5.6 Non-normal Systems

Consider a forward model,

$$\mathbf{x}(t) = \mathbf{A}\mathbf{x}(t-1), \tag{5.12}$$ {nonn1}

with t again an integer. In general, the underlying physics will fail to be self-adjoint and hence $\mathbf{A}$ will be non-normal, that is, $\mathbf{A} \neq \mathbf{A}^T$. We suppose the system is unforced, but is started with initial conditions $\mathbf{x}(0)$ which are a realization of white noise with variance σ^2. Thus at time t

$$\mathbf{x}(t) = \mathbf{A}^t\mathbf{x}(0) \tag{5.13}$$

Recalling the discussion in Chapter 3, P. 161, it follows immediately that the eigenvalues and right eigenvectors of $\mathbf{A}^t$ satisfy,

$$\mathbf{A}^t\mathbf{g}_i = \lambda_i^t\mathbf{g}_i. \tag{5.14}$$

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Expanding $\mathbf{x}(0)$ in the right eigenvectors of $\mathbf{A}$,

$$\mathbf{x}(0) = \sum_{i=1}^{N} \alpha_i(0)\,\mathbf{g}_i \tag{5.15}$$

and,

$$\mathbf{x}(t) = \sum_{i=1}^{N} \lambda_i^t \alpha_i(0)\,\mathbf{g}_i \tag{5.16}$$

Stability of the model demands that all $|\lambda_i| \leq 1$. But the lack of orthogonality of the $\mathbf{g}_i$ means that some of the α_i may be very large, despite the white noise properties of $\mathbf{x}(0)$. This result implies that some elements of $\mathbf{x}(t)$ can become very large, even though the limit $\lambda_i^t \to 0$, $t \to \infty$ means that they are actually transients. To an onlooker, the large response of the system to a bounded initial disturbance may make the system look unstable. Furthermore, the disturbance may become so large that the system becomes non-linear, and possibly non-linearly unstable.[172] That is, stable fluid systems may well appear to be unstable owing to the rapid growth of transients, or linearly stable systems may become unstable in the finite amplitude sense if the transients of the linearized system become large enough.

Now consider the forced situation with time-independent $\mathbf{A}$,

{nonnmod2}
$$\mathbf{x}(t) = \mathbf{A}\mathbf{x}(t-1) + \mathbf{q}(t-1)\,. \tag{5.17}$$

Take the Fourier transform of the difference equation (5.17), using the result[173] that if the transform of $\mathbf{x}(t)$ is,

$$\hat{\mathbf{x}}(s) = \sum_{t=0}^{\infty} \mathbf{x}(t)\,e^{-2\pi i s t}, \tag{5.18}$$

then the transform of $\mathbf{x}(t-1)$ is $e^{-2\pi i s}\hat{\mathbf{x}}(s)$. Solving for $\hat{\mathbf{x}}(s)$, we obtain

{xhat}
$$\hat{\mathbf{x}}(s) = \left(e^{-2\pi i s}\mathbf{I} - \mathbf{A}\right)^{-1}\hat{\mathbf{q}}(s)\,. \tag{5.19}$$

We will call $\left(e^{-2\pi i s}\mathbf{I} - \mathbf{A}\right)^{-1}$, the "resolvent" of $\mathbf{A}$, in analogy to the continuous case terminology of functional analysis.[174] If the resolvent is infinite for real values of $s = s_i$ it implies $\hat{\mathbf{x}}(s_i)$ is an eigenvector of $\mathbf{A}$ and an ordinary resonance is possible. For the mass-spring oscillator of Chapter 2, the complex eigenvalues of $\mathbf{A}$ produce $s_{1,2} = \pm 0.0507 + 0.0008i$, and the damped oscillator has no true resonance. Should any eigenvalue have a negative imaginary part, leading to $\left|e^{-2\pi i s_i t}\right| > 1$, the system would be unstable.

Define $z = e^{-2\pi i s}$, to be interpreted as an analytic continuation of s into the complex plane. The unit circle $|z| = 1$ defines the locus of real frequencies. The gist of the discussion of what are called "pseudo-spectra" is the possibility that the norm of the resolvent $\left\|(z\mathbf{I} - \mathbf{A})^{-1}\right\|$ may become very large, but still finite, on $|z| = 1$ without there being either instability or resonance, giving the illusion of linear instability.

5.6.1 POPs and Optimal Modes

For any linear model in canonical form, the right eigenvectors of $\mathbf{A}$ can be used directly to represent fluid motions,[175] as an alternative e.g., to the singular vectors (EOFs). These eigenvectors were called "principal oscillation patterns," or POPs by K. Hasselmann. Because $\mathbf{A}$ is usually not symmetric (not self-adjoint), the eigenvalues are usually complex, and there is no guarantee that the eigenvectors are a spanning set. But assuming that they provide an adequate expansion basis—usually tested by trying them—the right eigenvectors are used in pairs when there are complex conjugate eigenvalues. The expansion coefficients of the time-evolving field are readily shown to be the eigenvectors of $\mathbf{A}^T$—that is, the eigenvectors of the adjoint model. Assuming that the eigenvectors are not grossly deficient as a basis, and/or one is interested in only a few dominant modes of motion, the POP approach gives a reasonably efficient representation of the field.

Alternatively, even when $\mathbf{A} \neq \mathbf{A}^T$, it always has an SVD and one can try to use the singular vectors of $\mathbf{A}$—directly—to represent the time evolving field. The complication is that successive multiplications by non-symmetric $\mathbf{A}$ transfers the projection from the $\mathbf{U}$ vectors to the $\mathbf{V}$ vectors and back again. Write $\mathbf{A} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{V}^T$ and assume, as is normally true of a model, that it is full rank $K = N$ and $\boldsymbol{\Lambda}$ is square. Using Eqs. (4.97, 4.99), in the absence of observations,

$$\mathbf{x}(t) = \mathbf{A}\mathbf{x}(t-1), \tag{5.20}$$

$$\boldsymbol{\mu}(t-1) = \mathbf{A}^T\boldsymbol{\mu}(t), \tag{5.21}$$

one can always write,

$$\mathbf{x}(t) = \mathbf{V}\boldsymbol{\alpha}(t), \tag{5.22}$$ {69004}

where a is a set of vector coefficients. Write the adjoint solution as,

$$\boldsymbol{\mu}(t) = \mathbf{U}\boldsymbol{\beta}(t). \tag{5.23}$$ {69005}

Multiply (5.20) by $\boldsymbol{\mu}(t-1)^T$, and (5.21) by $\mathbf{x}(t)^T$ and subtracting,

$$\boldsymbol{\mu}(t-1)^T\mathbf{A}\mathbf{x}(t-1) = \mathbf{x}(t)^T\mathbf{A}^T\boldsymbol{\mu}(t) = \boldsymbol{\mu}(t)^T\mathbf{A}\mathbf{x}(t), \tag{5.24}$$ {en1}

or using (5.22, 5.23),

$$\boldsymbol{\beta}(t-1)^T\boldsymbol{\Lambda}\boldsymbol{\alpha}(t-1) = \boldsymbol{\alpha}(t)^T\boldsymbol{\Lambda}\boldsymbol{\beta}(t)^T, \tag{5.25}$$ {69006}

which can be interpreted as an energy conservation principle, summed over modes, if the $\mathbf{u}_i$, $\mathbf{v}_i$ are regarded as eigenmodes of the model.

Assume $\|\mathbf{A}\| < 1$ so that the system is fully stable. We can ask what disturbance of unit magnitude at time $t-1$, say, would lead to the largest magnitude of $\mathbf{x}(t)$? That is, we maximize

$\|\mathbf{A}\mathbf{q}(t-1)\|$ subject to $\|\mathbf{q}(t-1)\| = 1$. This requirement is equivalent to solving the constrained maximization problem for the stationary values of,

$$J = \mathbf{q}(t-1)^T \mathbf{A}^T \mathbf{A}\mathbf{q}(t-1) - 2\mu\left(\mathbf{q}(t-1)^T \mathbf{q}(t-1) - 1\right), \tag{5.26}$$

where μ is a scalar Lagrange multiplier, and which leads to the normal equations,

{nneig1}
$$\mathbf{A}^T \mathbf{A}\mathbf{q}(t-1) = \mu\mathbf{q}(t-1), \tag{5.27}$$

{nneig2}
$$\mathbf{q}(t-1)^T \mathbf{q}(t-1) = 1 \tag{5.28}$$

Eq. (5.27) shows that the solution is $\mathbf{q}(t-1) = \mathbf{v}_1$, $\mu = \lambda_1^2$ the first singular vector and value of $\mathbf{A}$ with (5.28) automatically satisfied. the particular choice of μ assures we obtain a maximum rather than a minimum. With $\mathbf{q}(t-1)$ proportional to the $\mathbf{v}_1$ singular vector of $\mathbf{A}$, it maximizes the growth rate of $\mathbf{x}(t)$.[176] The initial response would be just $\mathbf{u}_1$, the corresponding singular vector. If the time step is very small compared to the growth rates of model structures, the analysis can be applied instead to $\mathbf{A}^{t_1}$, that is, the transition matrix after t_1 time steps. The next largest singular value will give the second fastest growing mode, etc.