2.12 Appendix 3 Recursive Least-Squares and Gauss-Markov Solutions

The recursive least-squares solution Eq. (2.425) is appealingly simple. Unfortunately, obtaining it from the concatenated least-squares form (Eq. 2.424),

$$\begin{aligned}\tilde{\mathbf{x}}(2) &= \left\{\mathbf{E}(1)^T\mathbf{R}_{nn}(1)^{-1}\mathbf{E}(1)+\mathbf{E}(2)^T\mathbf{R}_{nn}(2)^{-1}\mathbf{E}(2)\right\}\times \\ &\quad\left\{\mathbf{E}(1)^T\mathbf{R}_{nn}(1)^{-1}\mathbf{y}(1)+\mathbf{E}(2)^T\mathbf{R}_{nn}(2)^{-1}\mathbf{y}\right\}\end{aligned}$$

is not easy at all. First note that

$$\begin{aligned}\tilde{\mathbf{x}}(1) &= \left[\mathbf{E}(1)^T\mathbf{R}_{nn}(1)^{-1}\mathbf{E}(1)\right]^{-1}\mathbf{E}(1)^T\mathbf{R}_{nn}(1)^{-1}\mathbf{y}(1) \\ &= \mathbf{P}(1)\,\mathbf{E}(1)^T\mathbf{R}_{nn}(1)^{-1}\mathbf{y}(1)\,,\end{aligned} \tag{2.461}$$

where,

$$\mathbf{P}(1) = \left[\mathbf{E}(1)^T\mathbf{R}_{nn}(1)^{-1}\mathbf{E}(1)\right]^{-1},$$

are the solution and uncertainty of the overdetermined system from the first set of observations alone. Then

$$\tilde{\mathbf{x}}(2) = \left\{\mathbf{P}(1)^{-1} + \mathbf{E}(2)^T\mathbf{R}_{nn}(2)^{-1}\mathbf{E}(2)\right\}^{-1} \times \left\{\mathbf{E}(1)^T\mathbf{R}_{nn}(1)^{-1}\mathbf{y}(1) + \mathbf{E}(2)^T\mathbf{R}_{nn}(2)^{-1}\mathbf{y}(2)\right\}.$$

Now we apply the matrix inversion lemma, in the form Eq. (2.36), to the first bracket (using $\mathbf{C} \to \mathbf{P}(1)^{-1}, \mathbf{B} \to \mathbf{E}(2), \mathbf{A} \to \mathbf{R}_{nn}(2)$)

$$\begin{aligned}
\tilde{\mathbf{x}}(2) &= \left\{\mathbf{P}(1) - \mathbf{P}(1)\mathbf{E}(2)^T\left[\mathbf{E}(2)\mathbf{P}(1)\mathbf{E}(2)^T + \mathbf{R}_{nn}(2)\right]^{-1}\mathbf{E}(2)\mathbf{P}(1)\right\}\left\{\mathbf{E}(1)^T\mathbf{R}_{nn}(1)^{-1}\mathbf{y}(1)\right\} + \\
&\quad + \left\{\mathbf{P}(1) - \mathbf{P}(1)\mathbf{E}(2)^T\left[\mathbf{E}(2)\mathbf{P}(1)\mathbf{E}(2)^T + \mathbf{R}_{nn}(2)\right]^{-1}\mathbf{E}(2)\mathbf{P}(1)\right\}\left\{\mathbf{E}(2)^T\mathbf{R}_{nn}(2)^{-1}\mathbf{y}(2)\right\} \\
&= \tilde{\mathbf{x}}(1) - \mathbf{P}(1)\mathbf{E}(2)^T\left[\mathbf{E}(2)\mathbf{P}(1)\mathbf{E}(2)^T + \mathbf{R}_{nn}(2)\right]^{-1}\mathbf{E}(2)\tilde{\mathbf{x}}(1) + \\
&\quad + \mathbf{P}(1)\mathbf{E}(2)^T\left\{\mathbf{I} - \left[\mathbf{E}(2)\mathbf{P}(1)\mathbf{E}(2)^T + \mathbf{R}_{nn}(2)\right]^{-1}\mathbf{E}(2)\mathbf{P}(1)\mathbf{E}(2)^T\right\}\mathbf{R}_{nn}(2)^{-1}\mathbf{y}(2)
\end{aligned}$$

using (2.461) and factoring $\mathbf{E}(2)^T$ in the last line. Using the identity,

$$\left[\mathbf{E}(2)\mathbf{P}(1)\mathbf{E}(2)^T + \mathbf{R}_{nn}(2)\right]^{-1}\left[\mathbf{E}(2)\mathbf{P}(1)\mathbf{E}(2)^T + \mathbf{R}_{nn}(2)\right] = \mathbf{I},$$

and substituting for $\mathbf{I}$ in the previous expression, factoring, and collecting terms, we have finally,

$$\tilde{\mathbf{x}}(2) = \tilde{\mathbf{x}}(1) + \mathbf{P}(1)\mathbf{E}(2)^T\left[\mathbf{E}(2)\mathbf{P}(1)\mathbf{E}(2)^T + \mathbf{R}_{nn}(2)\right]^{-1}\left[\mathbf{y}(2) - \mathbf{E}(2)\tilde{\mathbf{x}}(1)\right] \tag{2.462}$$ {recurs16}

which is the desired expression. The new uncertainty is given by (2.426) or (2.428).

Manipulation of the recursive Gauss-Markov solution (2.441) or (2.442) is similar, involving repeated use of the matrix inversion lemma. Consider Eq. (2.441) with $\mathbf{x}_b$ from Eq. (2.446),

$$\begin{aligned}
\tilde{\mathbf{x}}^+ &= \left(\mathbf{E}(2)^T\mathbf{R}_{nn}^{-1}\mathbf{E}(2)\right)^{-1}\left[\mathbf{P}_a + \mathbf{E}(2)^T\mathbf{R}_{nn}^{-1}\mathbf{E}(2)\right]^{-1}\tilde{\mathbf{x}}_a + \\
&\quad \mathbf{P}_a\left[\mathbf{P}_a + \mathbf{E}(2)^T\mathbf{R}_{nn}^{-1}\mathbf{E}(2)\right]^{-1}\left(\mathbf{E}(2)^T\mathbf{R}_{nn}\mathbf{E}(2)\right)^{-1}\mathbf{E}(2)^T\mathbf{R}_{nn}^{-1}\mathbf{y}(2).
\end{aligned}$$

Using Eq. (2.37) on the first term (with $\mathbf{A} \to \left(\mathbf{E}(2)^T\mathbf{R}_{nn}^{-1}\mathbf{E}(2)\right)^{-1}, \mathbf{B} \to \mathbf{I}, \mathbf{C} \to \mathbf{P}_a$), and on the second term with $\mathbf{C} \to \left(\mathbf{E}(2)\mathbf{R}_{nn}^{-1}\mathbf{E}(2)\right), \mathbf{A} \to \mathbf{P}_a, \mathbf{B} \to \mathbf{I}$, this last expression becomes,

$$\tilde{\mathbf{x}}^+ = \left[\mathbf{P}_a^{-1} + \mathbf{E}(2)^T\mathbf{R}_{nn}^{-1}\mathbf{E}(2)\right]^{-1}\left[\mathbf{P}_a^{-1}\tilde{\mathbf{x}}_a + \mathbf{E}(2)^T\mathbf{R}_{nn}^{-1}\mathbf{y}(2)\right],$$

yet another alternate form. By further application of the matrix inversion lemma,[60] this last expression can be manipulated into Eq. (2.448), which is necessarily the same as (2.462).

These expressions have been derived assuming that matrices such as $\mathbf{E}(2)^T\mathbf{R}_{nn}^{-1}\mathbf{E}(2)$ are non-singular (full-rank overdetermined). If they are singular, they can be inverted using a generalized inverse, e.g. replacing $\tilde{\mathbf{x}}(1)$ with the particular SVD solution, but taking care that $\mathbf{P}(1)$ includes the nullspace contribution (e.g., from Eq. (2.271)).