3.4 Linear Programming

In a number of important geophysical fluid problems, the objective functions are linear rather than quadratic functions. Fluid property fluxes such as heat—for example, scalar properties, C_i are carried by a fluid flow at rates $\sum C_i x_i$, which are linear functions of $\mathbf{x}$. If one sought the extreme fluxes of C, it would require finding the extremal values of the corresponding linear function. Least squares does not produce useful answers in such problems because linear objective functions achieve their minima or maxima only at plus or minus infinity—unless the elements of $\mathbf{x}$ are bounded. The methods of *linear programming* are generally directed at finding extremal properties of linear objective functions subject to bounding constraints. In general

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terms, such problems can be written as,

$$\text{minimize: } J = \mathbf{c}^T\mathbf{x}\,,$$

$$\mathbf{E}_1\mathbf{x} = \mathbf{y}_1, \tag{3.41}$$
{eq:52001}

$$\mathbf{E}_2\mathbf{x} \geq \mathbf{y}_2, \tag{3.42}$$
{eq:52002}

$$\mathbf{E}_3\mathbf{x} \leq \mathbf{y}_3\,, \tag{3.43}$$
{eq:52003}

$$\mathbf{a} \leq \mathbf{x} \leq \mathbf{b}\,, \tag{3.44}$$
{eq:52004}

that is, as a collection of equality and inequality constraints of both greater than or less than form, plus bounds on the individual elements of $\mathbf{x}$. In distinction to the least squares and minimum variance equations we have hitherto been discussing, these are hard constraints; they cannot be violated at all in an acceptable solution.

Linear programming problems are normally reduced to what is referred to as a *canonical form*, although different authors use different definitions of what it is. But all such problems are reducible to,

$$\text{minimize: } J = \mathbf{c}^T\mathbf{x}\,, \tag{3.45}$$
{eq:52005}

$$\mathbf{E}\mathbf{x} \leq \mathbf{y} \tag{3.46}$$
{eq:52006}

$$\mathbf{x} \geq \mathbf{0}\,. \tag{3.47}$$
{eq:52007}

The use of a minimum rather than a maximum is readily reversed by introducing a minus sign, and the inequality is similarly readily reversed. The last relationship, requiring purely positive elements in $\mathbf{x}$, is obtained without difficulty by simple translation.

Linear programming problems are widespread in many fields including, especially, financial and industrial management where they are used to maximize profits, or minimize costs, in, say, a manufacturing process. Necessarily then, the amount of a product of each type is positive, and the inequalities reflect such things as the need to consume no more than the available amounts of raw materials. In some cases, J is then literally a "cost" function. General methodologies were first developed during World War II in what became known as "operations research" ("operational research" in the U.K.)[78], although special cases were known much earlier. Since then, because of the economic stake in practical use of linear programming, immense effort has been devoted both to textbook discussion and efficient, easy-to-use software.[79] Given this highly accessible literature and software, we will not actually describe the methodologies of solution, but merely make a few general points.

The original solution algorithm invented by G. Dantzig is usually known as the “simplex method” (a simplex is a convex geometric shape). It is a highly efficient search method conducted along the bounding constraints of the problem. In general, it is possible to show that the outcome of a linear programming problem falls into several distinct categories: (1) The system is “infeasible,” meaning that it is contradictory and there is no solution; (2) the system is unbounded, meaning that the minimum lies at negative infinity; (3) there is a unique minimizing solution; and (4) there is a unique finite minimum, but it is achieved by an infinite number of solutions $\mathbf{x}$.

The last situation is equivalent to observing that if there are two minimizing solutions, there must be an infinite number of them because then any linear combination of the two solutions is also a solution. Alternatively, if one makes up a matrix from the coefficients of $\mathbf{x}$ in Equations (3.45)–(3.47), one can ask whether it has a nullspace. If one or more such vectors exists, it is also orthogonal to the objective function, and it can be assigned an arbitrary amplitude without changing J. One distinguishes between *feasible solutions*, meaning those that satisfy the inequality and equality constraints but which are not minimizing, and *optimal solutions*, which are both feasible and minimize the objective function.

An interesting and useful feature of a linear programming problem is that equations (3.45)–(3.47) have a "dual":

{eq:52008}
$$\text{maximize:} \;\; J_2 = \mathbf{y}^T \boldsymbol{\mu}\,, \tag{3.48}$$

{eq:52009}
$$\mathbf{E}^T \boldsymbol{\mu} \geq \mathbf{c} \tag{3.49}$$

{eq:52010}
$$\boldsymbol{\mu} \geq \mathbf{0}\,. \tag{3.50}$$

It is possible to show that the minimum of J must equal the maximum of J_2. The reader may want to compare the structure of the original (the “primal”) and dual equations with those relating the Lagrange multipliers to $\mathbf{x}$ discussed in Chapter 2. In the present case, the important relationship is,

{eq:52011}
$$\frac{\partial J}{\partial y_i} = \mu_i\,. \tag{3.51}$$

That is, in a linear program, the dual solution provides the sensitivity of the objective function to perturbations in the constraint parameters $\mathbf{y}$. Duality theory pervades optimization problems, and the relationship to Lagrange multipliers is no accident.[80] Some simplex algorithms, called the “dual simplex,” take advantage of the different dimensions of the primal and dual problems to accelerate solution. In recent years much attention has focused upon a new, nonsimplex method of solution[81] known as the “Karmackar” or “interior set” method.

Linear programming is also valuable for solving estimation or approximation problems in which norms other than the 2-norms, which have been the focus of this book, are used. For example, suppose that one sought the solution to the constraints $\mathbf{Ex}+\mathbf{n}=\mathbf{y}$, $M>N$, but subject not to the conventional minimum of $J=\sum_i n_i^2$, but that of $J=\sum_i |n_i|$ (a 1-norm). Such norms are less sensitive to outliers than are the 2-norms and are said to be "robust." The maximum likelihood idea connects 2-norms to Gaussian statistics, and similarly, 1-norms are related to maximum likelihood in exponential statistics.[82] Reduction of such problems to linear programming is carried out by setting, $n_i = n_i^+ - n_i^-$, $n_i^+ \geq 0$, $n_i^- \geq 0$, and the objective function is,

$$\min\colon\ J=\sum_i \left(n_i^+ + n_i^-\right) \tag{3.52}$$ {eq:52012}

Other norms, the most important[83] of which is the so-called infinity norm, which minimizes the maximum element of an objective function, are also reducible to linear programming.