

12.864 Inference from Data and Models 28 February 2005
Problem Set No. 2 Due: 7 March 2005

1. You have a single vector $\mathbf{v}_1 = [1, 1/2, 1/2, 1]^T$, but need three more vectors in order to have a basis. Starting with \mathbf{v}_1 , construct a complete orthonormal basis set.

2. For the tracer balance model shown on P. 9 of the notes, there are three reservoirs with tracer concentrations $C_i, 1 \leq i \leq 3$, where $C_1 = 1.1045, C_2 = 1.2385, C_3 = 1.2900$. Reservoir 0 is observed to have concentration, $C_0 = 1.2$. The problem is to make some inference about the values of the exchange coefficient ratios, $x_i = J_i/J_0$, given also that mass is conserved ($J_0 = J_1 + J_2 + J_3$).

(a) Find the solution of minimum norm, $J = \mathbf{x}^T \mathbf{x}$ in two ways, one of which uses Lagrange multipliers.

(b) Find the change—by direct calculation—in J when C_0 is changed from 1.2 to 1.18. Compare it to the value inferred from the Lagrange multipliers of part (a).

(c) What would your conclusion be if $C_0 = 1.3$?

(d) Repeat (a), (b) except find the solution minimizing $H = (x_2 - x_1)^2 + 4(x_3 - x_2)^2$.

3. A set of simultaneous equations takes the form,

$$\begin{aligned} 2x - y + z &= 1 \\ 3x + 2y + z &= 0 \end{aligned}$$

(a) Find the least-squares solution of minimum norm.

(b) Find the singular value decomposition of the coefficient matrix (using eigenvalue/eigenvector software, but not a packaged one for the SVD) and use it to find the solution. Is it the same as in (a)? What are solution and data resolution matrices and what do they tell you?

(c) Now consider the simultaneous equations

$$\begin{aligned} 2x + 3y + n_1 &= 1 \\ -x + 2y + n_2 &= 2 \\ x + y + n_3 &= -1 \end{aligned}$$

Using least-squares, find the solution such that $\mathbf{n}^T \mathbf{n}$ is a minimum, where $\mathbf{n} = [n_1, n_2, n_3]^T$. Compare it to the solution obtained from the SVD. Find the solution and data resolution matrices and describe their operation.

(d) Solve both sets of equations, using the SVD, so as to minimize $J = \mathbf{n}^T \mathbf{n} + \alpha^2 \mathbf{x}^T \mathbf{x}$, where $\mathbf{x} = [x, y, z]^T$ for $\alpha^2 = 0.01, 1, 100$ and discuss the relationship to the previous solutions.

5. Now consider the $N \times N$ set of equations,

$$\begin{pmatrix} 2.0354 & -0.20091 & 0.80773 & -1.3911 \\ 0.43550 & 0.66780 & 0.38084 & 0.40634 \\ -0.18426 & 1.1031 & 0.74877 & 1.6839 \\ -0.13230 & 0.18869 & 0.44984 & 1.0696 \end{pmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}.$$

Using an ordinary eigenvalue/eigenvector expansion (not an SVD) try to solve for \mathbf{x} . Is it possible? Compare it to the solution by SVD