1.3 More Examples

e:boxmodel1} *A Tracer Box Model*

In scientific practice, one often has observations of elements of the solution of the differential system or other model. Such situations vary enormously in the complexity and sophistication of both the data and the model. A useful and interesting example of a simple system, with applications in many fields, is one in which there is a large reservoir (Figure 1.2) connected to a number of source regions which provide fluid to the reservoir. One would like to determine the rate of mass transfer from each source region to the reservoir. getracerfig}

We suppose that some chemical tracer or dye, C_0 is measured in the reservoir, and that the concentrations of the dye, C_i, in each source region are known. Let the unknown transfer rates be J_{i0} (transfer from source i to reservoir 0). Then we must have,

{tracer1}

$$C_1 J_{10} + C_2 J_{20} + ... + C_N J_{N0} = C_0 J_{0\infty}, \tag{1.18}$$

which says that for a steady-state, the rate of transfer in, must equal the rate of transfer out (written $J_{0\infty}$). To conserve mass,

{tracer2}

$$J_{10} + J_{20} + ... + J_{N0} = J_{0\infty}. \tag{1.19}$$

This model has produced for us two equations in $N+1$ unknowns, $[J_{10,} J_{20}, ...J_{N0}, J_{0\infty}]$ which evidently is insufficient information if $N > 1$. The equations have also been written as though

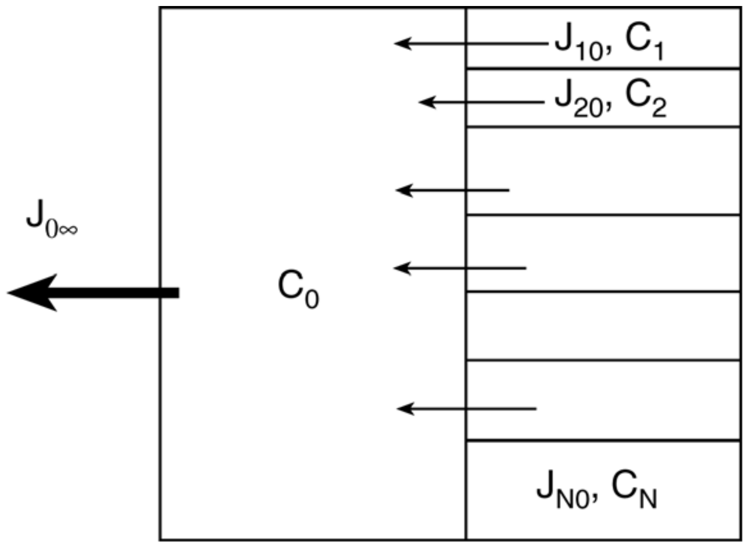

Figure 1.2: A simple reservoir problem in which there are multiple sources of flow, at rates J_{i0}, each carrying an identifiable property C_i, perhaps a chemical concentration. In the forward problem, given J_{i0}, C_i one could calculate C_0. One form of inverse problem provides C_0 and the C_i and seeks the values of J_{i0}.

{reserv1.tif}

everything were perfect. If, for example, the tracer concentrations C_i were measured with finite precision and accuracy (they always are), one might try to accommodate the resulting inaccuracy as,

$$C_1 J_{10} + C_2 J_{20} + ... + C_N J_{N0} + n = C_0 J_{0\infty} \tag{1.20}$$ {tracer3}

where n represents the resulting error in the equation. Its introduction, of course, produces another unknown. If the reservoir were capable of some degree of storage or fluctuation in level, one might want to introduce an error term into (1.19) as well. One should also notice, that as formulated, one of the apparently infinite number of solutions to Eqs. (6.1, 1.19) includes $J_{i0} = J_{0\infty} = 0$—no flow at all. More information is required if this null solution is to be excluded.

To make the problem slightly more interesting, suppose that the tracer C is radioactive, and decays with a decay constant λ. Eq. (6.1) becomes

$$C_1 J_{10} + C_2 J_{20} + ... + C_N J_{N0} - C_0 J_{0\infty} = -\lambda C_0 \tag{1.21}$$

Now if $C_0 > 0$, the zero solution for J_{ij} is no longer possible, but we still have many more unknowns than equations. These equations are once again in the canonical linear form $\mathbf{Ax} = \mathbf{b}$.

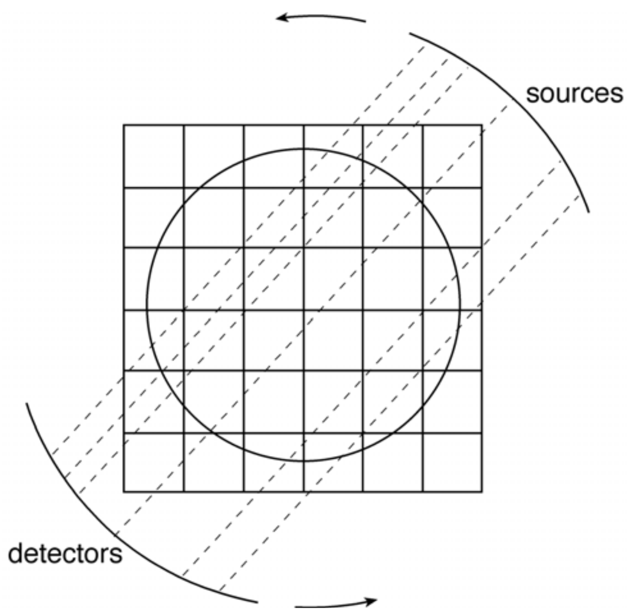

Figure 1.3: Generic tomographic problem in two dimensions. Measurements are made by integrating through an otherwise impenetrable solid between the transmitting sources and receivers using x-rays, sound, radio waves, etc. Properties can be anything measurable, including travel times, intensities, group velocities etc. The tomographic problem is to reconstruct the interior from these integrals. In the particular configuration shown, the source and receiver are supposed to revolve so that a very large number of paths can be built up. It is also supposed that the division into small rectangles is an adequate representation. In principle, one can have many more integrals than the number of squares defining the unknowns. {tomog1.tif}

A Tomographic Problem

So-called tomographic problems occur in many fields, most notably in medicine, but also in materials testing, oceanography, meteorology and geophysics. Generically, they arise when one is faced with the problem of inferring the distribution of properties inside an area or volume based upon a series of integrals through the region. Consider Fig. 1.3., where to be specific, suppose we are looking at the top of the head of a patient lying supine in a so-called CAT-scanner. The two external shell sectors represent in (a) a source of x-rays and, in (b) a set of x-ray detectors. X-rays are emitted from the source and travel through the patient along the indicated lines where the intensity of the received beam is measured. Let the absorptivity/unit length within the patient be a function, $c(\mathbf{r})$, where $\mathbf{r}$ is the vector position within the patient's head. Consider one source at $\mathbf{r}_s$ and a receptor at $\mathbf{r}_e$ connected by the path as indicated. Then the intensity measured at the receptor is,

$$I(\mathbf{r}_s, \mathbf{r}_r) = \int_{\mathbf{r}_s}^{\mathbf{r}_e} c(\mathbf{r}(s))\, ds, \qquad (1.22)$$ {tomog1}

where s is the arc-length along the path. The basic tomographic problem is to determine $c(\mathbf{r})$ for all $\mathbf{r}$ in the patient, from measurements of I. In the medical problem, the shell sectors rotate around the patient, and an enormous number of integrals along (almost) all possible paths are obtained. An analytical solution to this problem, as the number of paths becomes infinite, is produced by the Radon transform.[4] Given that tumors and the like have a different absorptivity than does normal tissue, the reconstructed image of $c(\mathbf{r})$ permits physicians to "see" inside the patient. In most other situations, however, the number of paths tends to be much smaller than the formal number of unknowns and other solution methods must be found.

Note first, however, that we should modify Eq. (1.22) to reflect the inability of any system to produce a perfect measurement of the integral, and so more realistically we write,

$$I(\mathbf{r}_s, \mathbf{r}_r) = \int_{\mathbf{r}_s}^{\mathbf{r}_e} c(\mathbf{r}(s))\, ds + n(\mathbf{r}_s, \mathbf{r}_r), \qquad (1.23)$$ {tomog2}

where n is the measurement noise.

To proceed, surround the patient with a bounding square (Fig. 1.4)—simply to produce a simple geometry—and divide the area into sub-squares as indicated, each numbered in sequence, $1 \leq j \leq N$. These squares are supposed sufficiently small that $c(\mathbf{r})$ is effectively constant within them. Also number the paths, $1 \leq i \leq M$. Then Eq. (1.23) can be approximated with arbitrary accuracy (by letting the sub-square dimensions become arbitrarily small) as,

$$I_i = \sum_{j=1}^{N} c_j \Delta r_{ij} + n_i. \qquad (1.24)$$ {tomog3}

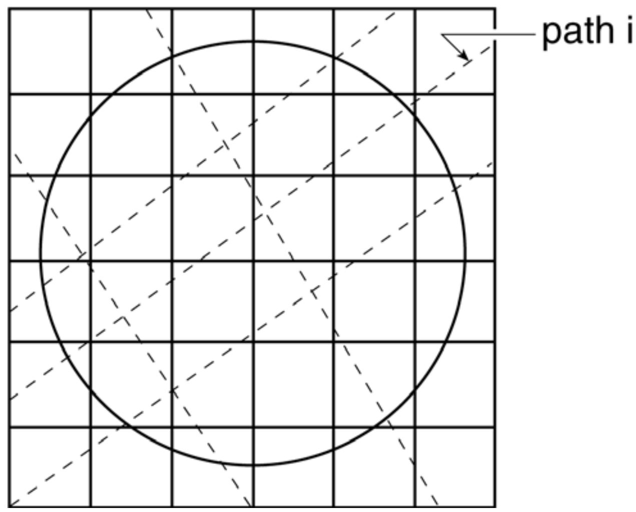

Figure 1.4: Simplified geometry for defining a tomographic problem. Some squares may have no integrals passing through them; others may be multiply-covered. Boxes outside the physical body can be handled in a number of ways, including the addition of constraints setting the corresponding $c_j = 0$. {tomog2.tif}

Here Δr_{ij} is the arc length of path i within square j (most of them will vanish for any particular path). Once again, these last equations are of the form

$$\mathbf{Ex} + \mathbf{n} = \mathbf{y}, \tag{1.25}$$ {canon1}

where here, $\mathbf{E} = \{\Delta r_{ij}\}$, $\mathbf{x} = [c_j]$, $\mathbf{n} = [n_i]$. Quite commonly there are many more unknown c_j than there are integrals I_i. (In the present context, there is no distinction between writing matrices $\mathbf{A}, \mathbf{E}$. $\mathbf{E}$ will generally be used where noise elements are present, and $\mathbf{A}$ where none are intended.)

Tomographic measurements do not always consist of x-ray intensities. In seismology or oceanography, for example, c_j is commonly $1/v_j$ where v_j is the speed of sound or seismic waves within the area; I is then a travel time rather than an intensity. The equations remain the same, however. This methodology also works in three-dimensions, the paths need not be straight lines and there are many generalizations.[5] A problem of great practical importance is determining what one can say about the solutions to Eqs. (4.34) even where many more unknowns exist than formal pieces of information y_i.

As with all these problems, many other forms of discretization are possible. For example,

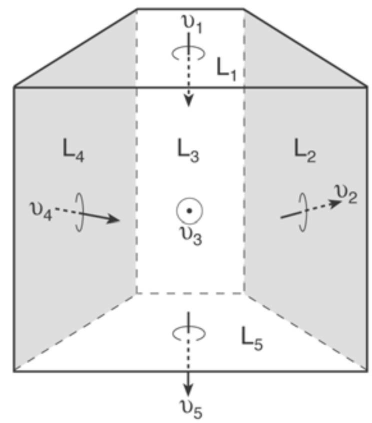

Figure 1.5: Volume of fluid bounded on four open sides across which fluid is supposed to flow. Mass is conserved, giving one relationship among the fluid transports v_i; conservation of one or more other tracers C_i leads to additional useful relationships. {track1.tif}

the continuous function $c(\mathbf{r})$ can be expanded,

$$c(\mathbf{r}) = \sum_q \sum_p a_{nm} T_n(r_x) T_m(r_y), \tag{1.26}$$

where $\mathbf{r} = (r_x, r_y)$, and the T_n are any suitable expansion functions (sines and cosines, Chebyschev polynomials, etc.). The linear equations (4.34) then represent constraints leading to the determination of the a_{nm}.

A Second Tracer Problem

Consider the closed volume in Fig. 1.5 enclosed by four boundaries as shown. There are steady flows, $v_i(z)$, $1 \le i \le 4$ either into or out of the volume, each carrying a corresponding fluid of constant density ρ_0. z is the vertical coordinate. If the width of each boundary is l_i, the statement that mass is conserved within the volume is simply,

$$\sum_{i=1}^{r} l_i \rho_0 \int_{-h}^{0} v_i(z)\, dz = 0, \tag{1.27}$$ {box1}

where the convention is made that flows into the box are positive, and flows out are negative. $z = -h$ is the lower boundary of the volume and $z = 0$ is the top one. If the v_i are unknown, Eq. (1.27) represents one equation (constraint) in four unknowns,

$$\int_{-h}^{0} v_i(z)\, dz,\ 1 \le i \le 4. \tag{1.28}$$

One possible, if boring, solution is $v_i(z) = 0$. To make the problem somewhat more interesting, we now suppose that for some mysterious reason, the vertical derivatives, $v_i'(z) = dv_i(z)/dz$, are known so that,

$$v_i(z) = \int_{-z_o}^{z} v_i'(z)\,dz + b_i(z_0), \tag{1.29}$$

where z_0 is a convenient place to start the integration (but can be any value). b_i are integration constants ($b_i = v_i(z_0)$) which remain unknown. Constraint (1.27) becomes,

$$\sum_{i=1}^{4} l_i \rho_0 \int_{-h}^{0} \left[\int_{-z_o}^{z} v_i'(z')\,dz' + b_i(z_0) \right] dz = 0, \tag{1.30}$$

or,

{box2}
$$\sum_{i=1}^{4} h l_i b_i(z_0) = -\sum_{i=1}^{4} l_i \int_{-h}^{0} dz \int_{-z_o}^{z} v_i'(z')\,dz' \tag{1.31}$$

where the right-hand side is known. Eq. (1.31) is still one equation in four unknown b_i, but the zero-solution is no longer possible, unless the right-hand side vanishes. Eq. (1.31) is a statement that the weighted average of the b_i on the left-hand-side is known. If one seeks to obtain estimates of the b_i separately, more information is required.

Suppose that information pertains to a tracer, perhaps a red-dye, known to be conservative, and that the box concentration of red-dye, C, is known to be in a steady-state. Then conservation of C becomes,

{box3}
$$\sum_{i=1}^{4} \left[h l_i \int_{-h}^{0} C_i(z)\,dz \right] b_i = -\sum_{i=1}^{4} l_i \int_{-h}^{0} dz \int_{-z_o}^{z} C_i(z')\, v_i'(z')\,dz', \tag{1.32}$$

where $C_i(z)$ is the concentration of red-dye on each boundary. Eq. (1.32) provides a second relationship for the four unknown b_i. One might try to measure another dye concentration, perhaps green dye, and write an equation for this second tracer, exactly analogous to (1.32). With enough such dye measurements, one might obtain more constraint equations than unknown b_i. In any case, no matter how many dyes are measured, the resulting equation set is of the form (1.9). The number of boundaries is not limited to four, but can be either fewer, or many more.[6]

Vibrating String

Consider a uniform vibrating string anchored at its ends $r_x = 0$, $r_x = L$. The free motion of the string is governed by the wave equation

$$\frac{\partial^2 \eta}{\partial r_x^2} - \frac{1}{c^2}\frac{\partial^2 \eta}{\partial t^2} = 0,\ c^2 = T/\rho, \tag{1.33}$$

where T is the tension, and ρ the density. Free modes of vibration (eigen-frequencies) are found to exist at discrete frequencies, s_q,

{vibrate2}
$$2\pi s_q = \frac{q\pi c}{L},\ q = 1, 2, 3, ..., \tag{1.34}$$

and which is the solution to a classical forward problem. A number of interesting and useful inverse problems can be formulated. For example, given $s_q \pm \Delta s_q$, $1 \leq q \leq M$, to determine L, or c. These are particularly simple problems, because there is only one parameter, either c or L to determine. More generally, it is obvious from Eq. (1.34) that one has information only about the ratio c/L—they could not be determined separately.

Suppose, however, that the density varies along the string, $\rho = \rho(r_x)$, so that $c = c(r_x)$. Then (it may be confirmed) that the observed frequencies are no longer given by Eq. (1.34), but by expressions involving the integral of c over the length of the string. An important problem is then to infer $c(r_x)$, and hence $\rho(r_x)$. One might wonder whether, under these new circumstances, L can be determined independently of c?

A host of such problems, in which the observed frequencies of free modes are used to infer properties of media in one to three dimensions exists. The most elaborate applications are in geophysics and solar physics, where the normal mode frequencies of the vibrating whole earth or sun are used to infer the interior properties for the earth (density and elastic parameters).[7] A good exercise is to render the spatially variable string problem in discrete form.