## 5.4 Reduced State Methods

The computational load of the Kalman filter and smoothers grows approximately as the cube of the state vector dimension. Thus either decoupling the problem into several smaller problems, or removing elements of the state vector, can have a very large payback in terms of the computational load reduction. (If one could solve the problem as two $(n/2)^3$ problems rather than as one-$n^3$ problem the difference in load is a factor of four.) One method for solving large fluid state problems is based upon the assumption that large spatial scales in a fluid flow evolve largely independent of small scales, and that it is the largest scales that are of primary interest.[166] Let $\mathbf{D}$ be a matrix operator that has the effect of averaging a vector spatially, so that $\mathbf{x}'(t) = \mathbf{D}\mathbf{x}(t)$ is a spatial average of $\mathbf{x}(t)$, with an equivalent reduced dimension, $N'$. (We refer to the "coarse-state" and "fine-state".) Then if $\mathbf{P}(t)$ is the error covariance of $\mathbf{x}(t)$,

$$\left\langle \left(\tilde{\mathbf{x}}' - \mathbf{x}'\right)\left(\tilde{\mathbf{x}}' - \mathbf{x}'\right)^T \right\rangle = \mathbf{P}'(t) = \mathbf{D}\mathbf{P}(t)\mathbf{D}^T.$$

will be of dimension $N' \times N'$ rather than $N \times N$. Now assume further that $\mathbf{D}$ has a left-inverse, $\mathbf{D}^+$, as described in Chapter 2, that would map the coarse state to the finer one. Suppose further that one has a coarse resolution model capable of propagating $\mathbf{x}'$. This model might be obtained from the fine-resolution model:

$$\begin{aligned} \mathbf{D}\mathbf{x}(t+1) &= \mathbf{D}\mathbf{A}(t)\mathbf{D}^+\mathbf{x}'(t) + \mathbf{D}\mathbf{B}\mathbf{D}^+\mathbf{u}'(t) + \mathbf{D}\Gamma\mathbf{D}^+\mathbf{q}'(t) \\ \text{or,} \quad \mathbf{x}'(t+1) &= \mathbf{A}'\mathbf{x}'(t) + \mathbf{B}'\mathbf{u}'(t), \end{aligned}$$

where $\mathbf{u}'(t) = \mathbf{D}\mathbf{u}(t), \mathbf{A}' = \mathbf{D}\mathbf{A}(t)\mathbf{D}^+$, $\mathbf{B}' = \mathbf{D}\mathbf{B}\mathbf{D}^+$, etc. Then the Kalman filter (and smoother) can be applied to $\mathbf{x}'(t)$ and the filtered data, $\mathbf{D}\mathbf{y}(t)$. One can estimate,

$$\tilde{\mathbf{x}}(t) = \mathbf{D}^+\mathbf{x}'(t),$$

and

$$\mathbf{P}(t) = \mathbf{D}^{+}\mathbf{P}'(t)\,\mathbf{D}^{+T.} \tag{5.11}$$ {reduced1}

Given $\mathbf{P}(t)$ one has $\mathbf{K}(t)$ for the fine-state, under the assumption that Eq. (5.11), based wholly upon the large-scales, is adequate. One can put any small scales in the fine-state observations into the data error of the coarse-state. A further reduction in computational load can be made by assuming a steady-state for $\mathbf{P}'(t), \mathbf{P}(t)$, and finding it using the doubling algorithm. In Chapter 7 we will describe an application of this method. The main issue with its general validity would lie with the assumption that errors in the fine state do not strongly influence the error budget of the coarse-state. This assumption cannot in general be correct (spatially averaged equations of fluid motion are not proper representations of the equations governing the averaged fields), and one must carefully assess the behavior of the algorithm as it evolves.

Determination of $\mathbf{D}$, $\mathbf{D}^{+}$ is important. In principle, the Gauss-Markov mapping procedures, as described in Chapter 2, would be appropriate (and would include error estimates should one choose to use them). Various strategies for reducing storage and computation are available.[167]

*Other approaches to state reduction.* The Eckart-Young-Mirsky Theorem, described in Chapter 2, shows that sometimes a comparatively small number of singular vectors can represent a field with considerable accuracy. Here "small" is measured relative to the number of grid points or basis functions used by the underlying model.[168] Suppose that the state vector $\mathbf{x}(t) = \mathbf{V}\boldsymbol{\alpha}(t)$, where $\mathbf{V}$ is the matrix of $\mathbf{v}_i$, the singular vectors of a large span of model—that is, the matrix to which the Eckart-Young-Mirsky theorem is applied is $\left[\begin{array}{cccc} \mathbf{x}(0) & \mathbf{x}(2) & . & \mathbf{x}(t_N) \end{array}\right]$—and then truncated to some acceptable sub-set,

$$\mathbf{x}(t) \approx \mathbf{V}_K \alpha(t).$$

Taking the canonical, full, model,

$$\mathbf{V}_K^T \mathbf{x}(t+1) = \mathbf{V}_K^T \mathbf{A}(t)\,\mathbf{V}_K \alpha(t) + \mathbf{V}_K^T \mathbf{B}\mathbf{V}_K \mathbf{u}(t) + \mathbf{V}_K^T \mathbf{\Gamma}\mathbf{V}_K \mathbf{q}(t)$$

or,

$$\boldsymbol{\alpha}(t+1) = \mathbf{A}'(t)\,\alpha(t) + \mathbf{B}'\mathbf{u}(t) + \Gamma'\mathbf{q}(t),$$

where, $\mathbf{A}'(t) = \mathbf{V}_K^T \mathbf{A}(t)\,\mathbf{V}_K$, $\mathbf{B}' = \mathbf{V}_K^T \mathbf{B}\mathbf{V}_K$, etc. is an evolution equation for the new state vector $\boldsymbol{\alpha}(t)$ whose dimension is $K << N$. (If $\mathbf{A}$ is time-independent, an alternative is to diagonalize it in the canonical equation by using its singular vector decomposition.[169]) Then each mode can be handled independently. As with the coarse-to-fine resolution transformation, one is assuming that errors in the suppressed singular vectors (those banished to the nullspace) do not significantly affect the errors of those retained. One must be vigilant, and test the original assumptions against the solution obtained.