Quantifying Uncertainty

Sai Ravela

M. I. T

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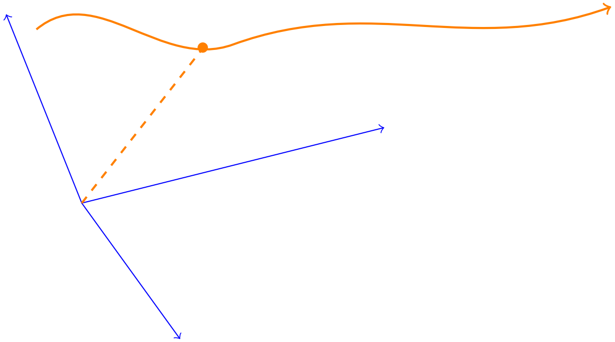
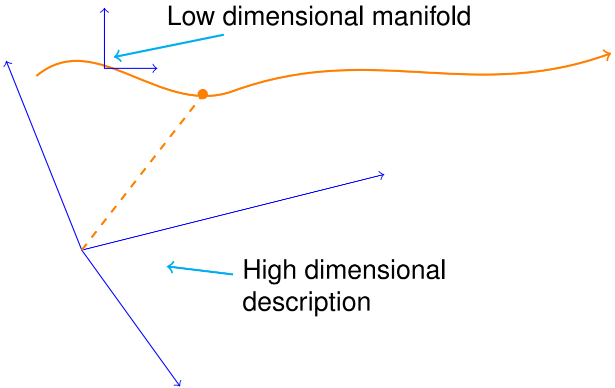


$$\dot{Y}(t) = f(Y(t), t) + \beta u$$

System with high dimensions



$\Rightarrow$ Can be described by few modes.

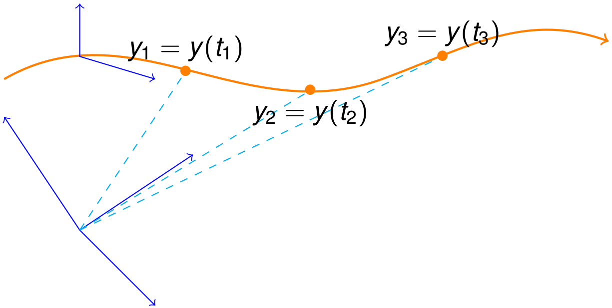
How to find subspace?
Projection Methods

POD-Proper Orthogonal Decomposition

- Galerkin Projection
- Snapshots

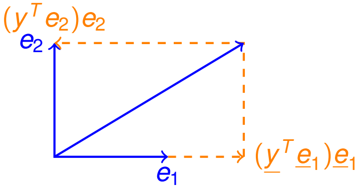
Krylov Subspace

Also Galerkin



$$\sum_{i=1}^{n} [y_n - \pi_d(y_n)]^2$$

Just minimize a projection error



$$\pi_d(y_n) = (\underline{y}^T \underline{e}_1)\underline{e}_1 + (\underline{y}^T \underline{e}_1)\underline{e}_1$$

$$\therefore \sum_{i=1}^{n} \left[y_n - \sum_{j=1}^{d} \left(y_n^T e_j \right) e_j \right]^2$$

Constraints ?

$$e_i^T e_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \textit{otherwise} \end{cases}$$

So, Lagrangian

$$J(e_1 \dots e_d) = \sum_{i=1}^{n} \left[\underline{y}_n - \sum_{j=1}^{d} \left(\underline{y}_n^T \underline{e}_j \right) \underline{e}_j \right]^2 + \sum_{k=1}^{d} \sum_{l=1}^{d} \lambda_{ij} \left(e_i^T e_j - \delta_{ij} \right)$$

$$\frac{\partial J}{\partial e_j} = \phi \Rightarrow \sum_{j=1}^{n} y_i(y_i^T e_j) = \lambda_{jj} e_j$$

$\lambda_{ij} = \phi$ for $i \neq j$.

Let y,

$$Y = [y_1, \ldots, y_n] \text{ and } E = [e_1, \ldots, e_n]$$

we have

$$\underline{\underline{\lambda}} = \begin{bmatrix} \lambda_{11} & \cdots & \lambda_{1d} \\ \vdots & \ddots & \vdots \\ \lambda_{d1} & \cdots & \lambda_{dd} \end{bmatrix}$$

$$\underline{\underline{YY}}^T \underline{\underline{E}} = \underline{\underline{\lambda E}}$$

$$YY^T \underline{e}_i = \underline{\lambda}_{ii} \underline{e}_i$$

An eigenvalue problem.

Notice

$$K(s,t) = Y(s)Y(t)$$
$$K = YY^T$$

$\Rightarrow$ Correlation matrix (or Covariance for centered process)
Still, how to do this?

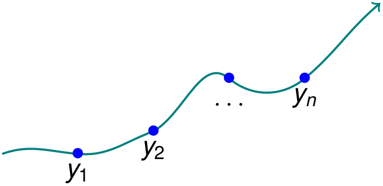
Via SVD

Letting

$$
\begin{aligned}
Y &= usv^T \\
YY^T &= us^2u^T \\
uu^T &= I, \quad u^Tu = I
\end{aligned}
$$

Where does Y come from?

Snapshots



$$Y = [y_1, \dots, y_n]$$

Typically $n << |y_i|$ and modes $d < n$.

But YY^T is huge!

Take 2

$$Y = \underbrace{usv^T}_{\substack{\text{svd via iterative methods} \\ \text{only `d' leading vectors} \\ \text{or modes}}}$$

Then: $e_i = U_i$, $\lambda_{ii} = S_{ii}^2$

Take 3

$$\underbrace{Y^T Y}_{small} = vs^2 v^T$$

$$YVS^{-1} = u$$
$$e_i = u_i$$
$$\lambda_{ii} = S_{ii}^2$$

Pod & Model reduction

Let's consider

$$\dot{y}(t) = f(y(t))$$

Let

$$\begin{aligned} y(t) &= u_d s r(t) \\ &= \underbrace{u_d}_{\textit{Basis}} \quad \underbrace{\chi(t)}_{\substack{\textit{time} \\ \textit{dependent} \\ \textit{decorrelated}}} \end{aligned}$$

$$u_d \dot{\chi}(t) = f(u_d \chi(t))$$

$$\dot{\chi}(t) = u_d^T f(\underbrace{u_d \chi(t)}_{y_d(t)})$$

With initial condition $\chi(0)$ calculated from y_0

$$y_d^{(i)}(0) = [y_0^T(0) u_i] u_i$$

Can this work?

a

Snapshots make it easy, but how badly does it work when dynamics not in steady state (statistical)?

b

How much to truncate by?

$$Residual = \sum_{i=d+1}^{n} \lambda_{ii}$$

$$\text{Information fraction} = \frac{\sum_{i=1}^{d} \lambda_{ii}}{\sum_{j=1}^{n} \lambda_{ii}}$$

c
Do PoD modes correspond to dynamical modes?

d
Is it sensitive to choice of the inner product?

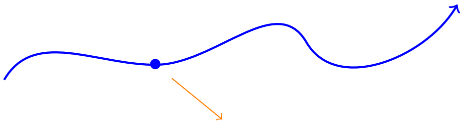
e
Is a PoD decomposition stable? e.g. $\dot{\chi} = Ax + Bu$ Rom may not have negative eigen values!

How to fix these issues?

- PoD+ Empirical Grammians (Balanced Truncation) e.g Wilcox
- Optimization for a stabilizing projection

A Trajectory Piecewise Approach

Via Krylov Subspace



$$\dot{y} = f(y) + B(y)u$$
$$Z = C^T Y$$

$$x' = f(x) + B(x)u$$
$$y = C^T u$$

Z $\Rightarrow$ Reduced state
$x \cong VZ$

Residual

$$f(VZ) + B(VZ) - \frac{d(VZ)}{dt}$$

Orthogonal to some reduced space, $V^T v = 0$

$$\frac{d}{dt}[V^T VZ(t)] = V^T f(VZ(t)) + V^T B(VZ(t))u(t)$$
$$y(t) = C^T VZ(t)$$

Is this supposed to be easy?

Linear Case

Say

$$f(x) = Fx$$
$$B(x) = B$$

$$\frac{d}{dt} V^T V Z(t) = V^T F V Z(t) + V^T B u(t)$$
$$y(t) = (V^T C)^T Z(t)$$

Linear Case contd.

$$w\frac{d}{dt}Z(t) = \tilde{A}Z(t) + \tilde{B}u(t)$$
$$y(t) = (V^T C)^T Z(t)$$

Suppose original system was linear and of general form:

$$E\dot{x} = Fx + Bu$$
$$y = C^T x$$

Introduction to Krylov Subspace methods

$$Ax = f$$

Letting $A = M - N(say)$

$$\begin{aligned}(M_N)u &= f \\ Mu^{n+1} &= Nu^n + f \\ u^{n+1} &= M^{-1}Nu^n + M^{-1}f \\ &= u^n + M^{-1}(F - Au^n)\end{aligned}$$

Letting

$$\gamma^n = f - Au^n$$

Iterative method:

$$u^{n+1} = u^n + M^{-1}\gamma^n$$
$$u^0 \to u^n \to u^?$$
$$e^n = u^n - u$$
$$e^{n+1} = (I - M^{-1}A)e^n$$

Examples

$$A = D + L + u$$

Jacobi: $M = D$, $-N = L + u$
Gauss-Sidel: $M = L + D$, $-N = u$
S0R:

$$M = \left(\frac{1}{w} D + L\right), \qquad -N = u - \frac{1-w}{w} D$$

etc.

Letting,

$$\tilde{A} = M^{-1}A \qquad \tilde{f} = M^{f}$$

$$\begin{aligned} u^{n+1} &= u^n + (\tilde{f} - \tilde{A}u^n) \\ &= u^n + \tilde{\gamma}^n \\ \tilde{\gamma}^{n+1} &= \tilde{\gamma}^n - \tilde{A}\tilde{\gamma}^n \\ &= (I - \tilde{A})\tilde{\gamma}^n \\ &= (I - \tilde{A})^{n+1}\tilde{\gamma}^0 \end{aligned}$$

$$\boxed{\tilde{\gamma}^n = (I - \tilde{A})^n\tilde{\gamma}^0}$$

$$u^{n+1} = u^0 + \sum_{i=0}^{n} (I - \tilde{A})^i \tilde{\gamma}^0$$
$$\in u^0 + \underbrace{Span\left\{\tilde{\gamma}^0, \tilde{A}\tilde{\gamma}^0, \widetilde{A^2}\tilde{\gamma}^0, \ldots, \widetilde{A^n}\tilde{\gamma}^0\right\}}_{Krylov \quad Subspace}$$

So:(after several steps)

$$E\hat{x} = Fx + Bu \qquad y = C^T x$$

we look for an orthonormal basis V in Krylov subspace

$$Span\left\{F^{-1}B, F^{-1}EF^{-1}B, \ldots, (F^{-1}E)^{v-1}F^{-1}B\right\}$$

Algo:(F, B, E, v)
Output: V

$$
\begin{aligned}
v_1 &= B/||B||_2 \\
\text{for } i &= 1 \text{ to } q \\
x &= Evi \\
\text{Solve } Av &= x \\
\textit{Orthogonalize} & \quad v \\
\text{for } j &= 1 \text{ to } i \\
\alpha &= v_j^T V \\
v &= v - \alpha v_j \\
v_i &= v/||v||_2
\end{aligned}
$$

Reduced System:$E = I$

$$w\widetilde{Z} = \widetilde{A}Z + \widetilde{B}u$$

$$y = \widetilde{C}^T Z$$

$$H_\gamma(s) = \widetilde{C}^T \left[sW - \widetilde{A}\right]^{-1} \widetilde{B}$$

Original

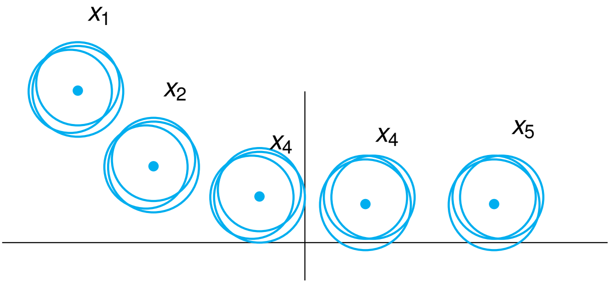
$$H(s) = C^T[sI - A]^{-1}B$$

$H_\gamma(s)$ will match first q moments of H.

However,

- Still an issue for nonlinear
- No provable error bounds
- No guarantee of stability

How does this compare to PoD?



Linearization & model reduction

$$\dot{x}_i = f(x_i) + B(x_i)u$$
$$\dot{x} = f(x_i) + \underbrace{A_i(x - x_i)}_{\text{Jacobian of f.}} + B_i u$$

A weighted combination of linearized models.

$$\sum_{i=1}^{S} w_i(x)\dot{x} = \sum_{j=1}^{S} w_j(x)\left[f(x_j) + A_j(x - x_j) + B_j u\right]$$

$\sum_i w_i(x) = 1$

So,

$$\sum_{s=1}^{S} w_i[VZ]V^T VZ = \sum_{i=1}^{S} w_i[VZ]\left[\overbrace{V^T f(x_i)}^{Precompute} + V^T A_i[VZ - x_i] + V^T B_i u\right]$$

Can be fast!

How to get weights?
Kernell estimated!

$$
\begin{aligned}
||x - x_i|| &= ||VZ - VZ_i|| \\
&= (Z - Z_i)V^T V(Z - Z_i) \\
&= ||Z - Z_i||_V
\end{aligned}
$$

$$d_i \in ||Z - Z_i||$$

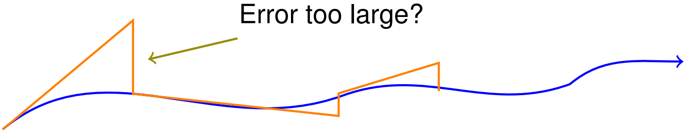
$$\widetilde{w}_i = e^{-Kd_i/\sigma^2}$$

$$\sigma^2 \sim \min d_i$$

$$\boxed{w_i = \frac{\widetilde{w}_i}{\sum \widetilde{w}_i}}$$

Next time: On the Grasmann Manifold
$\Rightarrow$ Model Interpolation

When to linearize?



Break when error too large
Interpolation also with response surface modeling.

Ideas

- Is there a way to combine PoD with krylov over MCMC?
- Can we assemble better collocation points?

$$\frac{\partial\theta}{\partial t}(x,t) = D\theta(x,t)$$

$$R(\theta) = \frac{\partial\theta}{\partial t} - D\theta$$

$$\theta = u\eta(t)$$

$$u^T R = 0$$

$$\frac{\partial\eta}{\partial t} = u^T D u\eta$$

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