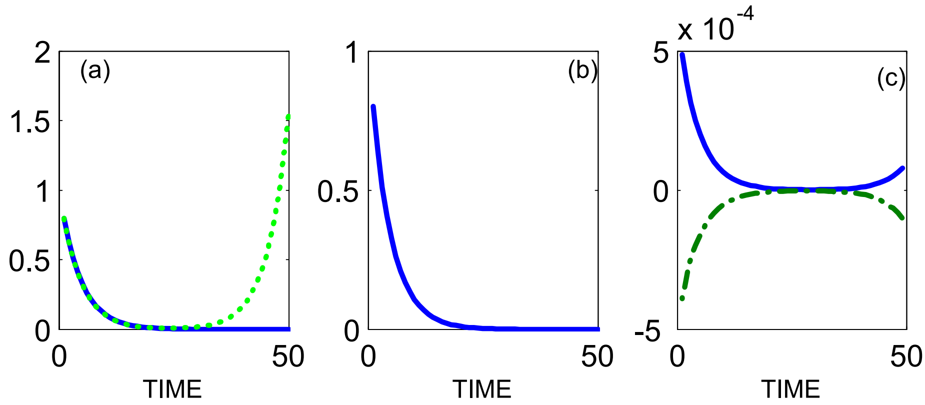

Figure 4.17: (a) Stable solution. $x_1(t)$ (solid curve) to Eq. (4.185) obtained by setting $k^2 = 0.05$, $\mathbf{x}(0) = [0.800, 1.00]^T$. Dashed line is an unstable solution (dotted) obtained by modifying the initial condition to $\mathbf{x}(0) = [0.80001, 1.00]^T$. The growing solution takes awhile to emerge, but eventually swamps the stable branch. (b) Solution obtained by overspecification, in which $\tilde{\mathbf{x}}(0) = [0.80001, 1.00]^T$, $\mathbf{P}(0) = .01\mathbf{I}_2$, $\tilde{\mathbf{x}}(50) = \left[1.4 \times 10^{-5}, 1\right]$, $\mathbf{P}(50) = \mathrm{diag}\left(\left[10^{-4}, 10^{4}\right]\right)$.(c) Lagrange multiplier values used to impose the initial and final conditions on the model. Solid curve is $\mu_1(t)$, and dash-dot is $\mu_2(t)$. {unstable1.eps

Lagrange multipliers (adjoint solution) are also shown in the figure, and imply that the system sensitivity is greatest at the initial and final times. For a full GCM, the technical details are much more intricate, but the principle is not in doubt.

This example can be thought of as the solution to a forward problem, albeit ill-posed, or as the solution to a more or less conventional inverse one. The distinction between forward and inverse problems has nearly vanished. Any forward model that is driven by observed conditions is ill-posed in the sense that there can again be no unique solution, only a most probable one, smoothest one, etc. As with an inverse solution, forward calculations no more produce unique solutions in these circumstances than do inverse ones. All problems involving observed parameters, initial or boundary conditions are necessarily ill-posed.