1.2 Partial Differential Equations

Finding the solutions of linear differential equations is equivalent, when discretized, to solving sets of simultaneous linear algebraic equations. Unsurprisingly, the same is true of partial differential equations. As an example, consider a very familiar problem:

Solve

$$\nabla^2\phi = \rho, \tag{1.12}$$ {eq:13001}

for ϕ, given ρ, in the domain $\mathbf{r} \in D$, subject to the boundary conditions $\phi = \phi_0$ on the boundary ∂D, where $\mathbf{r}$ is a spatial coordinate of dimension greater than 1.

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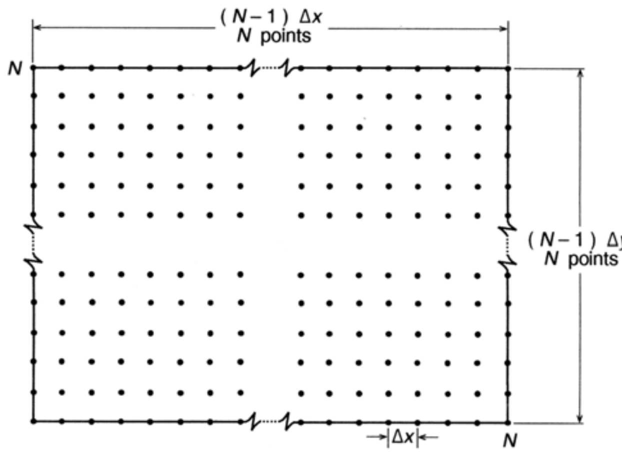

Figure 1.1: Square, homogeneous grid used for discretizing the Laplacian, thus reducing the partial differential equation to a set of linear simultaneous equations.

{grid1.tif}

This statement is the Dirichlet problem for the Laplace-Poisson equation, whose solution is well-behaved, unique, and stable to perturbations in the boundary data, ϕ_0, and the source or forcing, ρ. Because it is the familiar boundary value problem, it is by convention again labeled a forward or direct problem. Now consider a different version of the above:

Solve (1.12) for ρ given ϕ in the domain D.

This latter problem is even easier to solve than the forward problem: merely differentiate ϕ twice to obtain the Laplacian, and ρ is obtained directly from (1.12). Because the problem as stated is inverse to the conventional forward one, it is labeled, as with the ordinary differential equation, an *inverse problem*. It is inverse to a more familiar boundary value problem in the sense that the usual unknowns ϕ have been inverted or interchanged with (some of) the usual knowns ρ. Notice that both forward and inverse problems, as posed, are well-behaved and produce uniquely determined answers (ruling out mathematical pathologies in any of ρ, ϕ_0, ∂D, or ϕ). Again, there are many variations possible: one could, for example, demand computation of the boundary conditions, ϕ_0, from given information about some or all of ϕ, ρ.

Write the Laplace-Poisson equation in finite difference form for two Cartesian dimensions:

{eq:13002}

$$\phi_{i+1,j} - 2\phi_{i,j} + \phi_{i-1,j} + \phi_{i,j+1} - 2\phi_{i,j} + \phi_{i,j-1} = (\Delta x)^2 \rho_{ij}, \quad i, j \in D, \tag{1.13}$$

with square grid elements of dimension Δx. To make the bookkeeping as simple as possible, suppose the domain D is the square $N \times N$ grid displayed in Figure 1.1, so that ∂D is the four line segments shown. There are $(N-2)\times(N-2)$ interior grid points, and Equations (1.13) are then $(N-2)\times(N-2)$ equations in N^2 of the ϕ_{ij}. If this is the forward problem with ρ_{ij} specified, there are fewer equations than unknowns. But if we append to (1.13) the set of boundary conditions:

$$\phi_{ij}=\phi^0_{ij}\,,\quad i,\,j\in\partial D\,, \tag{1.14}$$ {eq:13003}

there are precisely $4N-4$ of these conditions, and thus the combined set (1.13) plus (1.14), which we write again as (1.9) with,

$$\mathbf{x}=\mathrm{vec}\{\phi_{ij}\}=\begin{bmatrix}\phi_{11}\\ \phi_{12}\\ .\\ .\\ \phi_{NN}\end{bmatrix},\qquad \mathbf{b}=\mathrm{vec}\left\{\rho_{ij},\phi^0_{ij}\right\}=\begin{bmatrix}\rho_{2,N+1}\\ \rho_{2,N+2}\\ .\\ .\\ \rho_{N-2,N-2}\\ \phi^0_{11}\\ .\\ \phi^0_{N,N}\end{bmatrix},$$

a set of $M=N^2$ equations in $M=N^2$ unknowns. (The operator, vec, forms a column vector out of the two-dimensional array ϕ_{ij}.) The nice properties of the Dirichlet problem can be deduced from the well-behaved character of the matrix $\mathbf{A}$. Thus the forward problem corresponds directly with the solution of an ordinary set of simultaneous algebraic equations.[3] One complementary inverse problem says: "Using (1.9) compute ρ_{ij} and the boundary conditions, given ϕ_{ij}," an even simpler computation—it involves just multiplying the known $\mathbf{x}$ by the known matrix $\mathbf{A}$.

But now let us make one small change in the forward problem, changing it to the Neumann one:

Solve,

$$\nabla^2\phi=\rho, \tag{1.15}$$ {eq:13005}

for ϕ, given ρ, in the domain $\mathbf{r}\in D$ subject to the boundary conditions $\partial\phi/\partial\hat{\mathbf{m}}=\phi'_0$ on the boundary ∂D, where $\mathbf{r}$ is again the spatial coordinate and $\hat{\mathbf{m}}$ is the unit normal to the boundary.

This new problem is another classical, much analyzed forward problem. It is, however, well-known that the solution is indeterminate up to an additive constant. This indeterminacy is clear in the discrete form: Equations (1.14) are now replaced by

$$\phi_{i+1,j} - \phi_{i,j} = \phi_{ij}^{0\prime\prime}, \qquad i,\, j \in \partial D' \tag{1.16}$$

{eq:13006}

etc., where $\partial D'$ represents the set of boundary indices necessary to compute the local normal derivative. There is a new combined set:

{eq:13007}

$$\mathbf{A}\mathbf{x} = \mathbf{b}_1,\ \mathbf{x} = \mathrm{vec}\left\{\phi_{ij}\right\},\ \mathbf{b}_1 = \mathrm{vec}\left\{\rho_{ij}, \phi_{ij}^{0\prime}\right\} \tag{1.17}$$

Because only *differences* of the ϕ_{ij} are specified, there is no information concerning the absolute value of $\mathbf{x}$. When we obtain some machinery in Chapter 2, we will be able to demonstrate automatically that even though (1.17) appears to be M equations in M unknowns, in fact only $M-1$ of the equations are independent, and thus the Neumann problem is an underdetermined one. This property of the Neumann problem is well-known, and there are many ways of handling it, either in the continuous or discrete forms. In the discrete form, a simple way is to add one equation setting the value at any point to zero (or anything else). Notice, however, that in all cases, the inverse problem of determining $\mathbf{b}_1$ from $\mathbf{x}$ remains simple and well-posed.