4.6 Controllability and Observability

In addition to determining whether there exists a steady-state solution either to the control or estimation Riccati equations, there are many reasons for examining in some detail the existence of many of the matrix operations that have been employed routinely. Matrix inverses occur throughout the developments above, and the issue of whether they exist has been ignored. Ultimately, however, one must face up to questions of whether the computations are actually possible. The questions are intimately connected to some very useful structural descriptions of models and data that we will now examine briefly.

Controllability

Consider the question of whether controls can be found to drive a system from a given initial state $\mathbf{x}(0)$ to an arbitrary $\mathbf{x}(t_f)$. If the answer is "yes," the system is said to be *controllable*. To find an answer, consider for simplicity,[134] a model with $\mathbf{B} = \mathbf{0}$ and with the control, u, a scalar.

Then the model time steps can be written

$$
\begin{aligned}
\mathbf{x}(1) &= \mathbf{A}\mathbf{x}(0) + \mathbf{\Gamma} u(0) \\
\mathbf{x}(2) &= \mathbf{A}\mathbf{x}(1) + \mathbf{\Gamma} u(1) \\
&= \mathbf{A}^2\mathbf{x}(0) + \mathbf{A}\mathbf{\Gamma} u(0) + \mathbf{\Gamma} u(1) \\
&\vdots \\
\mathbf{x}(t_f) &= \mathbf{A}^{t_f}\mathbf{x}(0) + \sum_{j=0}^{t_f-1} \mathbf{A}^{t_f-1-j}\mathbf{\Gamma} u(j) \\
&= \mathbf{A}^{t_f}\mathbf{x}(0) + \left[\mathbf{\Gamma}\ \ \mathbf{A}\mathbf{\Gamma}\cdots\mathbf{A}^{t_f-1}\mathbf{\Gamma}\right]\begin{bmatrix} u(t_f-1) \\ \vdots \\ u(0) \end{bmatrix}.
\end{aligned}
$$

To determine $u(t)$, we must be able to solve the system

{65002}
$$
\left[\mathbf{\Gamma}\ \ \mathbf{A}\mathbf{\Gamma}\cdots\mathbf{A}^{t_f-1}\mathbf{\Gamma}\right]\begin{bmatrix} u(t_f-1) \\ \vdots \\ u(0) \end{bmatrix} = \mathbf{x}(t_f) - \mathbf{A}^{t_f}\mathbf{x}(0)\,, \tag{4.151}
$$

or

$$
\mathbf{C}\mathbf{u} = \mathbf{x}(t_f) - \mathbf{A}^{t_f}\mathbf{x}(0)
$$

for $u(t)$. The state vector dimension is N; therefore the dimension of $\mathbf{C}$ is N by the number of columns, t_f (a special case—with $u(t)$ being scalar, $\mathbf{\Gamma}$ is $N \times 1$). Therefore, Equation (4.151) has no (ordinary) solution if t_f is less than N. If $t_f = N$ and $\mathbf{C}$ is nonsingular—that is, of rank N—there is a unique solution, and the system is controllable. If the dimensions of $\mathbf{C}$ are nonsquare, one could have a discussion, familiar from Chapter 2, of solutions for $u(t)$ with nullspaces present. If $t_f < N$, there is a nullspace of the desired output, and the system would not be controllable. If $t_f > N$, then there will still be a nullspace of the desired output, unless the rank is N, when $t_f = N$, and the system is controllable. The test can therefore be restricted to this last case.

This concept of controllability can be described in a number of interesting and useful ways[135] and generalized to vector controls and time-dependent models. To the extent that a model is found to be uncontrollable, it shows that some elements of the state vector are not connected to the controls, and one might ask why this is so and whether the model cannot then be usefully simplified.

Observability

The concept of "observability" is connected to the question of whether given N perfect observations, it is possible to infer all of the initial conditions. Suppose that the same model is used, and that we have (for simplicity only) a scalar observation sequence,

$$y(t) = \mathbf{E}(t)\mathbf{x}(t) + n(t)\,, \quad 0 \leq t \leq t_f\,. \tag{4.152}$$ {65003}

Can we find $\mathbf{x}(0)$? The sequence of observations can be written, with $u(t) \equiv 0$, as

$$\begin{aligned} y(1) &= \mathbf{E}(1)\mathbf{x}(1) = \mathbf{E}(1)\mathbf{A}\mathbf{x}(0) \\ &\vdots \\ y(t_f) &= \mathbf{E}(t_f)\mathbf{A}^{t_f}\mathbf{x}(0)\,, \end{aligned}$$

which is,

$$\begin{gathered} \mathbf{O}\mathbf{x}(0) = \begin{bmatrix} y\,(1) & \dots & y\,(t_f) \end{bmatrix}^T \\ \mathbf{O} = \begin{Bmatrix} \mathbf{E}(1)\mathbf{A} \\ \vdots \\ \mathbf{E}(t_f)\mathbf{A}^{t_f} \end{Bmatrix}. \end{gathered} \tag{4.153}$$ {65005}

If the "observability matrix" is square—that is, $t_f = N$ and is full rank—there is a unique solution for $\mathbf{x}(0)$, and the system is said to be observable. Should it fail to be observable, it suggests that at least some of the initial conditions are not determinable by an observation sequence and are irrelevant. Determining why that should be would surely shed light on the model one was using. As with controllability, the test (4.153) can be rewritten in a number of ways, and the concept can be extended to more complicated systems. The concepts of 'stabilizability," "reachability," "reconstructability," and "detectability" are closely related.[136], and there is a close connection between observability and controllability and the existence of a steady-state solution for the algebraic Riccati equations.

In practice, one must distinguish between mathematical observability and controllability and practical limitations imposed by the realities of observational systems. It is characteristic of fluids that changes occurring in some region at a particular time are ultimately communicated to all locations, no matter how remote, at later times, although the delay may be considerable, and the magnitudes of the signal may be much reduced by dissipation and geometrical spreading.

Nonetheless, one anticipates that there is almost no possible element of a fluid flow, no matter how distant from a particular observation, that is not in principle observable. A bit more is said about this subject in Chapter 5.