16.540 **Spring 2006**

VORTICITY AND CIRCULATION: Concepts and Applications

Lectures 4 - 9

OUTCOMES

Be able to:

- **Define vorticity in several ways**

- **Use vortex *kinematics* to provide insight into the structure of internal flow fields**

- **Use vortex *dynamics* to provide insight into the structure of internal flow fields**

- **Use circulation evolution to provide insight into the structure of internal flow fields**

- **Provide qualitative arguments for the generation of circulation and vorticity by baroclinic torque**

- **Give explicit arguments concerning the generation of vorticity at solid surfaces**

- **Provide quantitative linkages between thermodynamic and kinematic quantities in a rotational flow field**

WHY DO WE CARE ABOUT VORTICITY?

- **Analogy with descriptions of rigid body dynamics**
  - **Angular velocity, angular momentum natural quantities to use (rather than linear velocity, linear momentum)**
  - **No "new" information**
  - **Still Newton's laws but "repackaged" for problem of interest**
  - **Useful to describe fluid motion in terms of local angular velocity**
  - **Useful for insight - vorticity is sometimes easiest way to "explain" phenomena - especially with <u>swirl</u>**

PLAN OF SECTION ON VORTICITY

- Some definitions - vorticity, vortex line, vortex tube
- Physical interpretation of vorticity
- Vorticity kinematics
- Vorticity dynamics (evolution of the vorticity field)

} Development of concepts

- Description of flow fields in terms of velocity - applications of concepts

VORTICITY $\left(\vec{\omega} = \nabla \times \vec{\mathbf{V}}\right)$

- **External flows often irrotational**
  - $\nabla \times \vec{\mathbf{V}} = \mathbf{0} \quad \Rightarrow \quad \vec{\mathbf{V}} = \nabla\phi$
  - **One scalar equation:**

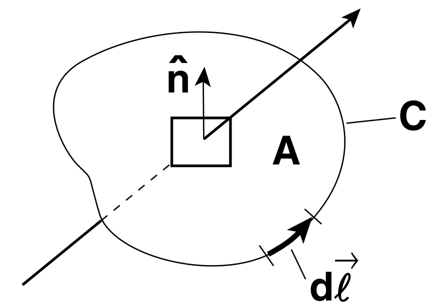
    Incompressible flow $\nabla \cdot \vec{\mathbf{V}} = \mathbf{0}$ **or** $\nabla^2\phi = \mathbf{0}$

- **Internal flows:** $\nabla \times \vec{\mathbf{V}} \neq \mathbf{0}$
  - **More surfaces**
  - **Differential energy addition to stream**

KINEMATICS: DEFINITIONS OF VORTICITY

- $\vec{\omega} = \nabla \times \vec{u}$
- **Stokes' Theorem** $\iint_{area} \nabla \times \vec{u} \cdot \hat{n}\, dA = \oint_{c} \vec{u} \cdot d\vec{\ell}$

  normal



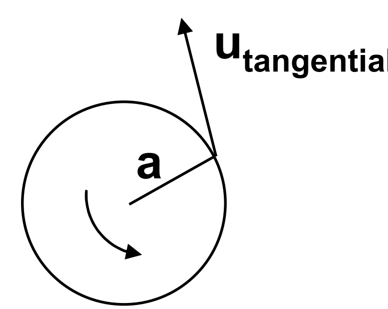
- **Simple case: Rotating cylinder of fluid**

  For small contour, $\omega dA = 2\pi a u_{tangential}$

  Rotating cylinder with angular velocity Ω

  $u_{tangential} = a\Omega$

  $\omega \underbrace{\pi a^2}_{dA} = 2\pi a \cdot a\Omega$



=> $\omega = 2\Omega$

- **Vorticity = twice local angular velocity of fluid**

- **Is this true only for plane?**

- **Take small contour in 3 $\perp$ planes -**
  - **3 components of vorticity - a vector**

- **Physical concept: Solidify small fluid sphere without change in angular momentum:**

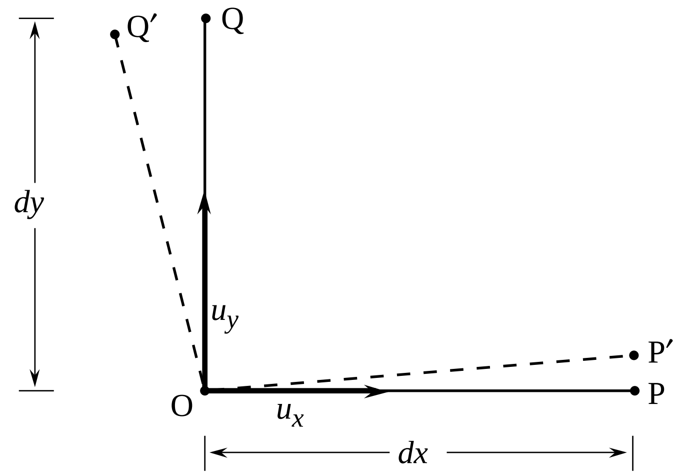
  Angular velocity = $\vec{\omega}/2$

EXAMINE ONE COMPONENT OF VORTICITY (z-component)

Consider two perpendicular fluid lines, OQ, OP :

Relative to O, the upward

velocity of P is $\frac{\partial u_y}{\partial x} \Delta x.$



Velocity of Q to left is $\left(-\frac{\partial u_x}{\partial y}\right) \Delta y.$

Rate of rotation is $\frac{\text{Tangential velocity}}{\text{Distance from center}}.$

Rate of rotation of OP is $\frac{\partial u_y}{\partial x}$**, rate of rotation of OQ is** $\left(-\frac{\partial u_x}{\partial y}\right).$

Sum of angular velocities of two perpendicular lines =

$$\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right) = \omega_z = \text{ z - component of vorticity}$$

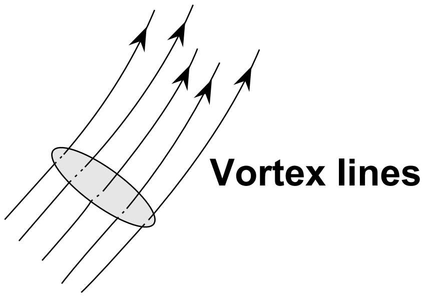
- **Average angular velocity = $\omega_z/2$**
  - **Has same value for any two perpendicular lines**

- **Generalize to 3-D: vorticity is a measure of fluid angular velocity**

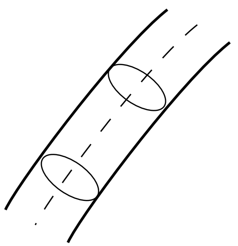
OTHER USEFUL CONCEPTS

- Vortex line
  - Line in fluid
  - Tangent to line has direction of vorticity vector

- Vortex tube
  - Vortex lines thru a small closed curve form a vortex tube



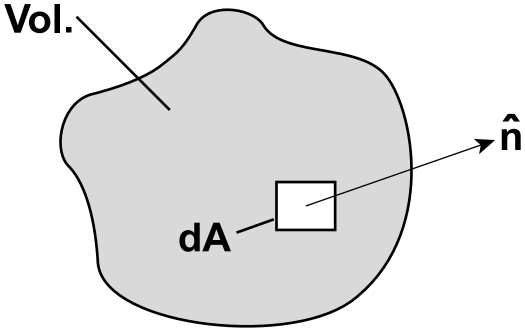
- Properties of vortex lines, vortex tubes
  - $\vec{\omega} = \nabla \times \vec{u}$
  - $\nabla \cdot \vec{\omega} = \nabla \cdot \left(\nabla \times \vec{u} \right)$



- **Vector identity** $\nabla \cdot (\nabla \times \vec{b}) = 0$
  - $\vec{b}$ **is any vector**

- **So** $\nabla \cdot \vec{\omega} = 0$

- **Divergence Theorem**



$$0 = \underset{\text{volume}}{\iiint} \nabla \cdot \vec{\omega}\, dV = \underset{\text{area}}{\iint} \vec{\omega} \cdot \hat{n}\, dA$$

- **Same number of vortex lines coming in as going out for a closed surface**

- **<u>Vortex lines cannot end in the fluid</u> (<u>Any</u> fluid)**

  - **Form closed loops, go to ∞, end on solid boundaries <u>in rotating flow</u>**

VORTEX TUBE

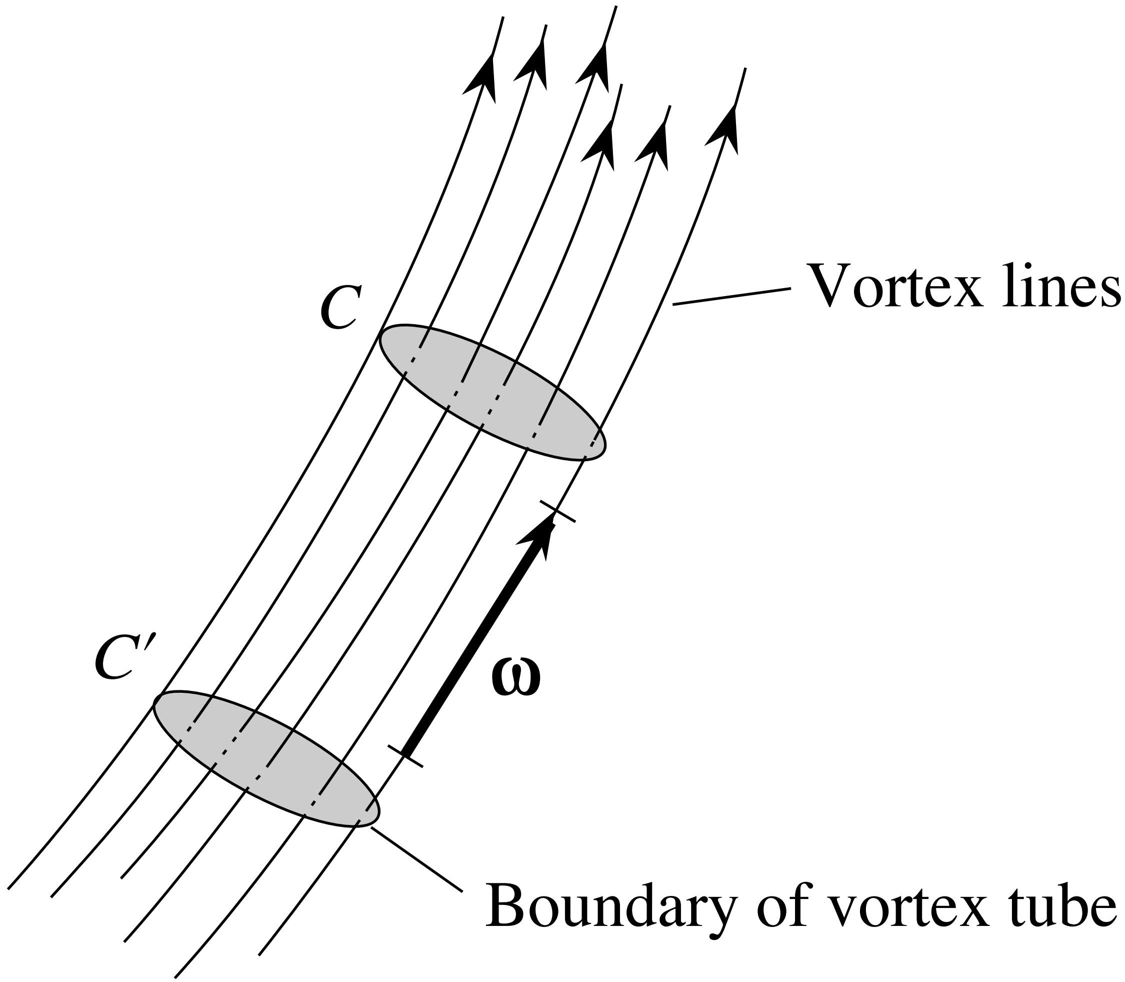
- **Area dA_n normal to tube, vorticity uniform across tube**
- **Quantity $\omega\, dA_n$ is called the "strength" of the vortex tube, is constant along the tube.**
- **Finite tube with vorticity not uniform; define *circulation***
- **Circulation is the total flux of vortex lines threading through the area A_n enclosed by the curve C**

$$\Gamma = \iint_A \vec{\omega} \cdot \hat{n}\, dA = \iint_{A_n} \omega\, dA_n$$

$$\Gamma = \oint \vec{u} \cdot d\vec{l}$$

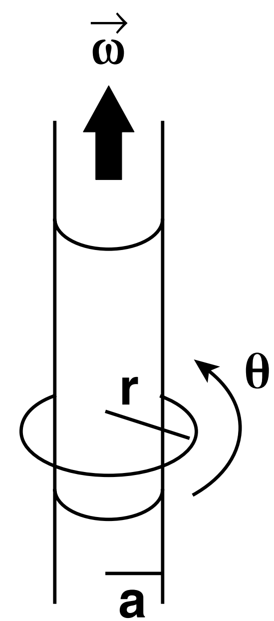
Γ **is constant along a vortex tube**

CIRCULATION FOR A VORTEX TUBE



VORTICITY-VELOCITY RELATIONSHIP

- **Straight, infinite vortex tube, constant vorticity,** $\vec{\omega}$
- **Radius of tube, a**
- **Circular contour of radius, r > a**



$$\Gamma = \iint \vec{\omega} \cdot \hat{\mathbf{n}}\, \mathbf{dA} = \pi \mathbf{a}^{2} \omega_{\mathbf{o}}$$

also

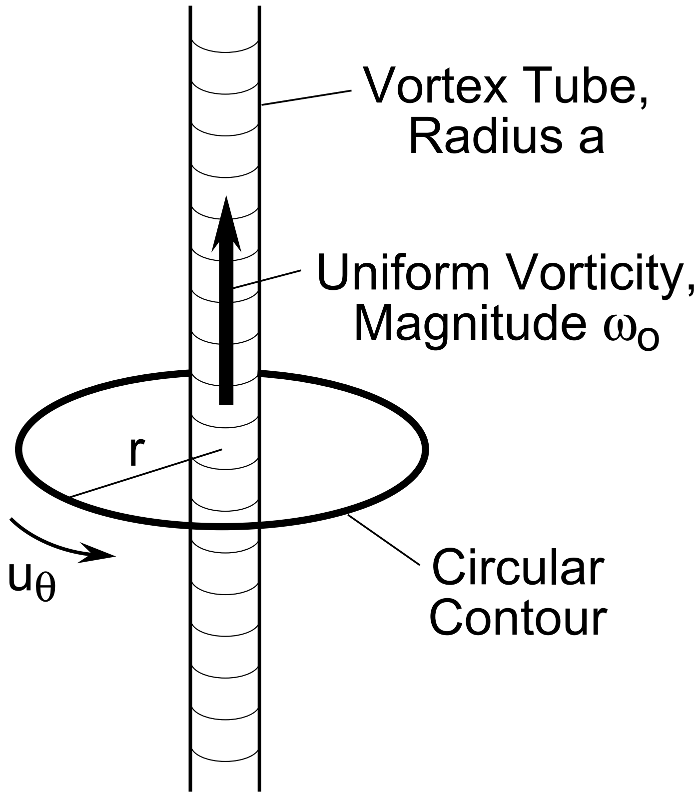
$$\Gamma = u_\theta \cdot 2\pi r = \pi a^2 \omega_o$$

so

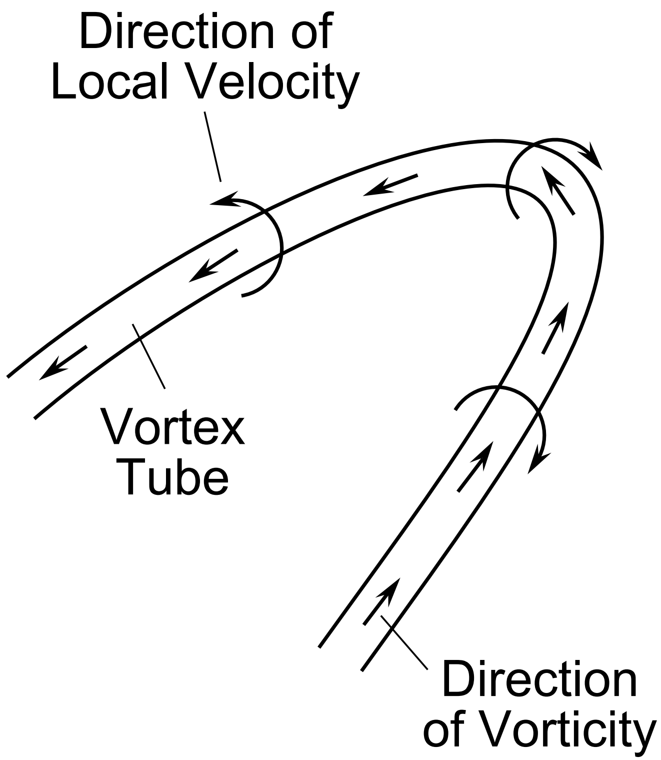
$$u_\theta = \frac{\omega_o a^2}{2r} = \frac{\Gamma}{2\pi r}$$

- **Velocity outside tube decays as 1/r**
- **Velocity inside tube (r ≤ a)**
  - $\Gamma = u_\theta \cdot 2\pi r = \pi r^2 \omega_o$
  - $u_\theta = \omega_o r / 2$
  - **Sense of velocity – right hand rule**

VELOCITY FIELD NEAR VORTEX TUBE



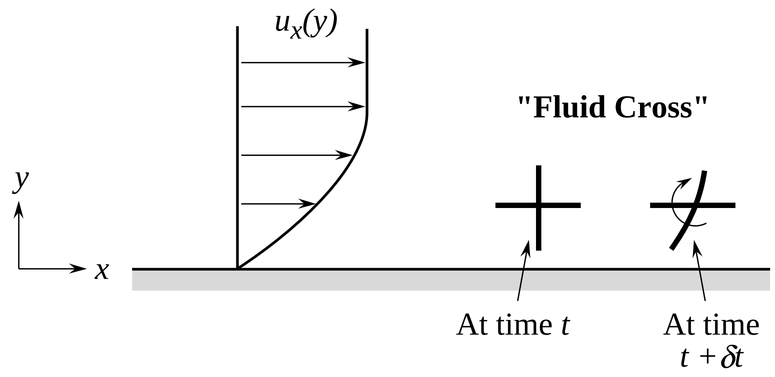
Velocity field associated
with straight vortex tube



Sketch of velocity associated
with curved vortex tube

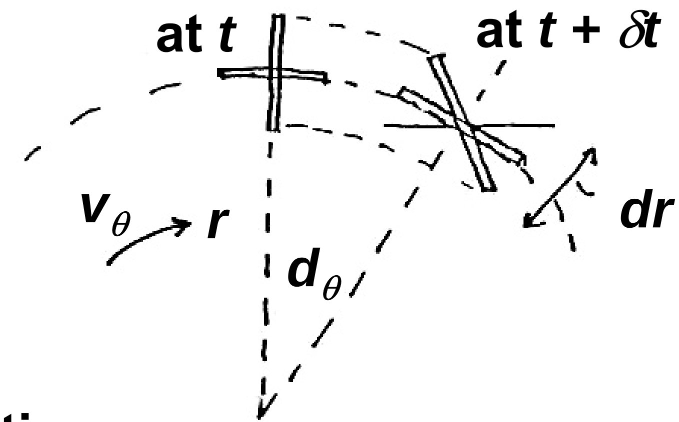
VORTICITY AND STREAMLINE CURVATURE

- Vorticity can be present *even if streamlines are straight*



- Have rotation rate of vertical fluid line of $-\frac{\partial u_y}{\partial x}$, no rotation horiz. line
- Total rotation of 2 perpendicular lines is $-\frac{\partial u_x}{\partial y} + 0 = \omega_z$

VORTICITY IS NOT NECESSARILY PRESENT IF STREAMLINES ARE CURVED



- **Consider 2-D flow with $u_\theta \propto 1/r,\quad u_r = 0$**

Horizontal leg rotates dθ in clockwise direction

$$d\theta = \frac{u_\theta}{r}\,\delta t$$

Vertical leg rotates $\dfrac{\partial u_\theta}{\partial r}\,\delta t$ in anti-clockwise direction

Net rotation in anti-clockwise direction is $\left(\dfrac{\partial u_\theta}{\partial r} - \dfrac{u_\theta}{r}\right)\delta t$

If $u_\theta \propto 1/r$, $\left(\dfrac{\partial u_\theta}{\partial r} - \dfrac{u_\theta}{r}\right) = 0$, NO NET ROTATION, $\omega = 0$

VORTICITY DYNAMICS: CHANGES IN VORTICITY IN A FLOW

- **Want to be able to describe how vorticity distribution evolves in a general situation**

- **Need to develop expression for rate of change of vorticity**

MOMENTUM EQUATION ⇒ EXPRESSION FOR RATE OF CHANGE OF VORTICITY

$$\frac{D\vec{u}}{Dt} = -\frac{1}{\rho}\nabla p + \vec{F}_{visc} + \vec{F}_{body}$$

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\nabla)\vec{u}$$

Viscous forces

Body forces, gravity, Coriolis, $\vec{J}\times\vec{B}, \ldots$

- **Vorticity is ∇ x so take ∇ x [Momentum Eq.]**

$$\nabla \times \left[\frac{D\vec{u}}{Dt} = -\frac{1}{\rho}\nabla p + \vec{F}_{visc} + \vec{F}_{body}\right]$$

- **Get:**

$$\frac{\partial \vec{\omega}}{\partial \mathbf{t}} + (\vec{\mathbf{u}} \cdot \nabla)\vec{\omega} - \vec{\omega}(\nabla \cdot \vec{\mathbf{u}}) = (\vec{\omega} \cdot \nabla)\vec{\mathbf{u}} - \nabla \times \left(\frac{\mathbf{1}}{\rho}\nabla \mathbf{p}\right) + \nabla \times \vec{\mathbf{F}}_{\mathbf{visc}} + \nabla \times \vec{\mathbf{F}}_{\mathbf{body}}$$

$$\frac{\mathbf{D}\vec{\omega}}{\mathbf{Dt}} = \text{lots of terms}$$

- **Different physical meaning for each term**

- **Build up general case by looking at simple situation and adding effects**

CASE 1 – INVISCID, CONSTANT DENSITY FLOW WITH CONSERVATIVE BODY FORCE

- **Note: Incompressible ≠ constant density**

$\vec{F}_{body} = \nabla\Psi$ **Force is gradient of a potential**

$$\nabla \times \nabla\Psi \equiv \mathbf{0}$$

$$\nabla \times \left(\frac{\mathbf{1}}{\rho}\nabla \mathbf{p}\right) = \frac{\mathbf{1}}{\rho}\nabla \times \nabla \mathbf{p} = \mathbf{0}$$

constant

$\nabla \cdot \vec{\mathbf{u}} = \mathbf{0}$ **incompressible**

- **Vorticity equation**

$$\frac{D\vec{\omega}}{Dt} - \vec{\omega}\left(\nabla \cdot \vec{u}\right)^{\nearrow 0} = \left(\vec{\omega} \cdot \nabla\right)\vec{u} - \nabla\left(\frac{1}{\rho}\nabla p\right)^{\nearrow 0} + \nabla \times \vec{F}_{visc}^{\nearrow 0} + \nabla \times \vec{F}_{body}^{\nearrow 0}$$

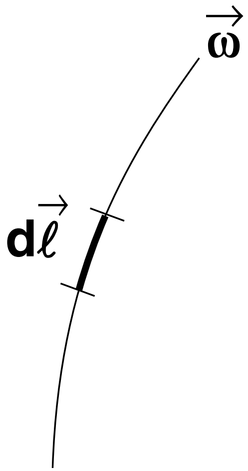
$$\boxed{\frac{D\vec{\omega}}{Dt} = \left(\vec{\omega} \cdot \nabla\right)\vec{u}}$$

- **Vorticity change in inviscid, constant density flow conservative**

- **Body forces**

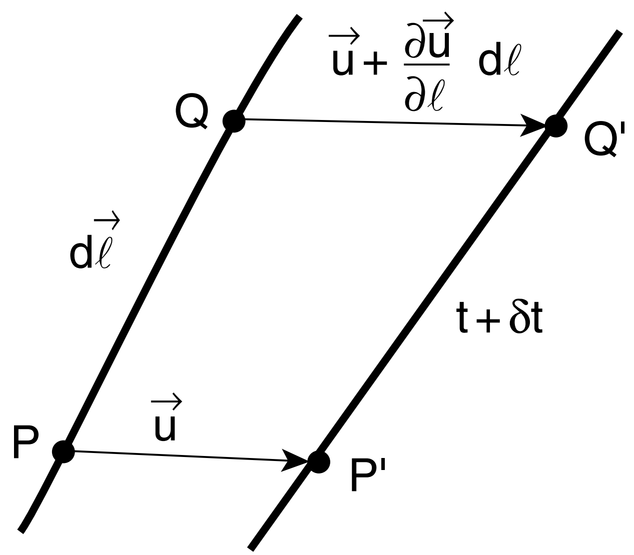
PHYSICAL INTERPRETATION OF VORTICITY CHANGE EQUATION

- **What does** $(\vec{\omega} \cdot \nabla)\vec{\mathbf{u}}$ **mean?**
  - ω **times derivative of** $\vec{\mathbf{u}}$ **in direction along** $\vec{\omega}$
  - $\mathbf{d}\vec{\ell}$ **is element of vortex line – quantity is** $\frac{\omega \partial \vec{\mathbf{u}}}{\partial \ell}$



- **If there is a velocity variation along a vortex line, the vorticity changes**

BEHAVIOR OF FLUID LINE (MATERIAL LINE)



- **At t, dl is $\overline{PQ}$**
- **P moves $\vec{u}\delta t$ in δt**
- **Q moves $\left(\vec{u} + \frac{\partial \vec{u}}{\partial \ell} d\ell\right)\delta t$**
- **Change in line element PQ is**

$$\left(\vec{u} + \frac{\partial \vec{u}}{\partial \ell} d\ell\right)\delta t - \vec{u}\delta t = \frac{\partial \vec{u}}{\partial \ell} d\ell \, \delta t$$

CHANGE OF LENGTH OF A FLUID LINE

- **Change of length over a time δt**

$$\delta\left(\mathbf{d}\vec{\ell}\right) = \frac{\partial \vec{\mathbf{u}}}{\partial \ell}\,\mathbf{d}\ell\,\delta\mathbf{t} \qquad \underline{\textbf{or}} \qquad \boxed{\frac{\delta\left(\mathbf{d}\vec{\ell}\right)}{\delta\mathbf{t}} = \frac{\partial \vec{\mathbf{u}}}{\partial \ell}\,\mathbf{d}\ell}$$

Rate of change of length of line element

- **Fractional rate of change of length:**

$$\frac{\mathbf{1}}{\mathbf{d}\ell}\frac{\mathbf{Dd}\vec{\ell}}{\mathbf{Dt}} = \frac{\partial \vec{\mathbf{u}}}{\partial \ell}$$

- **Vorticity equation**

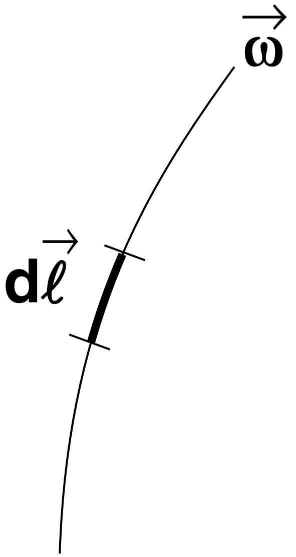
$$\frac{1}{\omega}\frac{D\vec{\omega}}{Dt} = \frac{\partial \vec{\mathbf{u}}}{\partial \ell}$$

- **So if $d\vec{\ell}$ is a line element on a vortex line**

$$\frac{1}{\omega}\frac{D\vec{\omega}}{Dt} = \frac{1}{d\ell}\frac{Dd\vec{\ell}}{Dt}$$

A solution is $\vec{\omega} = K\, d\vec{\ell}$

constant



- **Vortex lines and fluid lines behave the same way**

- **<u>VORTEX LINES MOVE WITH THE FLUID</u>**

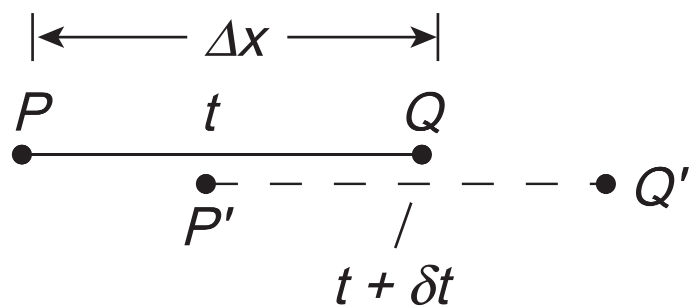
VORTEX LINES MOVE WITH THE FLUID

- **Vortex line stretched $\Rightarrow \vec{\omega}$ increases**
- **Vortex line "tipped" $\Rightarrow$ new component**
- **This is a true 3-D effect; not present in 2-D**
- **So:**
  - **Inviscid flow, incompressible, uniform density, conservative body forces**
  - **Vortex lines are "locked" to fluid particles**

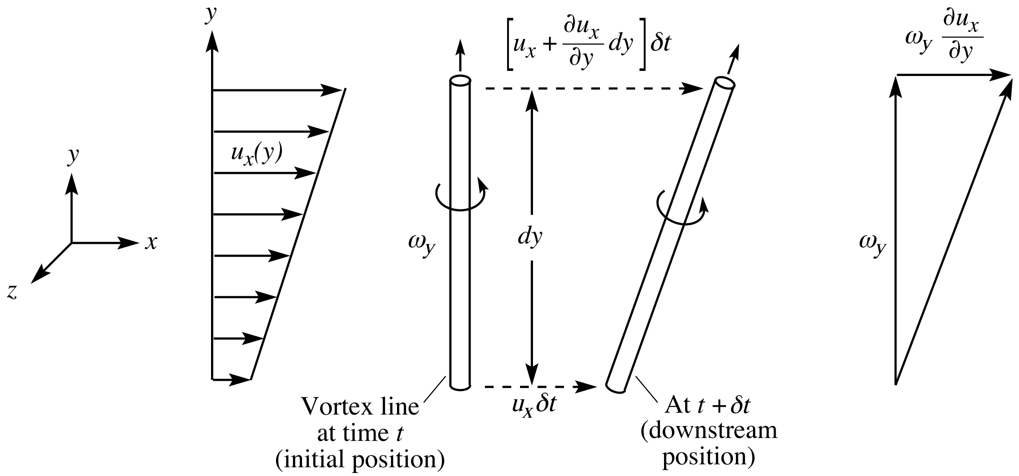
INTERPRETATION OF $(\vec{\omega}\cdot\nabla)\vec{u}$

- **Examine x-component of vorticity equation to see physical meaning of terms**

$$\frac{D\omega_x}{Dt} = \omega_x \frac{\partial u_x}{\partial x} + \omega_y \frac{\partial u_x}{\partial y} + \omega_z \frac{\partial u_x}{\partial z}$$



- **Change in length of PQ is** $\left(\frac{\partial u_x}{\partial x}\Delta x\right)\delta t$
- **Fractional change is 1/Δx times this**
- **Rate of change of x-vorticity is rate of change of length of vortex line element** $\frac{1}{\omega_x}\frac{D\omega_x}{Dt} = \frac{\partial u_x}{\partial x}$ **<u>vortex stretching</u>**



- **Vortex lines move with the fluid**

- **ω_y gets tipped into x-direction, creating an x-component of vorticity**

- **Rate of creation of x-vorticity can be found as follows:**

Relative to P, Q moves $\frac{\partial u_x}{\partial y} \Delta y \, \delta t$

Small angle α **is** $\tan\alpha \approx \alpha = \frac{\partial u_x}{\partial y}$

Small change in vorticity is $\frac{\delta\omega_x}{\omega_y} = \tan\alpha \approx \frac{\partial u_x}{\partial y}$

INTERPRETATION OF $(\vec{\omega} \cdot \nabla)\vec{u}$

- **This term represents the tipping of y (or z-) components of vorticity into the x-direction.**

•This is *TRULY A 3-D EFFECT*. It does not occur in 2-D flow.

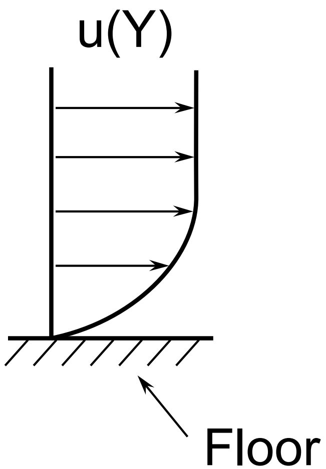
•FOR 2-D FLOW, the vorticity and the velocity are both functions of x and y. Plug into the vorticity equation and find that

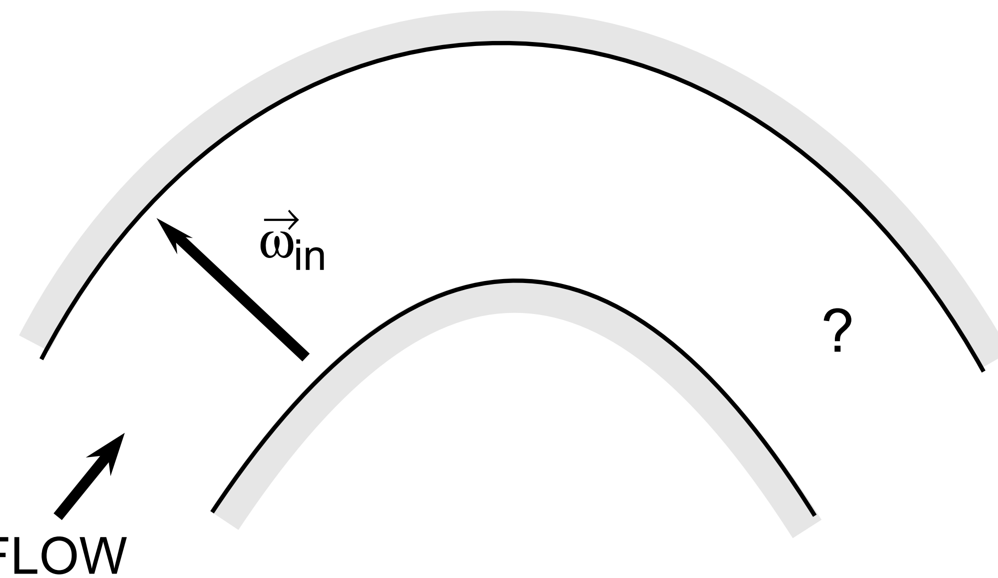
$$\frac{D\omega_y}{Dt} = \frac{D\omega_z}{Dt} = 0$$

$$\frac{D\omega_z}{Dt} = \boxed{\frac{D\omega}{Dt} = 0}$$

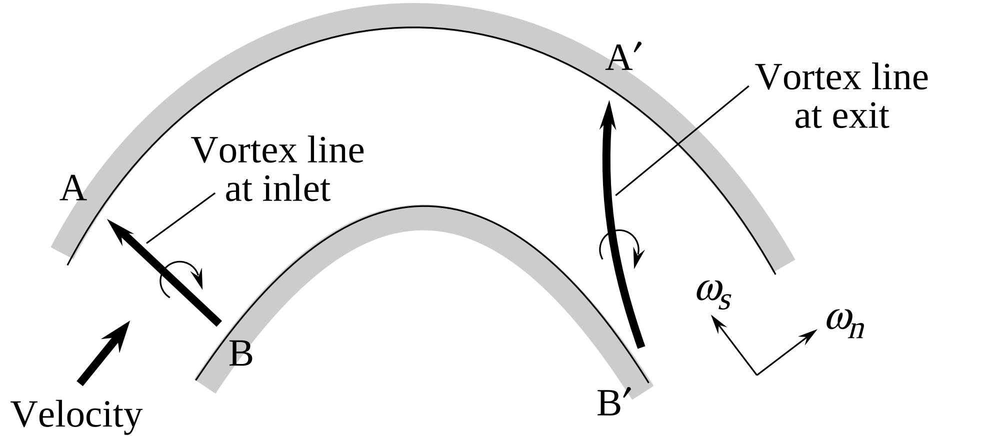
APPLICATIONS TO SOME RELEVANT FLOWS

- **Can predict changes in vorticity by examining kinematics of vortex lines**
- **Example: Secondary flow in a bend, blade passage**

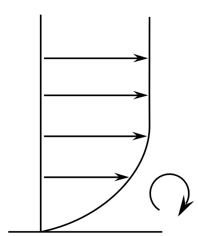




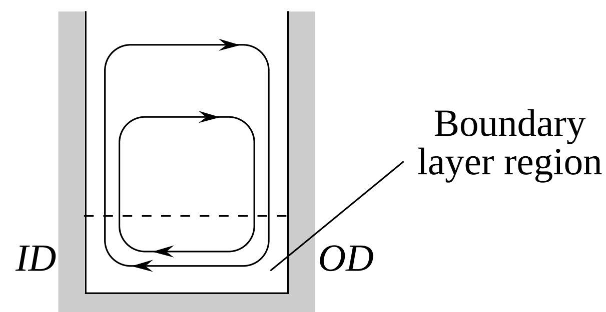
GENERATION OF STREAMWISE VORTICITY (AND SECONDARY FLOW) BY CONVECTION OF VORTEX LINES THROUGH A BEND



Top View of Passage



Inlet Streamwise Velocity



Secondary Streamlines at Passage Exit

- **Note: Can also understand in terms of pressures and accelerations**
- **In free stream**

$$\frac{\partial p}{\partial n} = \rho \frac{U_{freestream}^2}{r_c}$$

← **Local radius of curvature** (for r_c)

- **In boundary layer on floor,**

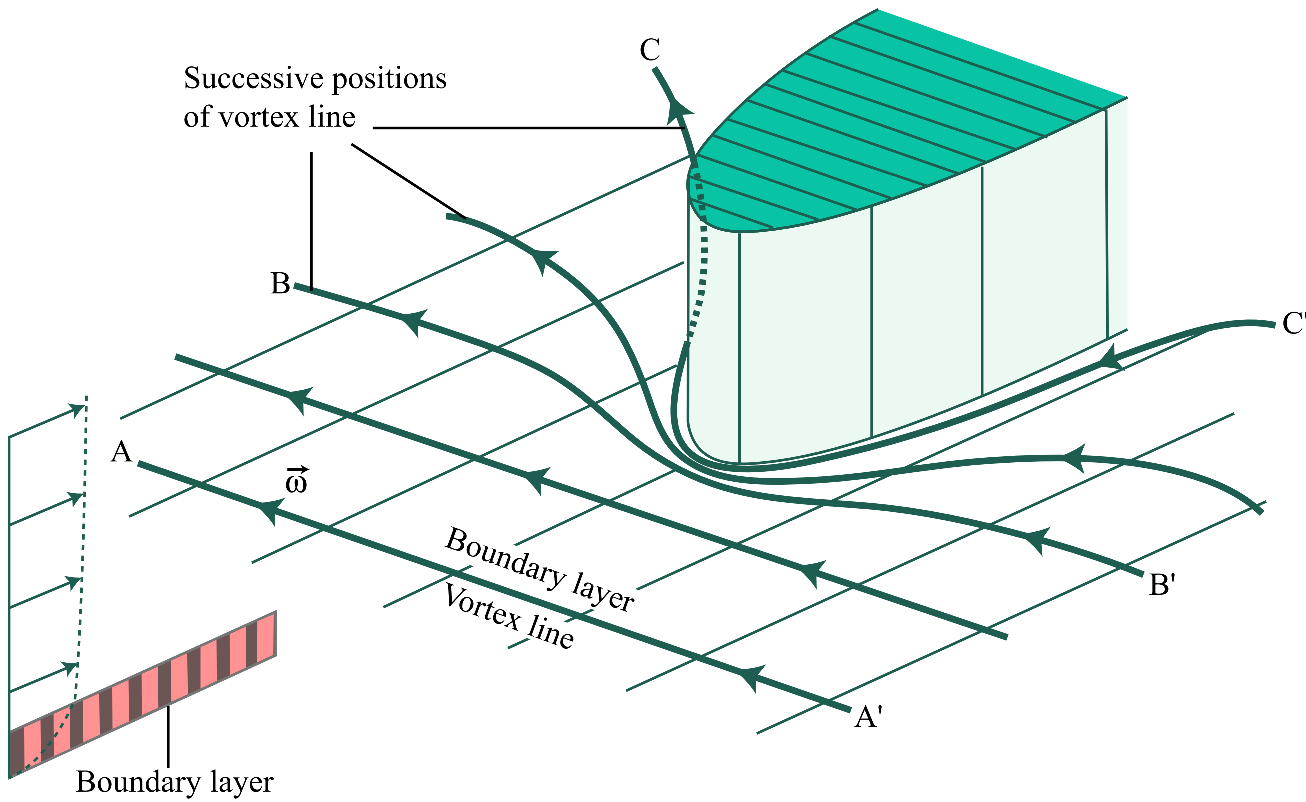
same $\frac{\partial p}{\partial n}$ $\qquad \rho \frac{u_{b.l.}^2}{r_{c_{b.l.}}} = \rho \frac{U_{freestream}^2}{r_c}$

$$\frac{r_{c_{b.l.}}}{r_{c_{freestream}}} \sim \left(\frac{u_{b.l.}}{U_{freestream}} \right)$$

- **So more curvature, sharper turn, in boundary layer ⇒ radially inward acceleration**
- **Boundary layer - less centrifugal force, same** $\frac{\partial p}{\partial n}$

⇒ Radially inward accelerationof fluid

HORSESHOE VORTEX
(Strut, Turbomachinery Blade)



Horseshoe vortex upstream of a strut; vortex lines wrapped around obstacle

HORSESHOE VORTEX UPSTREAM OF WEDGE [Schwind]

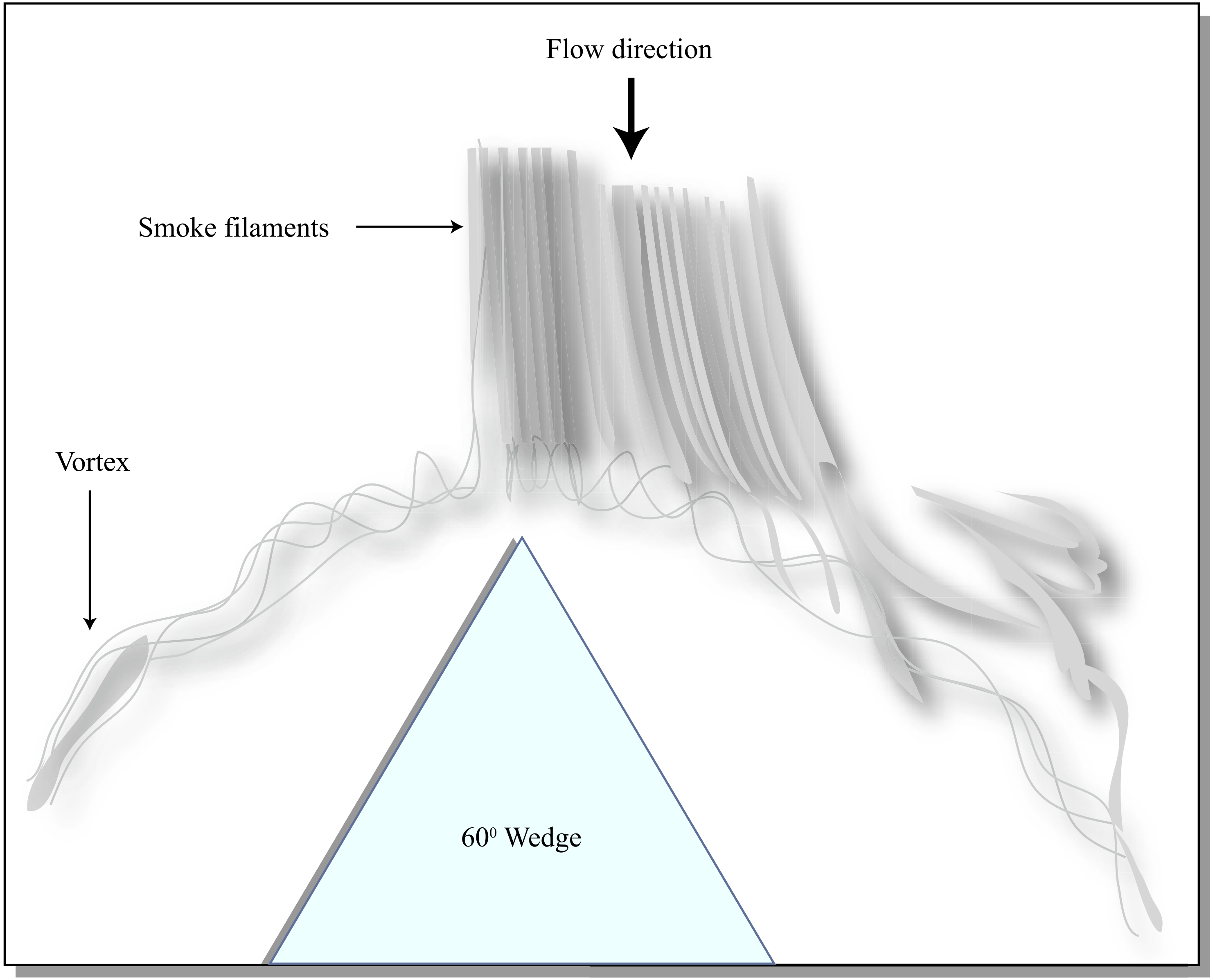
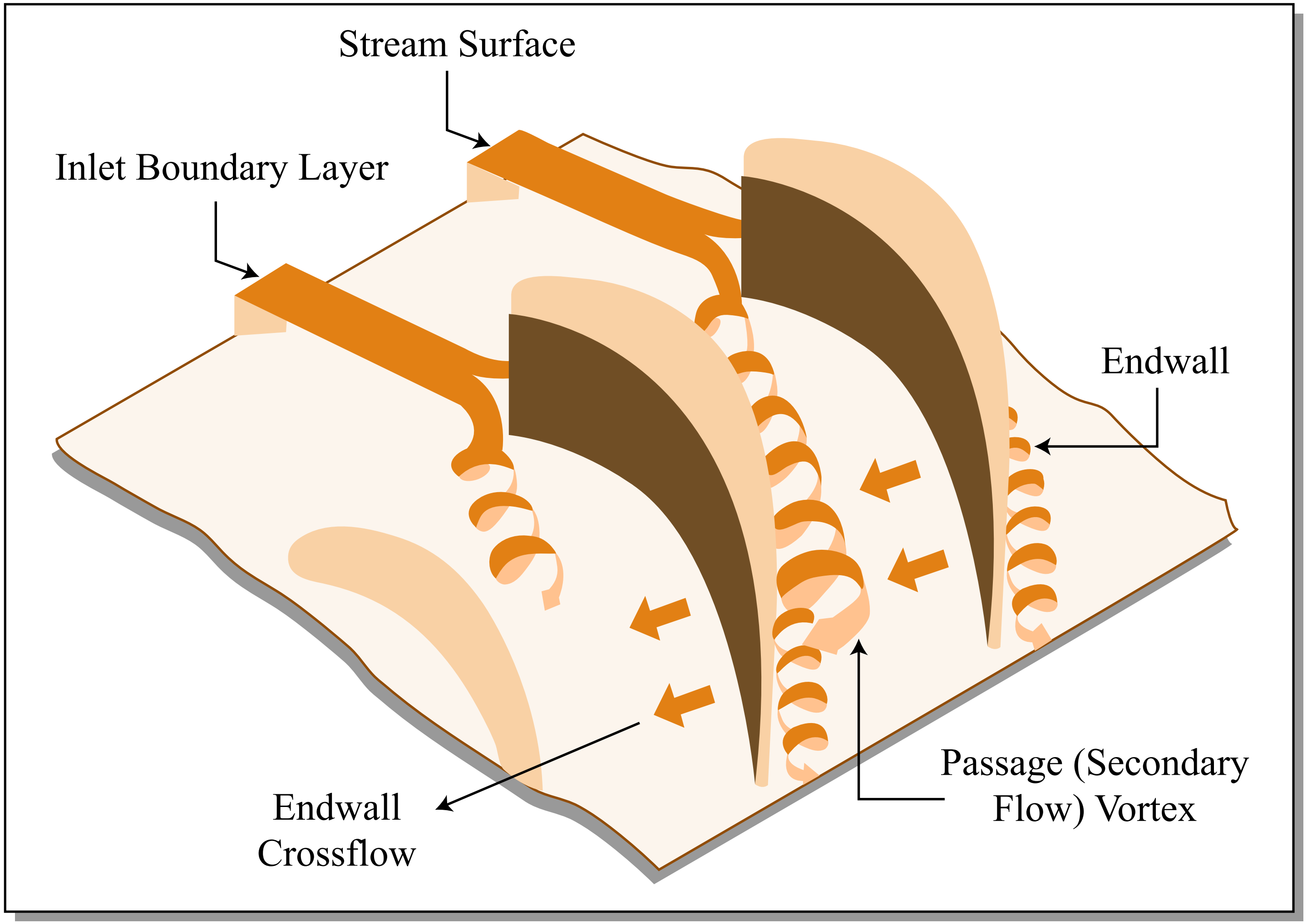


Figure by MIT OCW.

SKETCH OF TURBINE SECONDARY FLOW [Langston]



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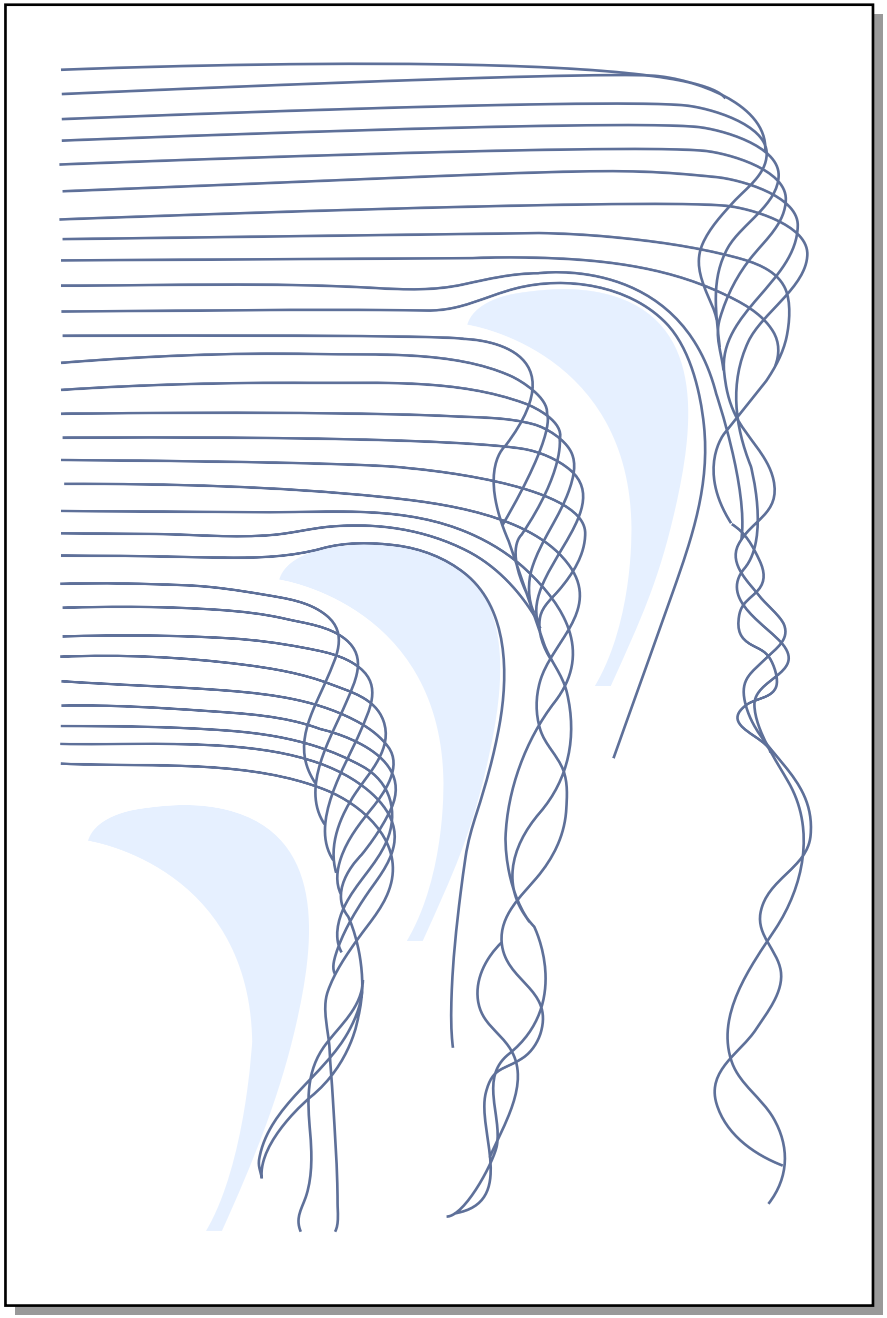
SECONDARY FLOW IN TURBINE BLADES [Gostelow]



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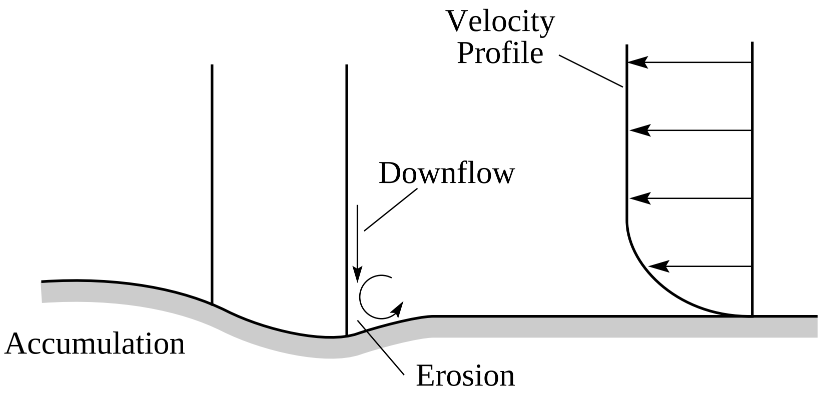
FLOW ROUND A LOG (MY BACK YARD)

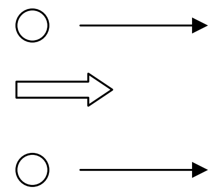


BEHAVIOR OF A VORTEX RING

- **Consider <u>two</u> infinite vortex tubes**

1

2

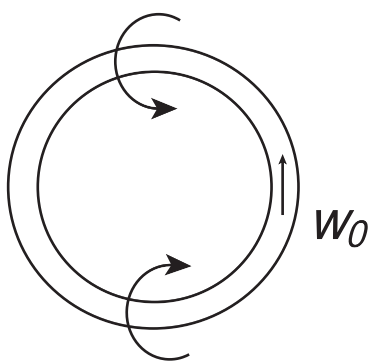


Velocity at 1 “due to” 2

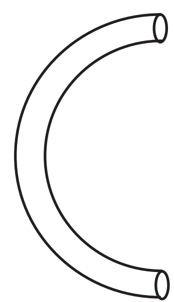
Velocity at 2 “due to” 1

Sense of vorticity

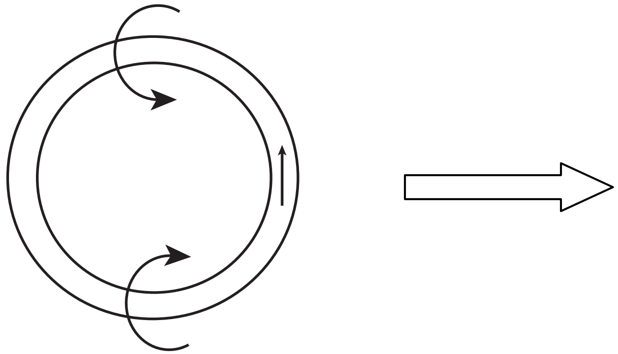
So two vortices will move with constant velocity, u = ?



Vortex ring
Slice thru ring



Vortex ring has some similarities to vortex pair



Translates along
With velocity = ?

Seems easier to "understand" using vorticity arguments than using pressure (force) description

VORTICITY MEASURES ANGULAR VELOCITY NOT ANGULAR MOMENTUM

Spherical fluid particle with $\omega_x = \omega_z = \omega_0$ at time *t*

Motion with u_z outwards, u_z, u_y inwards

Suppose $\frac{\partial u_z}{\partial z} = \varepsilon$ and symmetric about *z* - axis

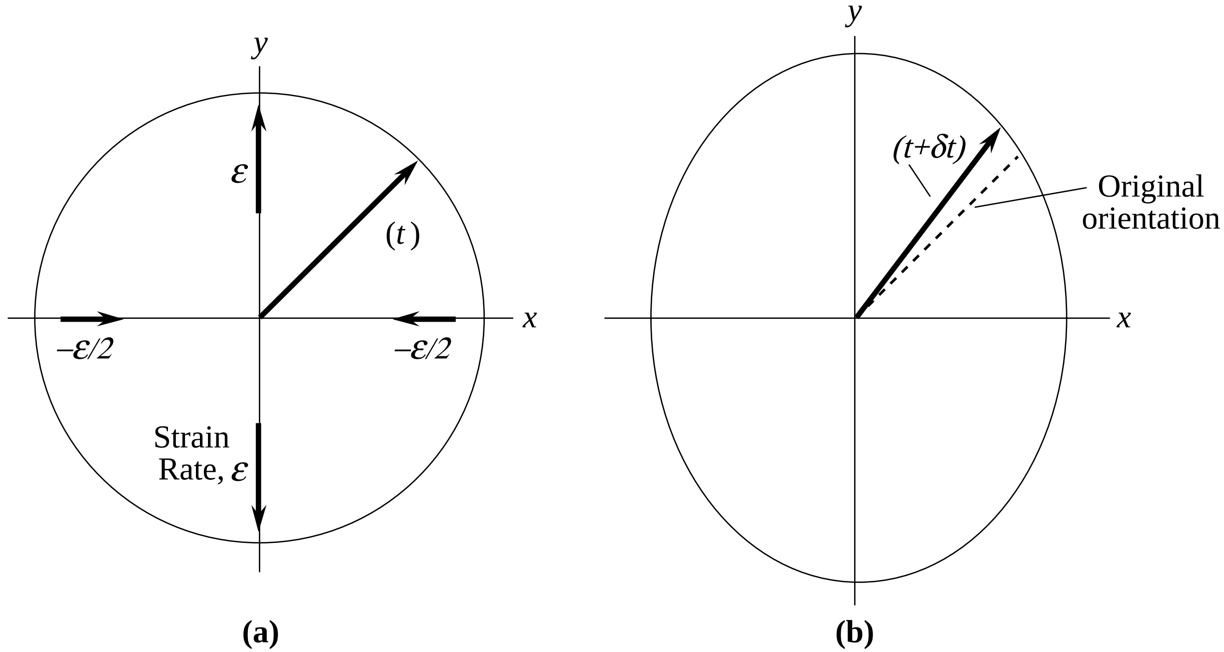
$$\frac{\partial u_x}{\partial x} = \frac{\partial u_y}{\partial y} = -\varepsilon / 2$$

Vorticity vector is at 45^o to *x, z*, axes initially

What happens to vorticity vector with time?

ω_z **is increased (stretched) by the motion**

ω_x **is decreased (contracted) by the motion**



$$\frac{D\vec{\omega}}{Dt} \neq 0$$

The vorticity vector is "tipped" by the deformation of the particle.

This is change in vorticity. What about change in angular momentum?

CHANGES IN ANGULAR MOMENTUM?

• **Only pressure forces act.**

• **Pressure forces are normal to surface of a spherical particle**

• **Pressure forces act through the center of mass of the particle and exert no torque**

• **No torque => No change in angular momentum**

Torque = rate of change of angular momentum

• **Conclusion is that vorticity $(\vec{\omega})$ changes but angular momentum $(\vec{H})$ does not.**

• **How does this happen? (What is going on physically?)**

• **To see this, let's look at the changes in angular velocity and angular momentum using the tools familiar from 3-D dynamics**

ANGULAR MOMENTUM AND VELOCITY CHANGES

$\vec{H} = \bar{\bar{I}}\vec{\omega}$; $\bar{\bar{I}}$ is the inertia tensor - 9 quantities

$$\bar{\bar{I}} = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix}$$ for a sphere, where $I_x = I_y = I_z = I$

$$\frac{d\vec{H}}{dt} = \vec{\omega}\frac{d\bar{\bar{I}}}{dt} + \bar{\bar{I}}\frac{d\vec{\omega}}{dt}$$

This is worked out in the notes in detail, but can see here one component, $H_x = I_x \omega_x$.

$$\frac{dH_x}{dt} = \omega_x \frac{dI_x}{dt} + I_x \frac{d\omega_x}{dt}$$

ω_x decreases and moment of inertia (I_x) about x-axis increases

As shown in notes, there are equal and opposite terms so that

$$\frac{dH_x}{dt} = 0$$

ANGULAR VELOCITY CHANGES

- For a 2-D flow we can calculate the change in angular momentum by considering torques

$$Torque = \frac{d(\text{Angular momentum})}{dt} = \frac{d(I\Omega)}{dt}$$

$$Torque = I\frac{d\Omega}{dt} + \Omega\frac{dI}{dt}$$

I for a small cylinder $= MR^2/2$; M = mass, R = radius

Cylinder deforms to an ellipse with I $= \frac{M}{4}\left(c^2 + b^2\right)$

$$b = R + \frac{\partial u_x}{\partial x}\delta t \quad ; \quad c = R - \frac{\partial u_y}{\partial y}\delta t$$

$$\delta I = \frac{M}{4}R^2\left[2R\left(\frac{\partial u_x}{\partial x}\right) + 2R\left(-\frac{\partial u_y}{\partial y}\right)\right]\delta t = 0 \quad ; \quad \text{Continuity}$$

CASE 2 - VORTICITY CHANGES IN INVISCID, INCOMPRESSIBLE FLOW WITH NON-UNIFORM DENSITY

$$\frac{D\vec{\omega}}{Dt} = (\vec{\omega} \cdot \nabla)\vec{V} - \nabla \times \left(\frac{1}{p} \nabla\rho \right)$$

$$\frac{D\vec{\omega}}{Dt} = (\vec{\omega} \cdot \nabla)\vec{V} + \frac{1}{\rho^2} \boxed{\nabla p \times \nabla\rho} \longleftarrow \text{new term}$$

- **Gradient is normal to surfaces having constant value**
- **If ∇p x $\nabla\rho \neq 0$** **∇p not parallel to $\nabla\rho$**

PHYSICAL MECHANISM FOR VORTICITY PRODUCTION

- **Constant density surfaces not aligned with constant pressure surfaces leads to vorticity production**

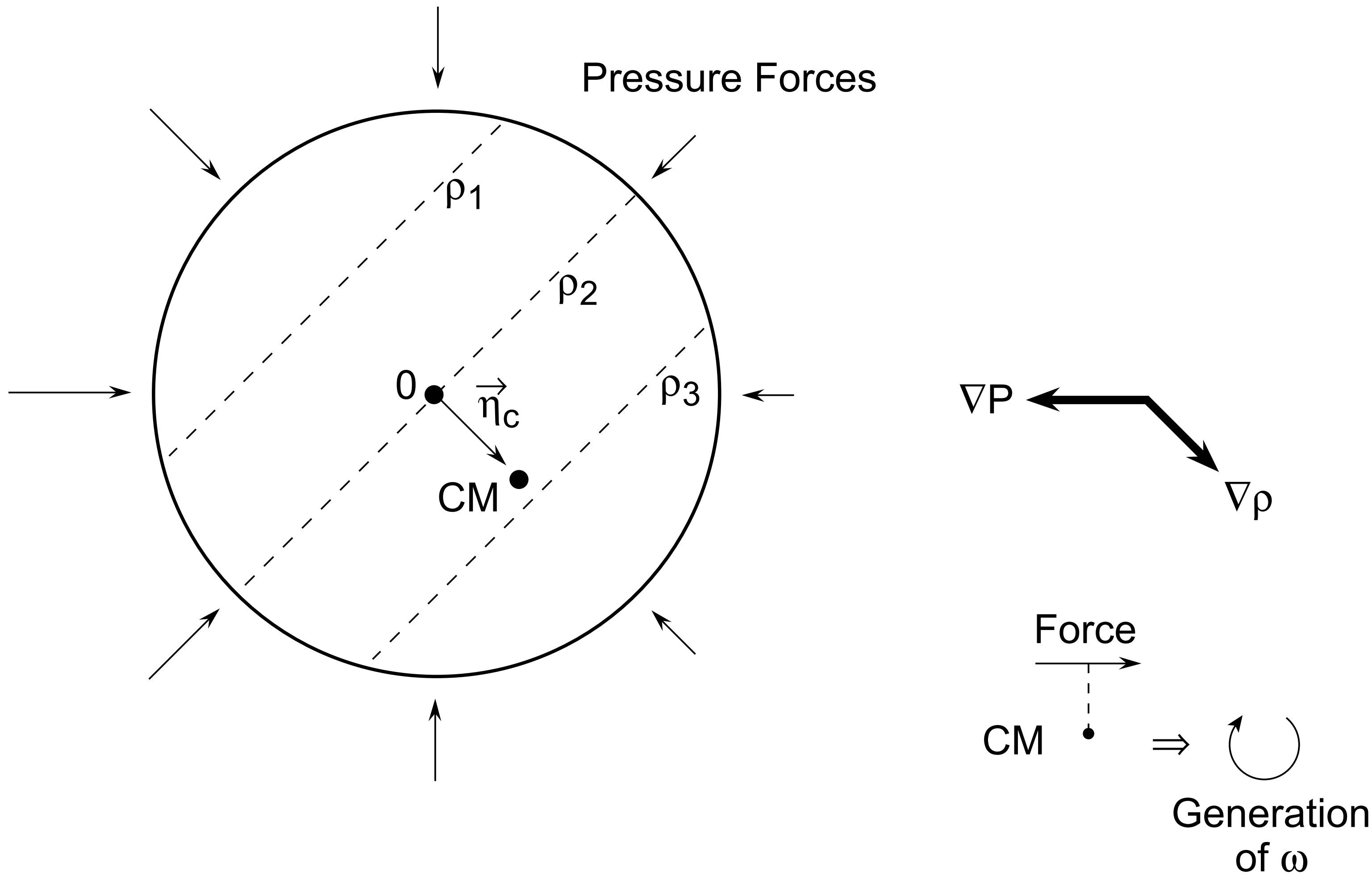
- **Look at two-dimensional example**

  (We already understand the $(\vec{\omega} \cdot \nabla)\vec{u}$ term)

$$\frac{D\omega}{Dt} = \frac{1}{\rho^2} \nabla\rho \times \nabla p$$

- **If surfaces of constant ρ and constant p are not aligned, there is a <u>torque</u> about the center of mass**

TORQUE IN A NON-UNIFORM DENSITY FLUID



Generation of vorticity due to the interaction of pressure and density gradients: pressure force torque about the center of mass of a fluid particle

$\delta I = 0$ for this problem (2 - D, incompressible spherical)

Torque directly related to changes in angular velocity

$$\text{Torque} = I\frac{d\Omega}{dt} = \vec{r} \times \vec{F}$$

$$\vec{r} = -\frac{1}{\rho_o}\nabla\rho R^2/4 \quad ; \quad \vec{F} = -\nabla p\left(\pi R^2\right)$$

$$\vec{r} \times \vec{F} = \left\{\frac{1}{\rho^2_o}\nabla\rho \times \nabla p\left[\left(\pi\rho_o R^2\right)\frac{R^2}{2}\right]\right\}$$

Underlined term is I

$$Torque = I\frac{d\Omega}{dt} \text{ becomes}$$

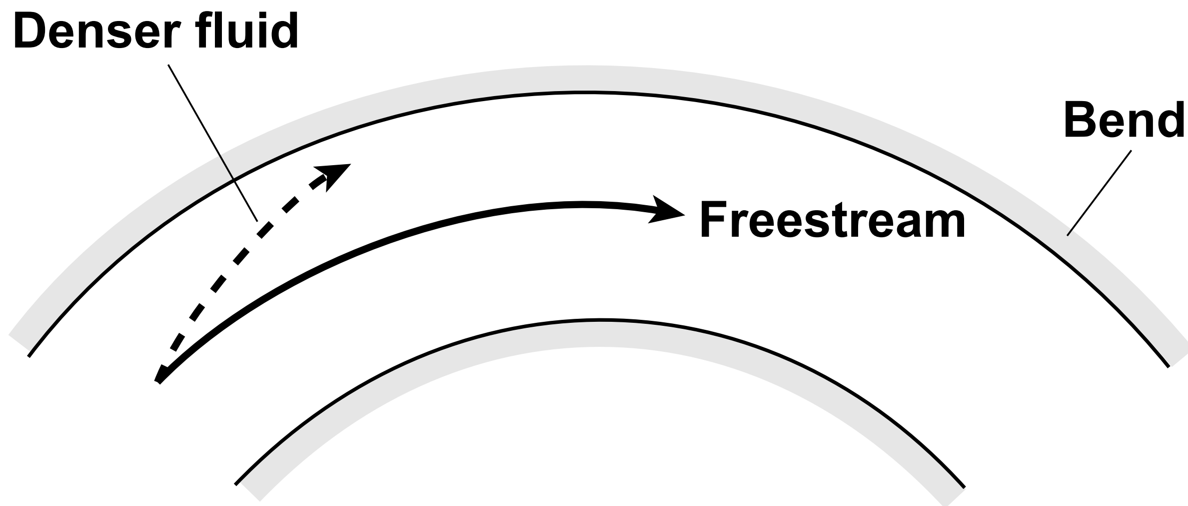
$$\frac{1}{\rho_o}\nabla\rho \times \nabla p = 2\frac{d\Omega}{dt} = \frac{d\omega}{dt}$$

Fluid dynamics is a branch of dynamics; the connections are useful and helpful

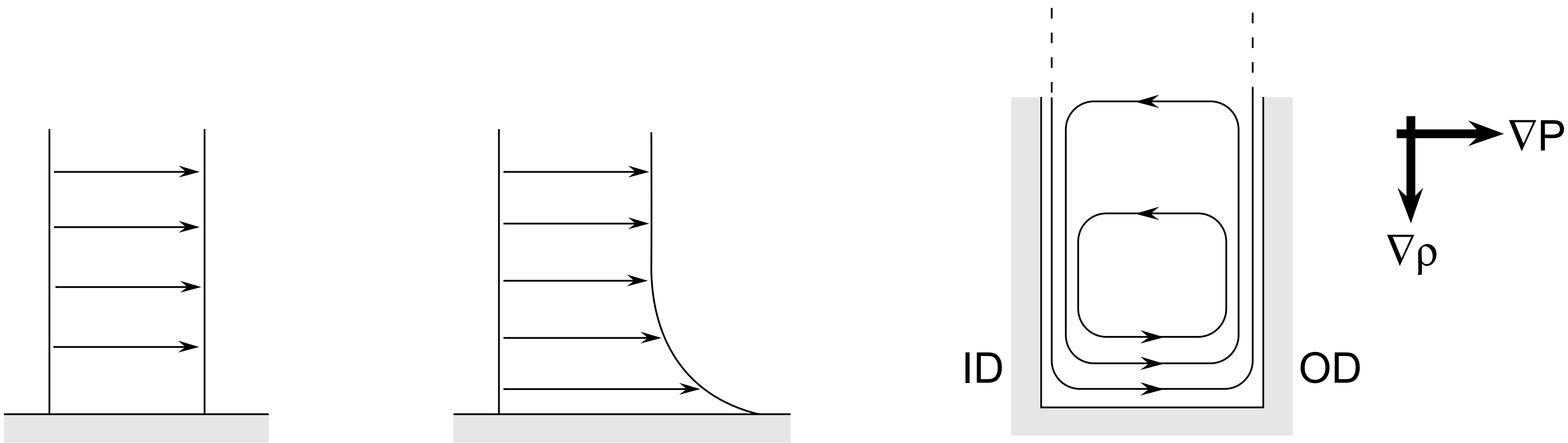
EXAMPLES OF VORTICITY PRODUCTION DUE TO "$\nabla\rho$ x ∇p"

1) **Flow round a bend**

- **Initial conditions: $\vec{u}$ = constant, $\vec{\omega} = 0$**
  - $\rho = \rho(z)$
  - $\nabla\rho$ **points down**
  - ∇p **points radially outward**
- $\nabla\rho$ **x** ∇p **is in streamwise direction; leads to secondary circulation**



- **“Primitive variable” explanation:**
  - **Pressure gradient set by free stream ρ; $\partial p/\partial r = \rho u^2$**
  - **Fluid near bottom is denser, won’t follow free streamlines (too much inertia to be turned), flows to outside of bend**



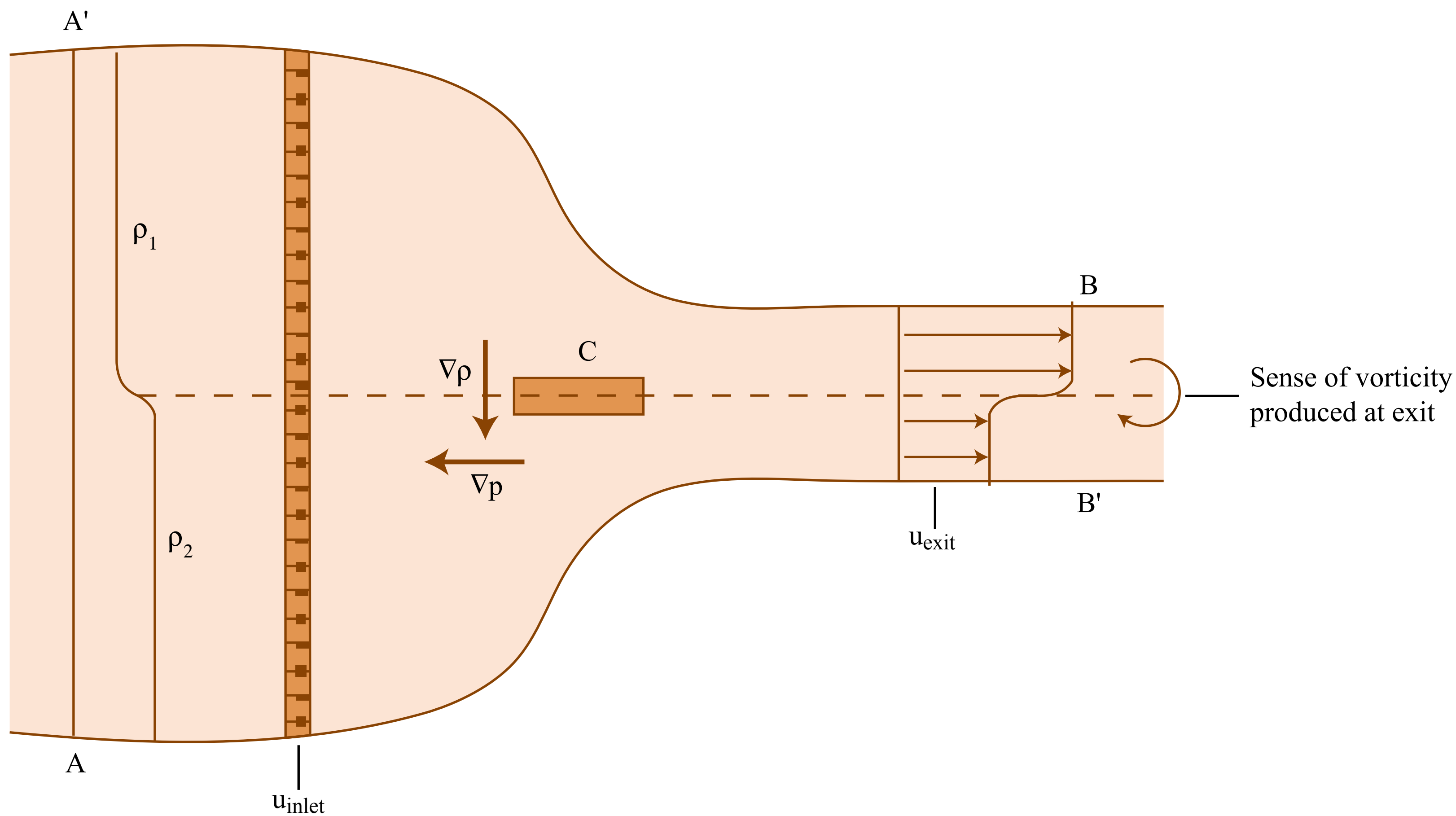
Inlet Streamwise Velocity:
$\vec{\omega}_{\mathbf{inlet}} = 0$

ρ**(z)**$_{\mathbf{inlet}}$

Secondary Streamlines at Passage Exit

Generation of streamwise vorticity (and secondary flow) due to interaction of pressure and density gradients

OUTFLOW FROM RESERVOIR OF THERMALLY STRATIFIED FLUID (COMBUSTOR)



Vorticity production in a fluid of non-uniform density; channel with inlet area >> exit area

Figure by MIT OCW.

VORTICITY PRODUCTION IN STRATIFIED FLOW

- **Alternative explanation**
  - **High and low density streams have same Δp - same force**
  - **Low density stream has less mass**
  - **Aa = F/m $\Rightarrow$ acceleration of low density stream is higher**
  - **Final velocity for ρ_1 stream > than for ρ_2 stream**

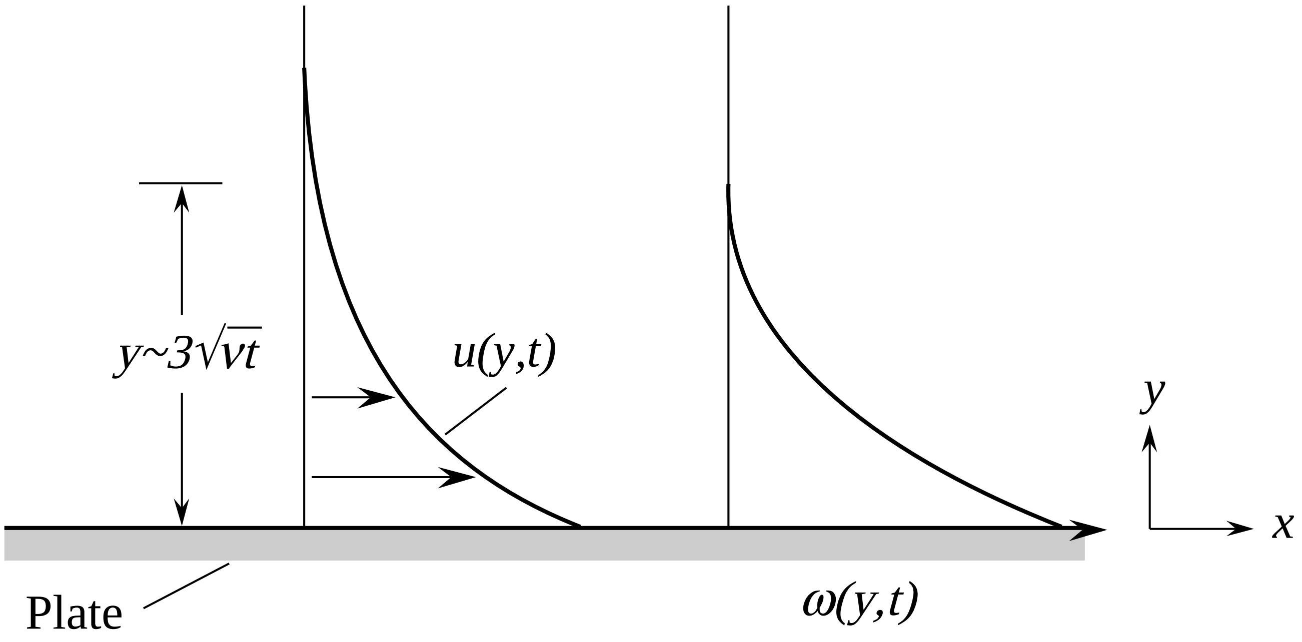
CASE 3 - VISCOUS FLOW

Incompressible, const ρ, conservative $\vec{F}_{body}$

Look at basic problem: Viscous flow near infinite flat plate which we impulsively start - 2-D flow $\frac{\partial}{\partial x} = 0$

Equations for velocity, vorticity

Generation of vorticity due to the action of viscous forces: impulsively started plate: *U (0,t) = 0, t < 0; U (0, t) = U, t > 0*



$$\frac{\partial u_x}{\partial t} = \nu \frac{\partial^2 u_x}{\partial y^2} \longleftarrow \text{ net viscous forces}$$

and

$$\frac{\partial \omega}{\partial t} = \nu \frac{\partial^2 \omega}{\partial y^2} \longleftarrow \text{ net viscous torque } \left(\nabla x \vec{F}_{visc}\right)$$

Vorticity is altered due to viscous effects
Viscous forces can exert a torque
Dynamic correspondence worked out in notes

Note time and length scales from form of solution $\frac{\omega}{U/\sqrt{\nu t}} \propto e^{y^2/2\nu t}$

δ ~ distance of appreciable vorticity:

$$\frac{y^2}{\nu t} \sim 1 \Rightarrow \delta \sim \sqrt{\nu t}$$

VISCOUS STRESSES AND TORQUES ON A FLUID ELEMENT

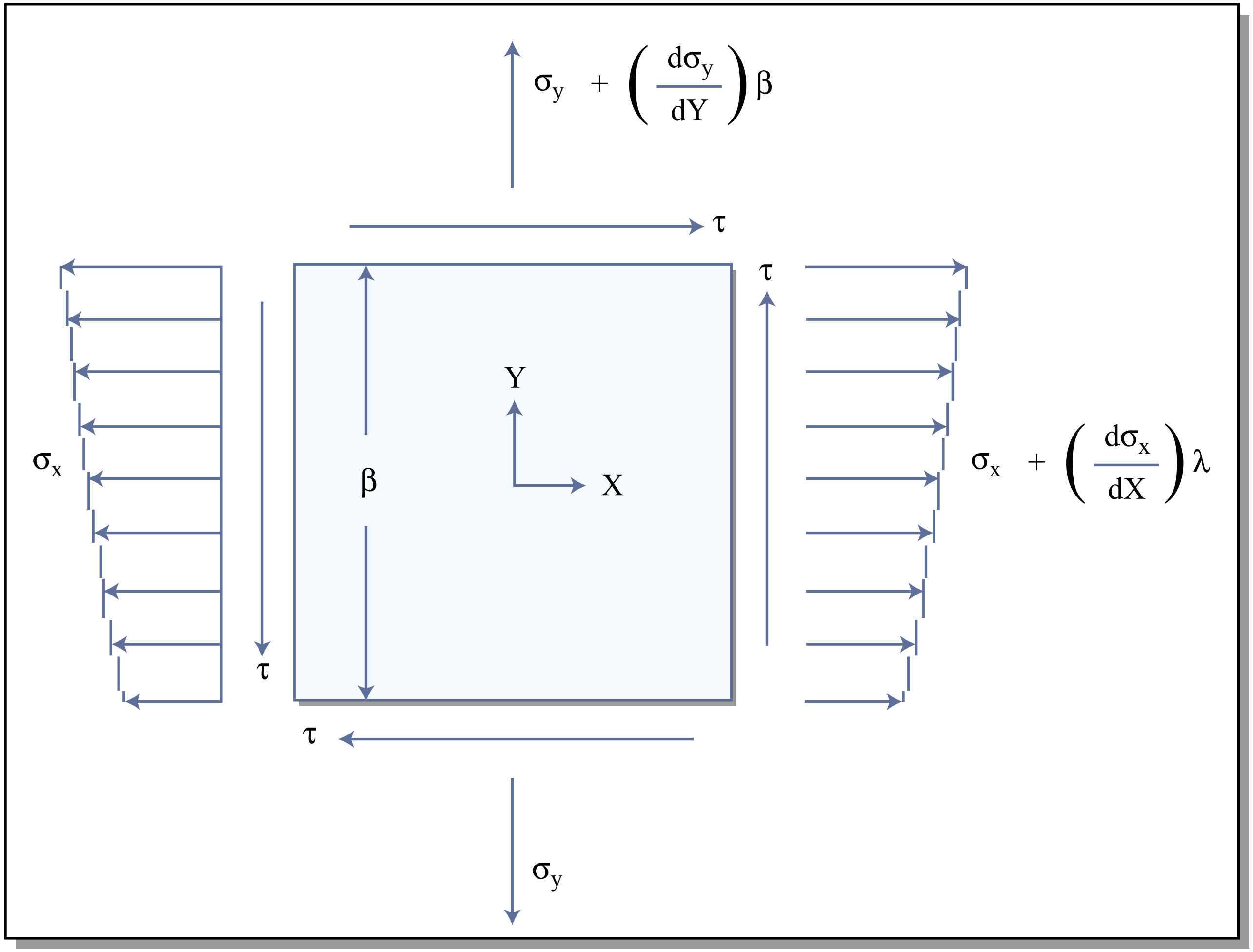


Figure by MIT OCW.

VISCOUS STRESSES AND TORQUES ON A FLUID ELEMENT

- **Region near wall of appreciable vorticity scales as** $\sqrt{\nu t}$ **.**

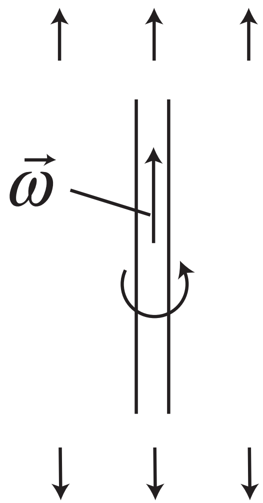
- **Flow along a stationary wall**

  $$t \sim x/U$$

  $$\delta \sim \sqrt{\nu x/U}$$

- **We have been looking at effects one-by-one. Now put two together: vortex stretching plus effects of viscosity**

VORTEX STRAINED (STRETCHED) ALONG ITS AXIS



Conditions

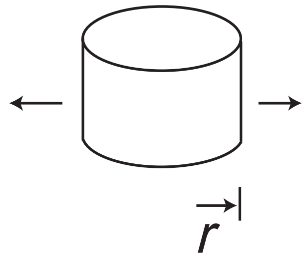
Axisymmetric flow

Constant strain rate, α

Use cylindrical coordinate system

Velocity components: $u_z = \alpha z$ **(Strain rate is** ***z*****)**

Continuity -



$$\left\lfloor \frac{\partial u_z}{\partial z} dz \right\rfloor \pi r^2 + 2\pi u_r r dz = 0$$

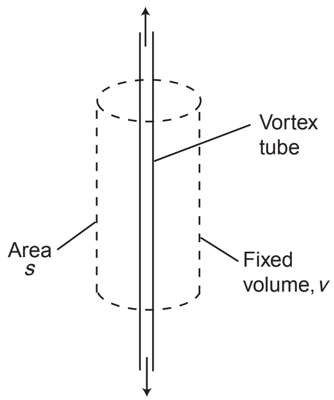
Flow thru top & bottom

Flow thru sides

$$u_r = -\alpha r/2$$

ALTERNATIVE VIEW OF PROCESS

Consider volume fixed in space



Write expression for changes of vorticity in fixed volume, *v*

$$\frac{\partial}{\partial t}\int_{\nu} \vec{\omega} d\nu = \int_{s} (\vec{\omega}\cdot\nabla)\vec{u} d\nu - \int_{s} (n'\cdot\vec{u})\vec{\omega} ds$$

$$-\int \hat{n}\times F_{visc} ds$$

Steady flow - volume surfaces away form viscous regions

$$\int_{s} (\hat{n}\cdot\hat{u})\vec{\omega} ds = \int_{\nu} (\omega\cdot\nabla)\vec{u} d\nu$$

Flux out **Produced inside**

$$u_\theta = \frac{\Gamma}{2\pi r}\left(1 - e^{-\alpha r^2/4\nu}\right)$$

Only vorticity is in *z* direction

$$\omega_z = \frac{\Gamma}{\pi} e^{-\alpha r^2/4\nu}$$

Appreciable vorticity only exists for $r\sqrt{4\nu/\alpha}$, say (whatever initial Distribution is)

Vorticity equation

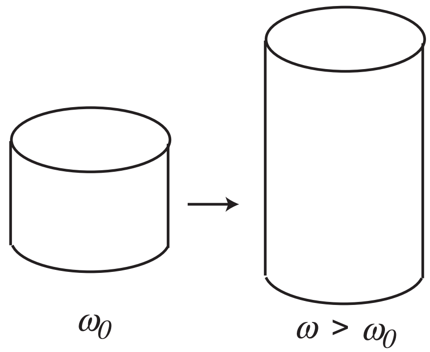
$$u_r \frac{\partial \omega_z}{\partial r} = \omega_z \frac{\partial}{\partial_z} + \nu\left[\frac{\partial^2 \omega_z}{\partial r^2} + \frac{1}{r}\frac{\partial (r\omega_z)}{\partial r}\right]$$

Change in vorticity as particle (ring of particles) is (are) convected inward due to:

- **a) Vortex stretching and vorticity production**
- **b) Viscous torques**

EXAMINE SMALL ELEMENT

a) As element's radius shrinks, angular velocity increases
(ang. mom. is const)



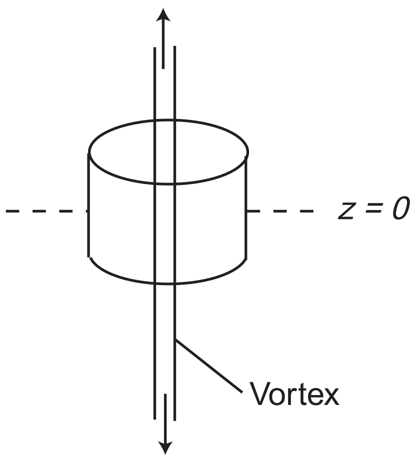
b) But as element "spins up" viscous torques try to decrease its ω.

• A balance between a) and b). Also, strain rate α sets size of vortex (sets radius)

This is a model problem with applications:

Horse shoe vortex
Inlet vortex

One other point: look at control volume of radius *r* $>> \sqrt{\nu/\alpha}$



No vorticity on sides - fluid comes in irrotational

Fluid continually leaving thru top and bottom with vorticity

Net outflow of vorticity because vorticity is produced inside by vortex stretching (straining)

INLET VORTEX

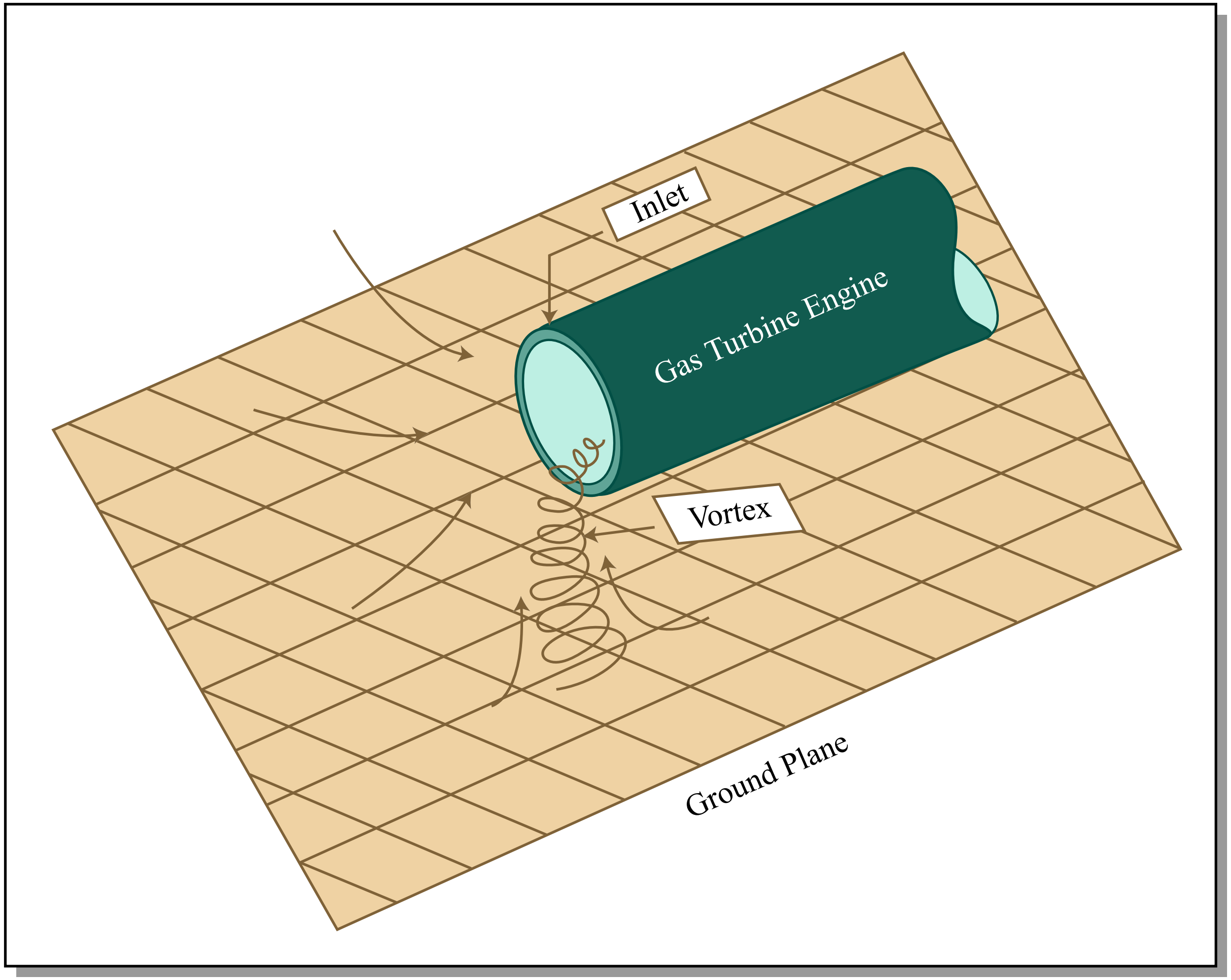


Figure by MIT OCW.

INGESTION OF VORTEX LINES INTO INLET

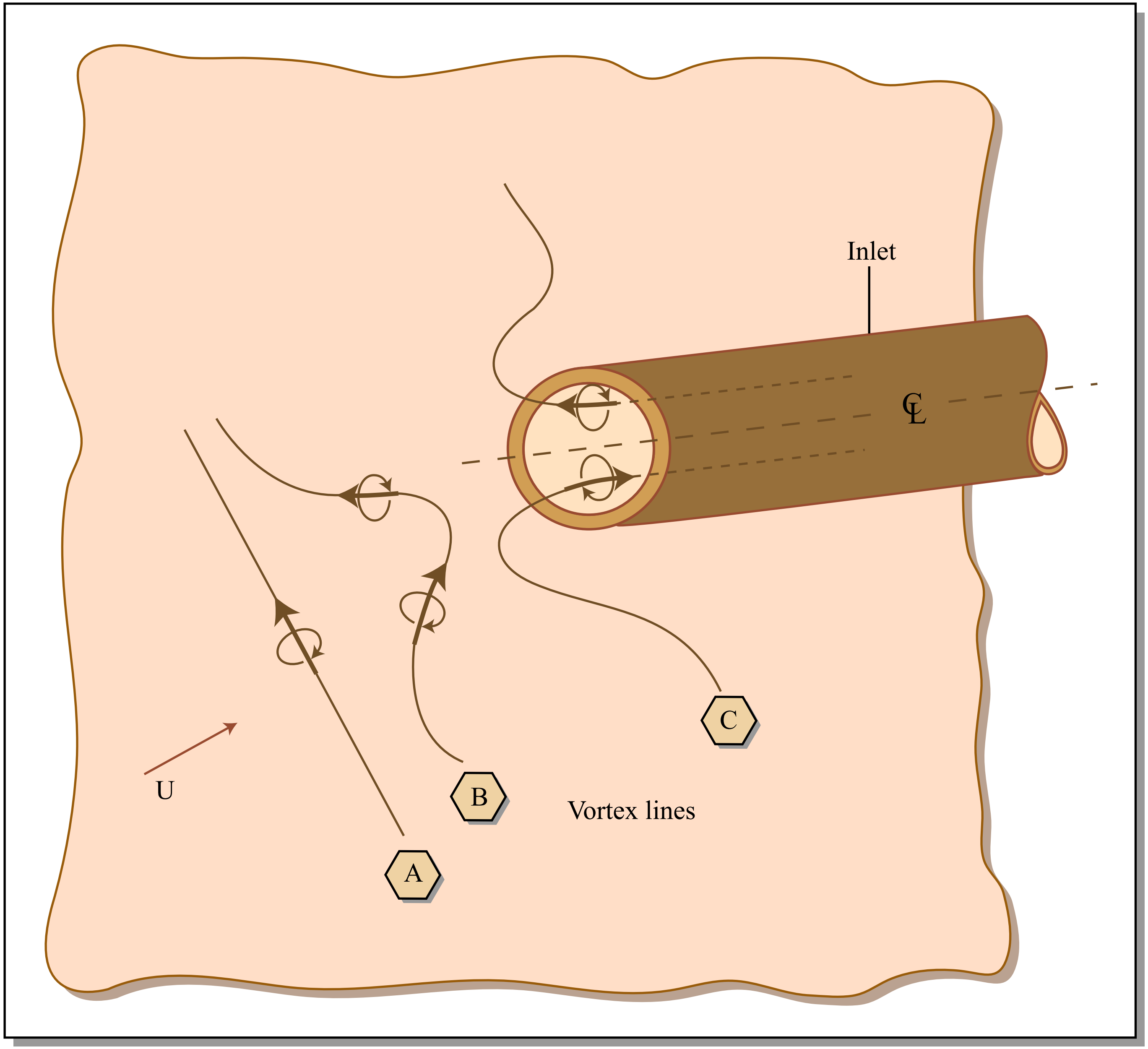


Figure by MIT OCW.

INGESTION OF VORTEX LINES INTO AN INLET: A BASIC QUESTION

- **Vortex lines cannot end in fluid. If a vortex line is ingested into an inlet, there are thus two legs of the line that "stick out".**

- **Only one vortex seems to be observed, however!**

CASE 4 – COMPRESSIBLE FLOW

- **Inviscid, conservative body force (these act as in incompressible case)**

- **Start with general vorticity equation**

$$\frac{D\vec{\omega}}{Dt} = (\vec{\omega} \cdot \nabla)\vec{u} + \vec{\omega}(\nabla \cdot \vec{u}) - \nabla \times \left(\frac{1}{\rho}\nabla p\right)$$

ANALOGY WITH INCOMPRESSIBLE FLOW

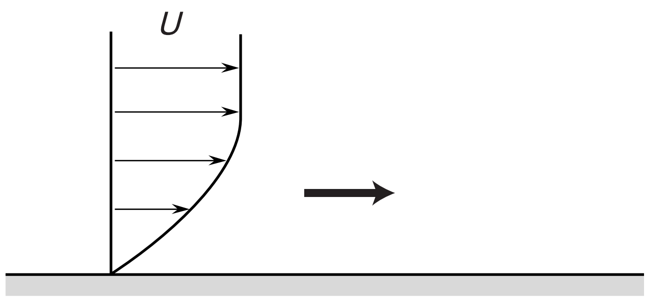
- $\vec{\omega}/\rho$ **for a compressible flow behaves like** $\vec{\omega}$ **for an incompressible flow**

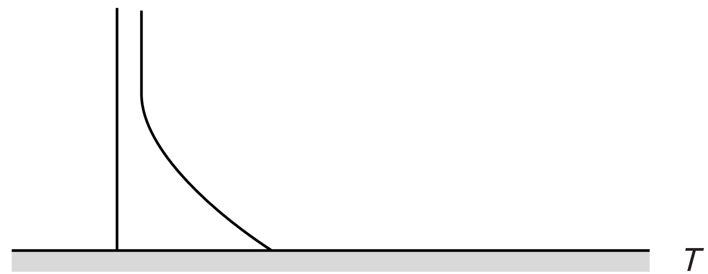
$$\frac{D(\vec{\omega}/\rho)}{Dt} = \left(\frac{\vec{\omega}}{\rho}\cdot\nabla\right)\vec{u} - \frac{1}{\rho}(\nabla T \times \nabla s)$$

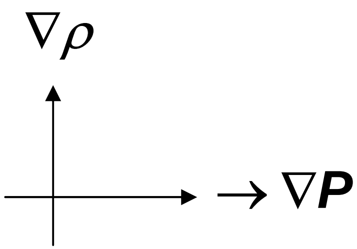
- **For a compressible flow,** $\vec{\omega}/\rho$ **can be altered if** $\rho \neq \rho(p)$ **or, equivalently, S ≠ S(T)**

- **2-D isentropic flow:** ω/ρ **= const ;** $\rho\uparrow \Rightarrow \omega\uparrow$

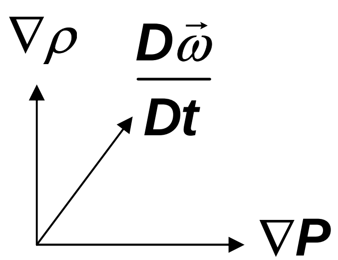
Flow in a high speed boundary layer with adverse pressure gradient (2-D)

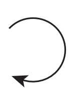






Fluid near wall has same *P*, higher *T* than outside boundary layer

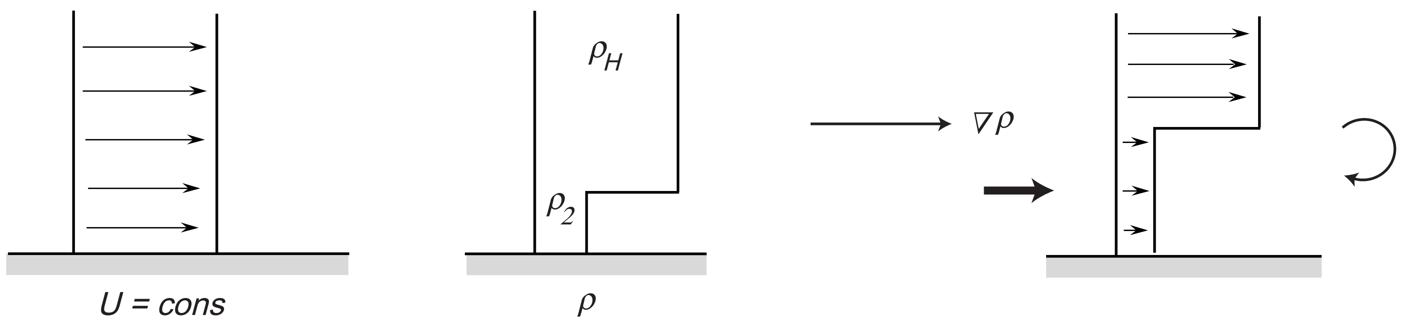


Vorticity 

$$\frac{D}{Dt}\vec{\omega}/\rho = \frac{1}{\rho^3}\nabla\rho \times \nabla P$$

<u>Shape</u> of boundary layer
Profile changes

Simpler model problem



Another view

$$dP = \rho_{fs} u_e du_e$$

$$dP_{b.l.} = dP_{fs} = \rho_{b.l.} u_{b.l.} du_{b.l.}$$

Neglecting visc.

$$\frac{du_{b.l.}}{du_e} = \frac{\rho_{fs} u_e}{\rho_{b.l.} u_{b.l.}}$$

"Double whammy" on deceleration

CIRCULATION AND VORTICITY

- **Circulation around a contour C is equal to the flux of vorticity through A bounded by C**
- **Circulation – a more global quantity than vorticity**
  - **Often are more interested in overall effects than in details**
- **Circulation – a scalar**
- **Wish to find rate of change of circulation for a <u>fluid contour</u> – closed curve composed of the same fluid particles**

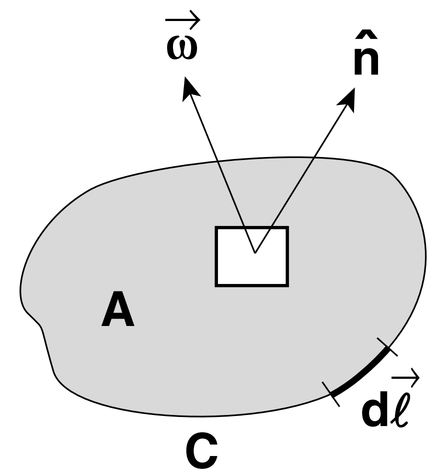
DEFINITION OF CIRCULATION

$$\Gamma_c = \oint_c \vec{u} \cdot d\vec{\ell}$$

Closed contour, C

$$= \iint_A \nabla \times \vec{u} \cdot \hat{n}\, dA$$

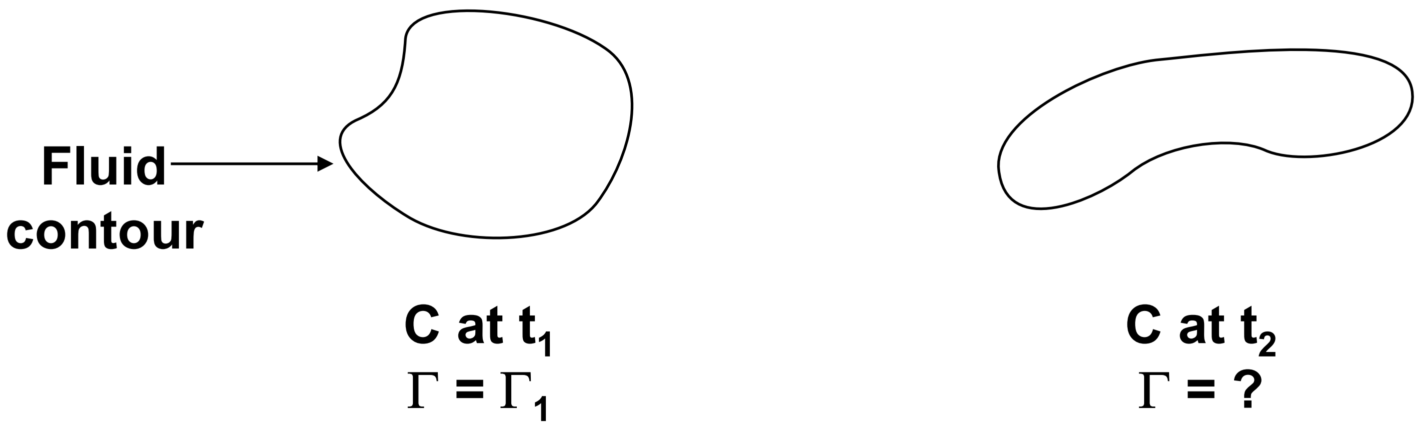
(Stokes)



$$\Gamma = \iint_A (\vec{\omega} \cdot \hat{n})\, dA$$

Flux of vorticity through area A, bounded by C

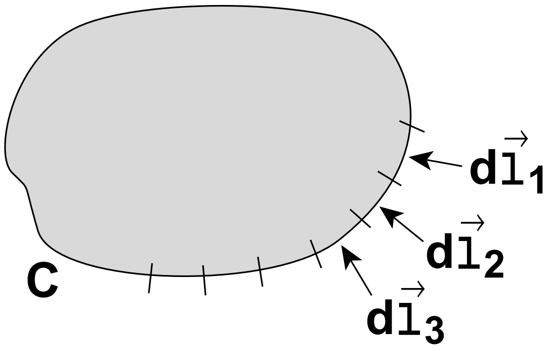
CHANGE IN CIRCULATION FOR A FLUID CONTOUR



Want to find $\frac{D\Gamma_c}{Dt}$

$$\frac{D\Gamma_c}{Dt} = \frac{D}{Dt}\oint \vec{u} \cdot d\vec{\ell}$$

CHANGE IN CIRCULATION (cont.)



- **Can think of this as**

$$\frac{D}{Dt}\sum_{i=1}^{N}\vec{u}_i \cdot d\vec{\ell}_i \cong \frac{D\Gamma_c}{Dt}$$

- **Always consider "same" N fluid particles, so can say**

$$\frac{D}{Dt}\sum u_i \cdot d\ell_i = \sum \frac{D}{Dt}\left(\vec{u}_i \cdot d\vec{\ell}_i\right)$$

- **Can do this with integral because consider contour of same particles**

$$\frac{D\Gamma_c}{Dt} = \oint_c \frac{D}{Dt}\left(\vec{u} \cdot d\vec{\ell}\right)$$

CHANGE IN CIRCULATION (cont.)

$$\frac{D\Gamma_c}{Dt} = \oint_c \frac{D\vec{u}}{Dt} \cdot d\vec{\ell} + \oint_c \vec{u} \cdot \frac{Dd\vec{\ell}}{Dt}$$

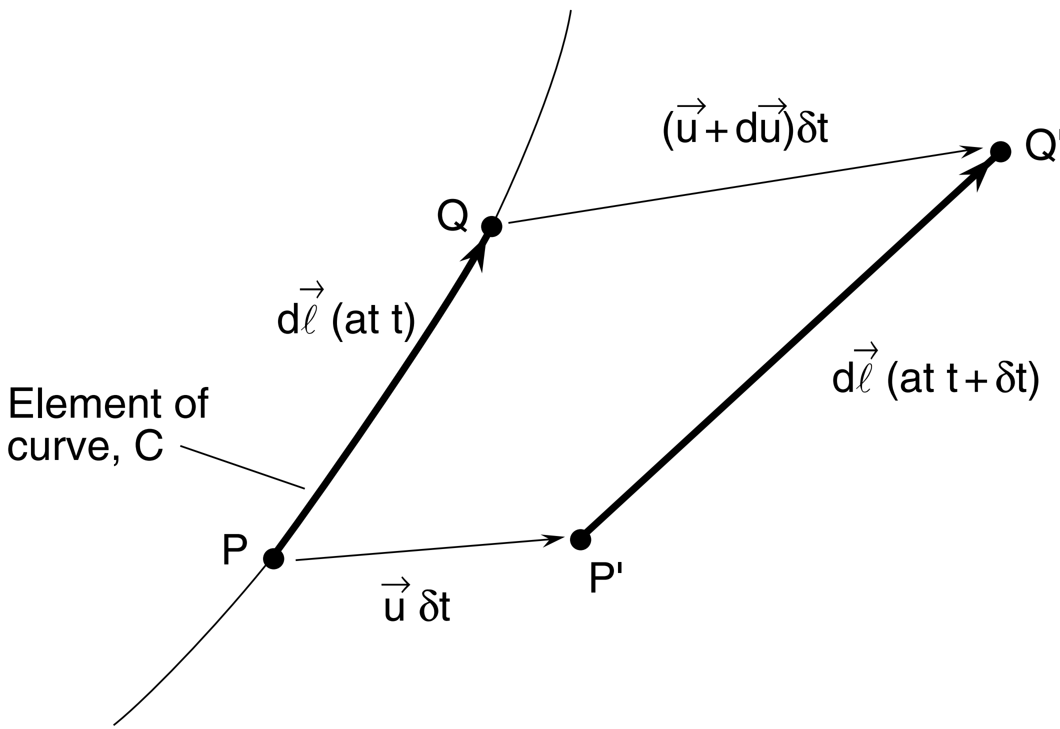
(1) (2)

- Look at (2):
  - $d\vec{\ell}$ is $\overrightarrow{PQ}$ at t
  - $d\vec{\ell}$ is P'Q' or $\overrightarrow{PQ} + \overbrace{d\vec{u}\,\delta t}^{\delta\, d\vec{\ell}}$ at t + δt
- Rate of change of $d\vec{\ell}$ is $\dfrac{\delta\left(d\vec{\ell}\right)}{\delta t}$

Or $d\vec{u}$ so:

$$\frac{Dd\vec{\ell}}{Dt} = d\vec{u}$$

CHANGE IN CIRCULATION (cont.)



Rate of change, in length and orientation, of a vortex line element $d\vec{\ell}$ of fluid contour

CHANGE IN CIRCULATION (cont.)

Hence $\quad (2) = \oint_c \vec{u} \cdot d\vec{u} = \oint_c d\left(\frac{\vec{u} \cdot \vec{u}}{2}\right)$

$= 0$ Integral of an exact differential round a closed contour

$$\therefore \frac{D\Gamma_c}{Dt} = \oint \frac{D\vec{u}}{Dt} \cdot d\vec{\ell}$$

RATE OF CHANGE OF CIRCULATION (concluded)

$$\frac{D\Gamma_c}{Dt} = \oint_c \left[-\frac{\nabla p}{\rho} + \vec{F}_{visc} + \vec{F}_{body} \right] \cdot d\vec{\ell}$$

→ **Kelvin's Theorem**

- **Rate of change of circulation round a fluid contour, C**
- **Constant density, inviscid, conservative body force**

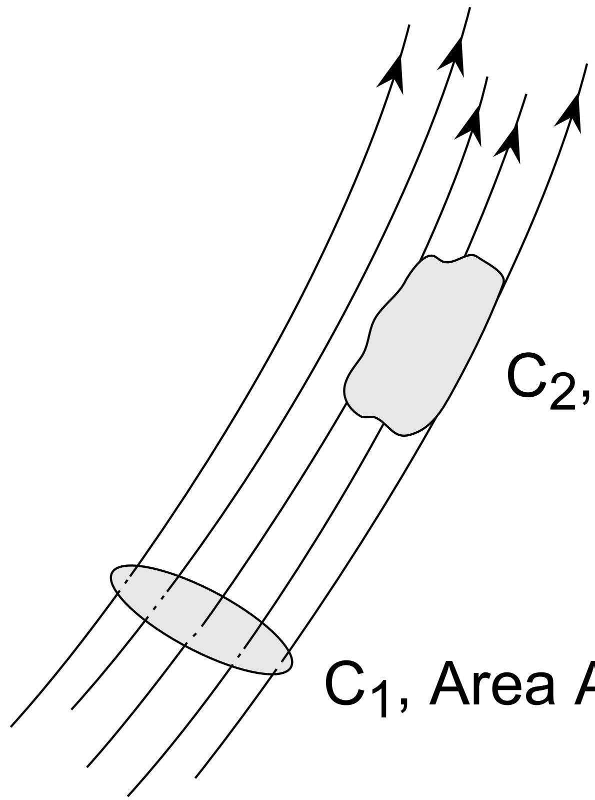
$$\frac{D\Gamma_c}{Dt} = 0$$

IMPLICATIONS OF KELVIN'S THEOREM
(Constant Density, Inviscid, Conservative Body Force)

- **If a fluid contour once has $\Gamma_c = 0$, it always has $\Gamma_c = 0$**
- **If fluid comes from reservoir with $\Gamma_c = 0$, then $\Gamma_c = 0$ everywhere**

  $\Rightarrow$ **Potential flow**

VORTEX TUBE



Vortex tube showing contour C_1, which encloses all vortex lines in tube, and C_2, which has zero circulation

VORTEX TUBE IN CONSTANT DENSITY FLOW

- Γ_{c_1} is constant, C_1 is a fluid contour
- C_1 always encloses vortex lines
- $\Gamma_{c_2} = 0$ (C_2 on wall of tube)
- Vortex lines never permeate A_2
  - Remain confined in tube

$\Rightarrow$ Vortex lines move with the fluid

EXTENSION TO COMPRESSIBLE FLOW

- $\frac{D\Gamma_c}{Dt} = 0$ for ρ = constant
- Suppose $\rho = \rho(p)$ (e.g. isentropic compressible flow)

  (still inviscid, conservative forces)

$$\boxed{\frac{D\Gamma_c}{Dt} = -\oint_c \frac{\nabla p}{\rho}}$$

"Kelvin's form" of Kelvin's Theorem

- If $\rho = \rho(p)$, r.h.s. is

$$\oint_c \frac{\nabla p}{\rho(p)} \cdot d\vec{\ell} = \oint_c \frac{dp}{\rho(p)} = 0$$

- **so if** $\rho = \rho(\mathbf{p})$

$$\frac{D\Gamma_c}{Dt} = 0 \quad \Rightarrow$$ **Vortex lines move with the fluid**

- **Also consider element in vortex tube**

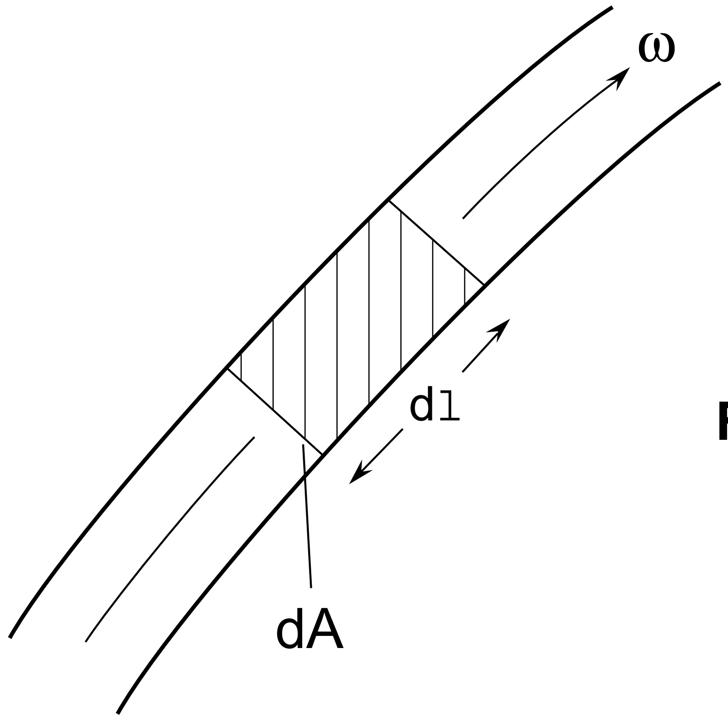
  a) ρ **dA dl = constant**

  b) ω **dA = constant**

$$\frac{b)}{a)} \Rightarrow \frac{\omega}{\rho d\ell} = \text{constant}$$

- ω/ρ **for compressible flow plays same role as** ω **in incompressible flow**

FLUID ELEMENT IN VORTEX TUBE

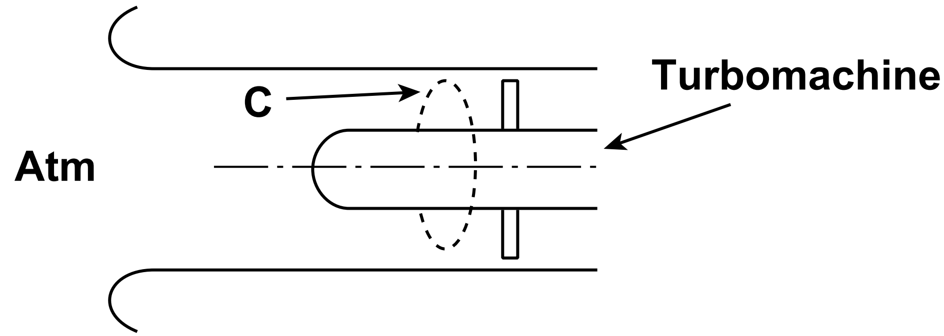


Flow in which $\rho = \rho(p)$

Fluid element in vortex tube; mass = ρ dA dl

EXAMPLES OF THE USE OF KELVIN'S THEOREM

- **Example 1: "Pre-whirl"**



Question: What is average C_θ upstream of a rotor?

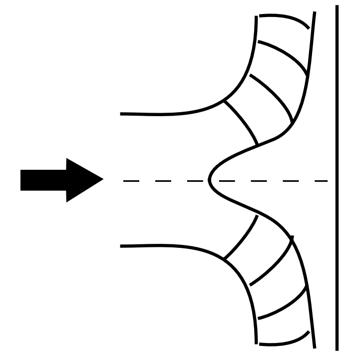
Consider contour, C.

Far upstream Gc = 0 so upstream of rotor

$$\Gamma_c = \int_c \vec{C}_\theta \cdot d\vec{\ell} = 0 \Rightarrow (C_\theta)_{av} = 0$$

Unless <u>backflow</u> from separation in rotor

- **Example 2: Relative eddy in centrifugal impeller**



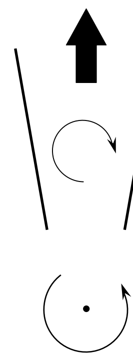
Flow in rotating passage:

$$\Gamma_c = 0 \text{ in absolute (fixed) system, } \frac{D\Gamma_c}{Dt} = 0$$

In rotating system

$$\vec{u}_{abs} = \vec{u}_{rel} + \vec{\Omega} \times \vec{r}$$

$$\oint \vec{u}_{rel} \cdot d\vec{\ell} = -\oint_c \vec{\Omega} \times \vec{r} \cdot d\vec{\ell}$$



RELATIVE CIRCULATION

$$\Gamma_{c_{rel}} = -2A_c\Omega \quad \left(A_c \perp \vec{\Omega}\right)$$

$$\omega_{rel} = \frac{\Gamma_{c_{rel}}}{A_c} = -2\Omega \quad \textbf{(relative vorticity)}$$

- **So-called "relative eddy"**

RELATIVE VELOCITY PROFILE IN A ROTATING STRAIGHT CHANNEL

$\omega_{rel} = -2\Omega$

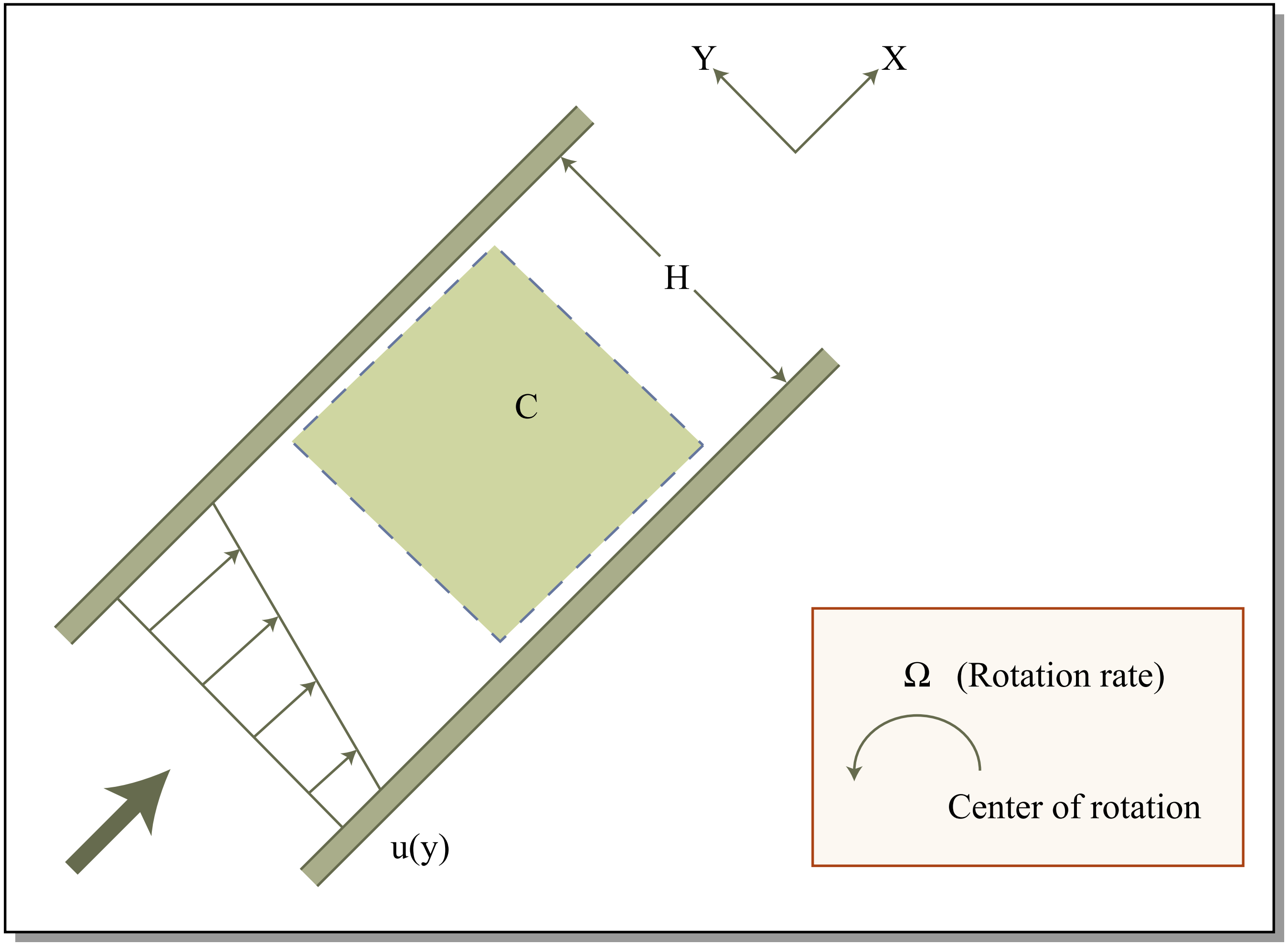


Figure by MIT OCW.

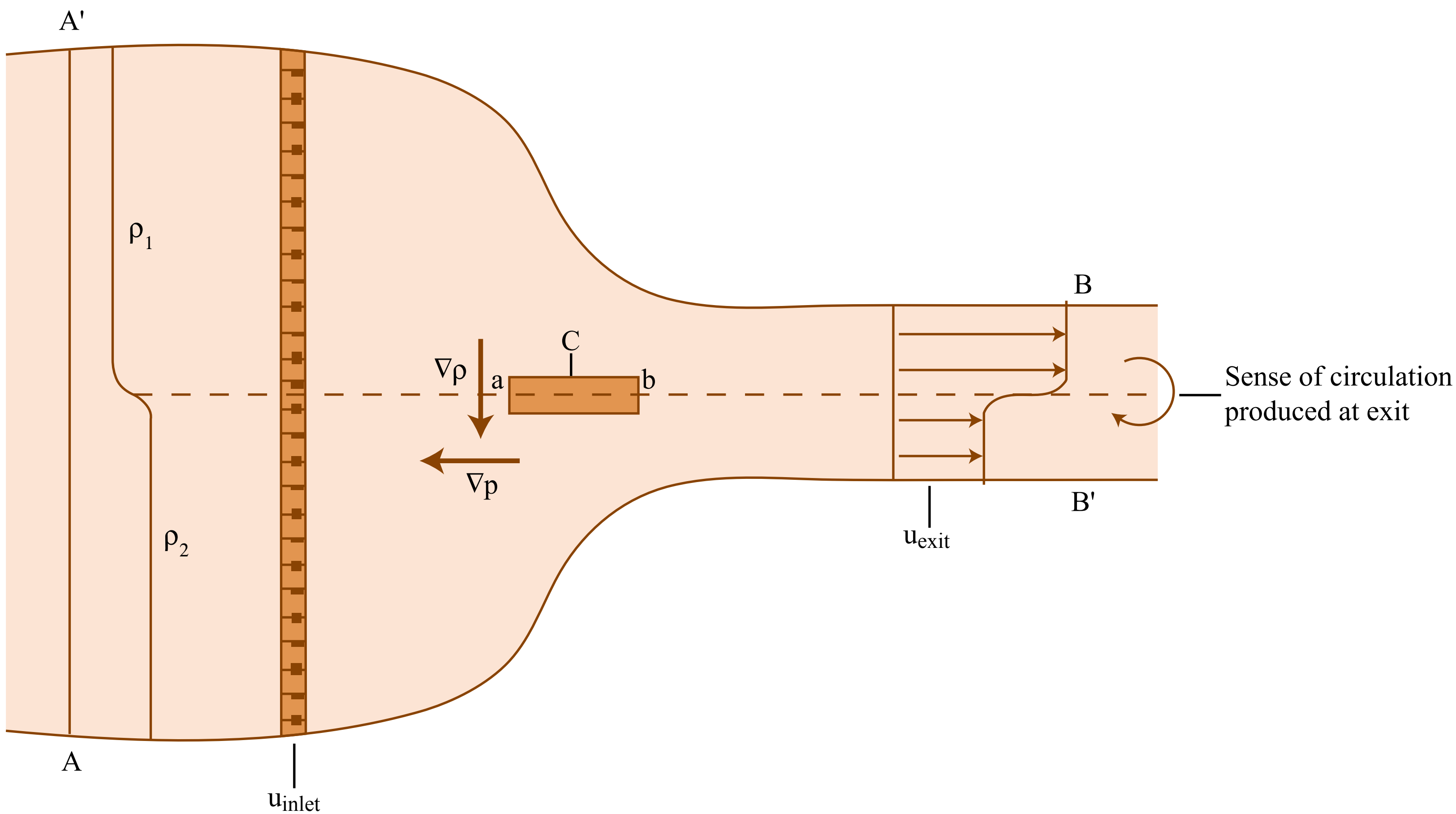
FLOWS WITH NON-UNIFORM DENSITY

$$\frac{D\Gamma_c}{Dt} = -\oint \frac{\nabla p}{\rho} \cdot d\vec{\ell} = \iint_A \frac{\nabla \rho \times \nabla p}{\rho^2} \cdot \hat{n}\, dA$$

- Circulation is produced when density gradients are not aligned with pressure gradients
- Example: Flow from reservoir

$$-\oint_c \frac{\nabla p}{\rho} \cdot d\vec{\ell} \cong \left(\frac{1}{\rho_2^2} - \frac{1}{\rho_1}\right) \int_a^b \nabla p \cdot d\vec{\ell} = \left(\frac{1}{\rho_2} - \frac{1}{\rho_1}\right) \Delta p_{ab}$$

- Where Δp_{ab} is change in pressure from one end of the contour to the other, $a \to b$



Change in circulation in a fluid of non-uniform density; channel with inlet area >> exit area

Figure by MIT OCW.

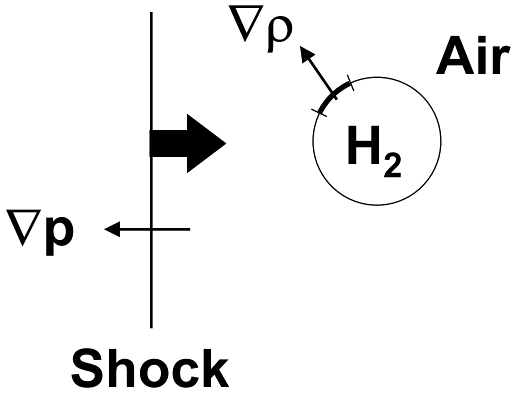
INVISCID COMPRESSIBLE FLOW

$$\frac{D\Gamma_c}{Dt} = -\oint \frac{\nabla p}{\rho}$$

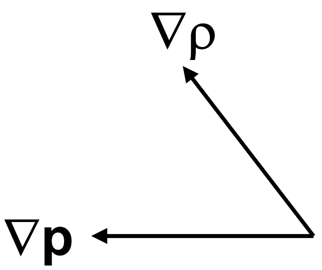
$$= \iint_A \frac{\nabla \rho \times \nabla p}{\rho^2} \cdot \hat{n}\, dA$$

$$= \iint_A \nabla T \times \nabla s \cdot \hat{n}\, dA$$

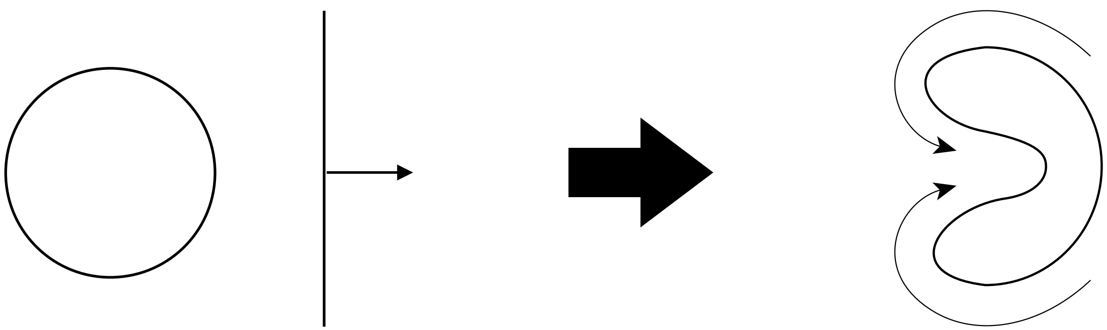
EXAMPLE: SHOCK-ENHANCED MIXING



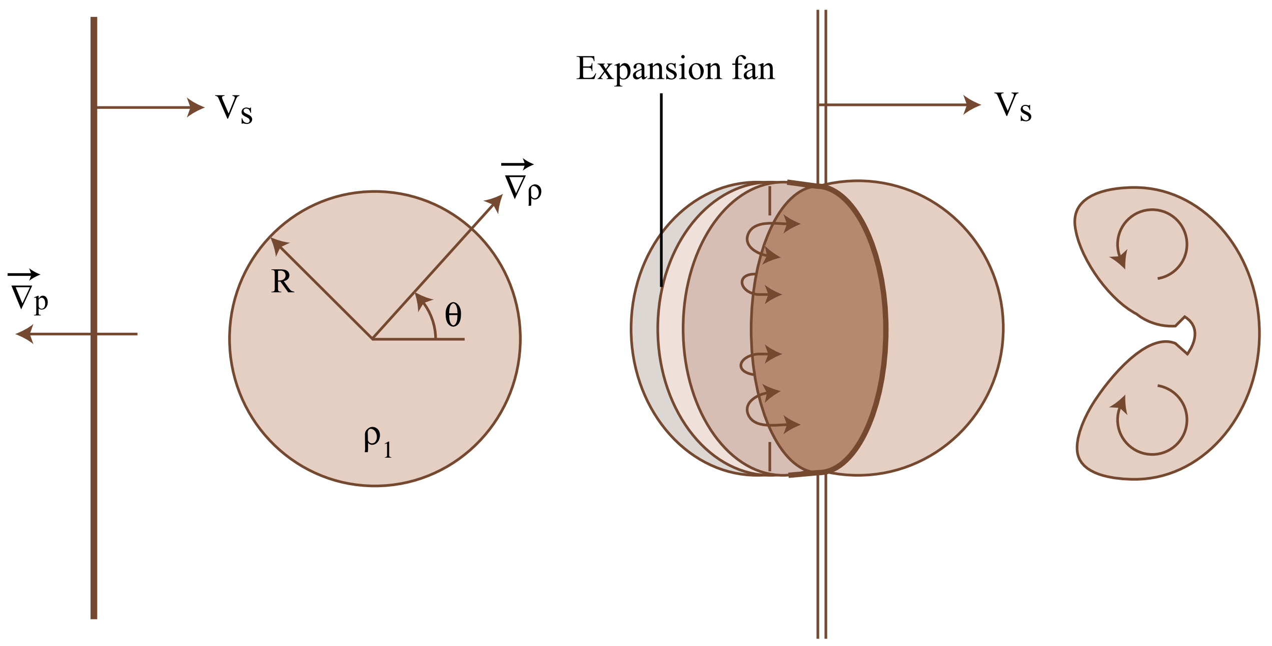
On surface
have a density
discontinuity



Generate
vorticity at
interface



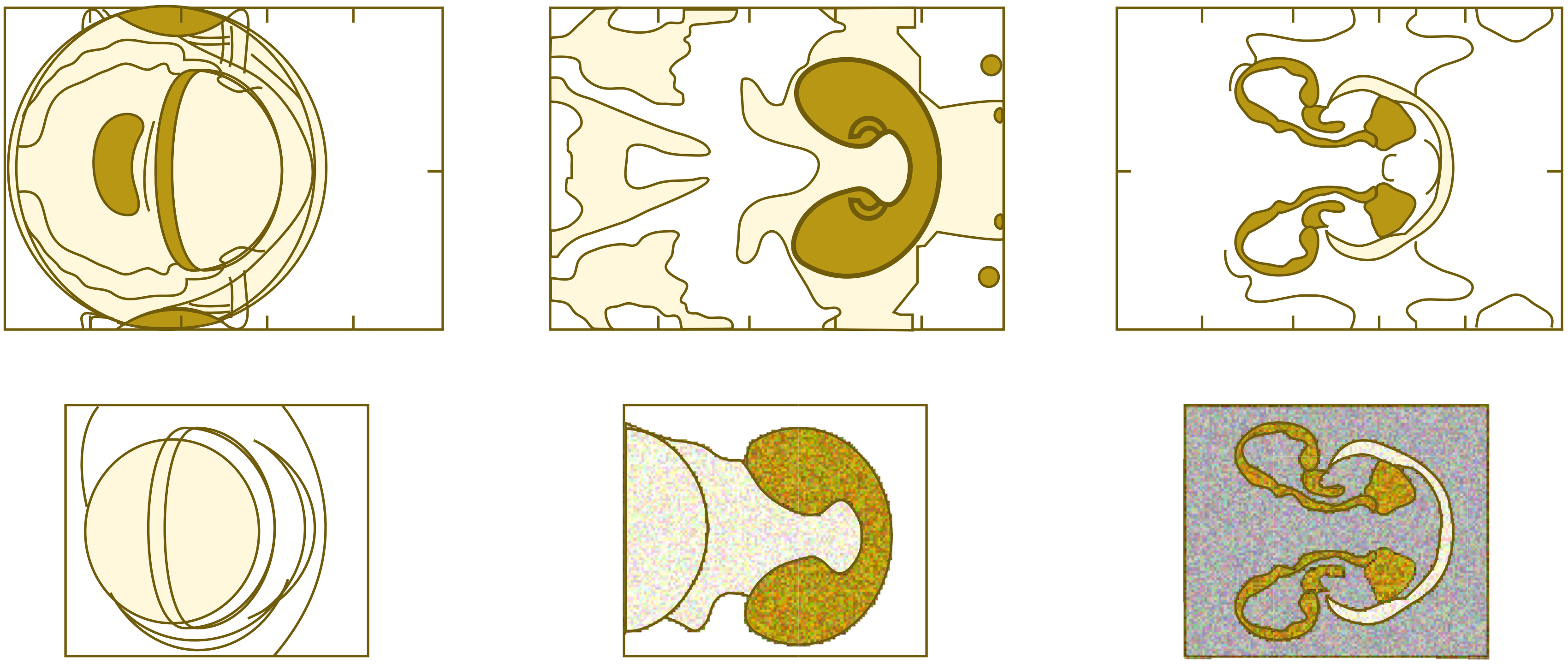
“Rolls up”



Two-dimensional, unsteady interaction of a shock with a light gas inhomogeneity

Figure by MIT OCW.

COMPARISON OF NUMERICAL AND COMPUTATIONAL RESULTS (JACOBS)



(a) t = 0.102ms
t* = 1.39

(b) t = 0.427ms
t* = 5.81

(c) t = 0.983ms
t* = 13.4

Figure by MIT OCW.

THREE-DIMENSIONAL, STEADY INTERACTION OF A COLUMN OF LIGHT GAS WITH AN OBLIQUE SHOCK

[Waitz et al]

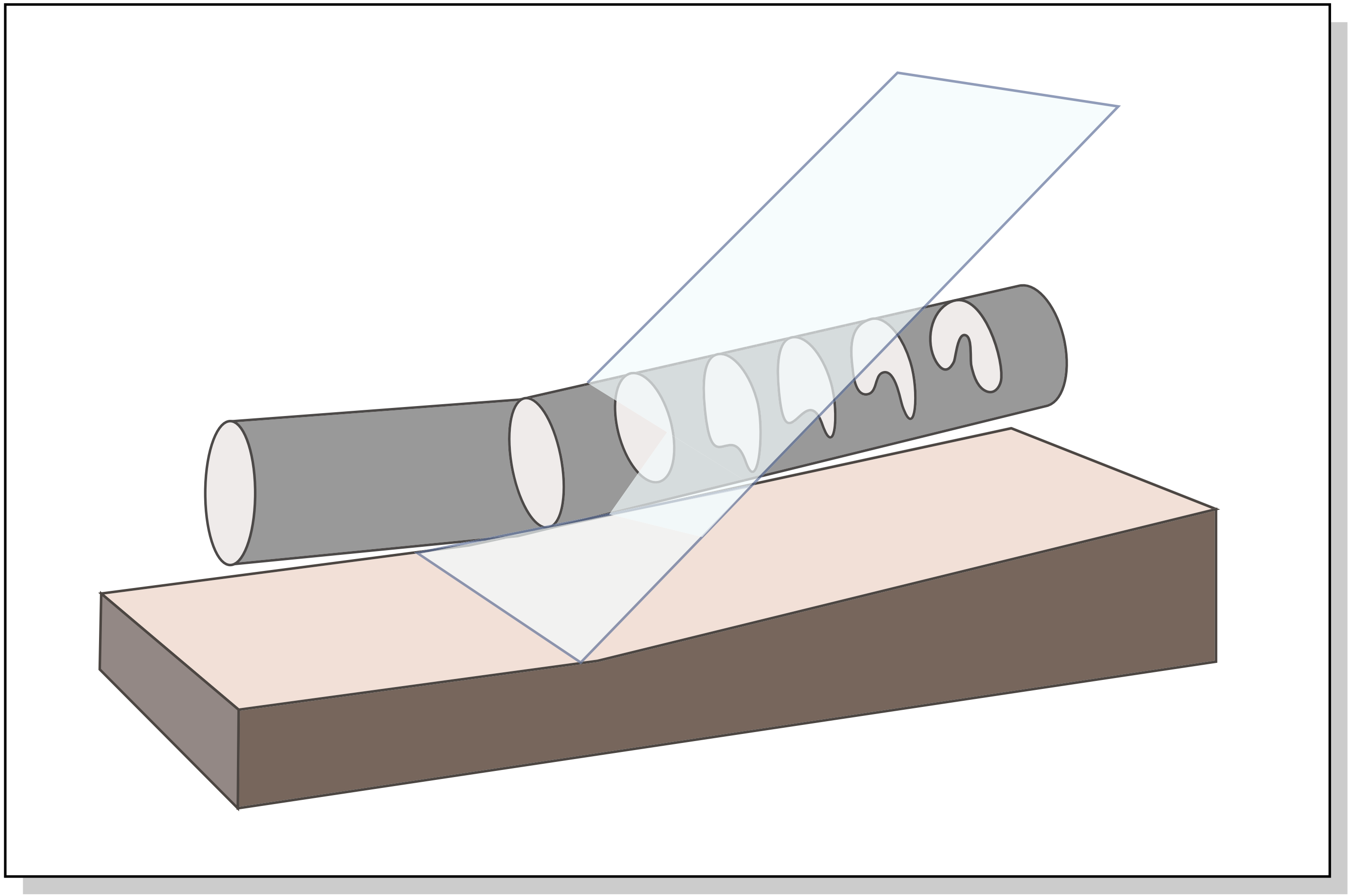


Figure by MIT OCW.

CONTOURS OF HELIUM MASS FRACTION DOWNSTREAM OF A SCRAMJET INJECTOR FOR *M = 1.7* INJECTION INTO *M - 6* AIR [53], [54]

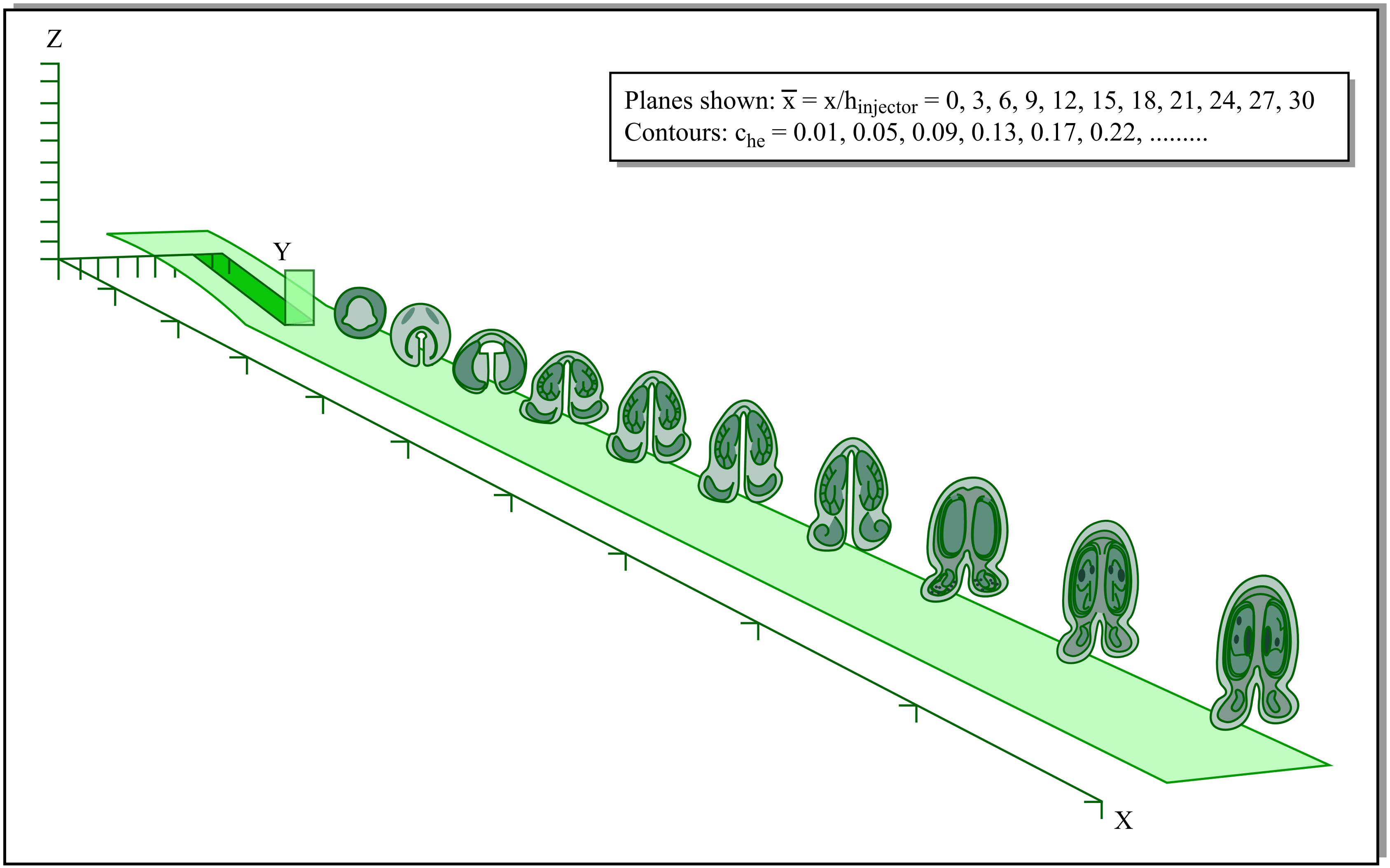
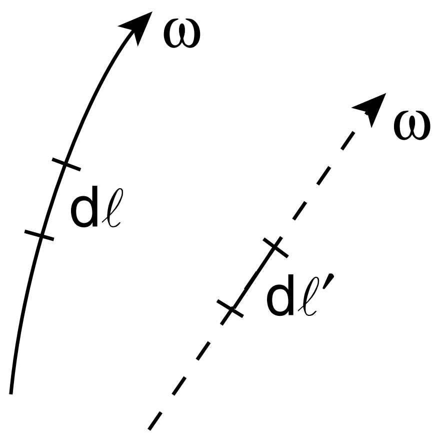


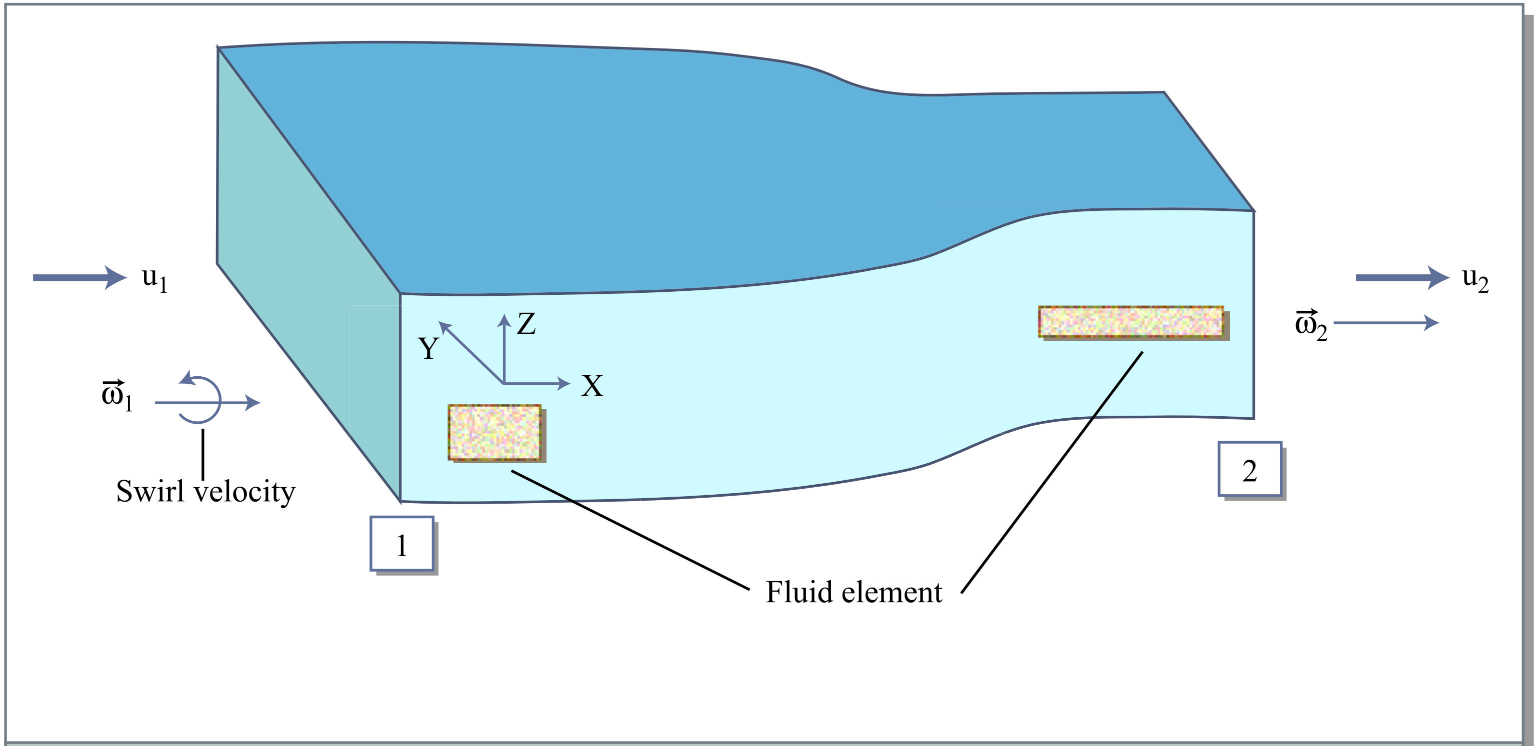
Figure by MIT OCW.

FLOW DESCRIPTION IN TERMS OF VORTICITY AND CIRCULATION

- **Inviscid, incompressible flow**
- **We have derived ω/dl = const**
  - **ω is vorticity magnitude**
  - **dl is length of line element on vortex line (vortex tube)**
- **Apply to non-uniform flow in diffuser or nozzle**



STREAMWISE VORTICITY IN NOZZLE

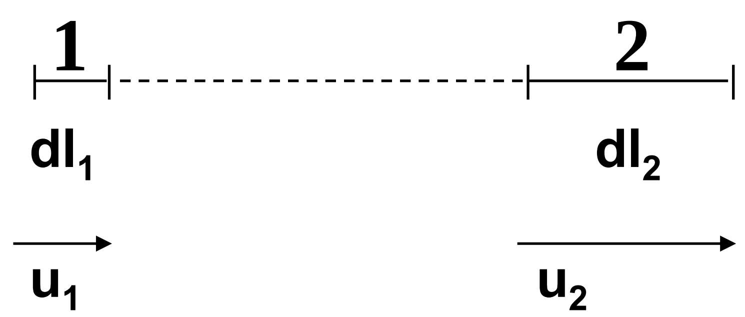


$\vec{\omega} \sim \omega_z \hat{e}_z$ (streamwise vorticity)

Figure by MIT OCW.

STREAMWISE VORTICITY

- Component of vorticity in <u>streamwise direction</u> (swirl non-uniformity)
- Assume vortex filaments carried (convected) by mean (background) flow
- What happens $1 \rightarrow 2$?
- Along a streamline



$$\frac{d\ell_1}{d\ell_2} = \frac{u_1}{u_2}$$

STREAMWISE VORTICITY CHANGE IN NOZZLE

- So $\frac{\omega_2}{\omega_1} = \frac{u_2}{u_1}$ (streamwise vorticity)
- ω increases
- What is often of more interest is relative uniformity of flow - swirl angle

$$\tan \alpha_1 \sim \frac{\text{swirl velocity}}{\text{axial velocity}}$$

FLOW ANGLE CHANGE IN NOZZLE

- Suppose vortex tube is circular, radius r

$$\alpha_1 \sim \frac{\omega_1 r_1}{2u_1} \qquad (\alpha_1 << 1)$$

- Continuity: r^2u = constant along a streamtube

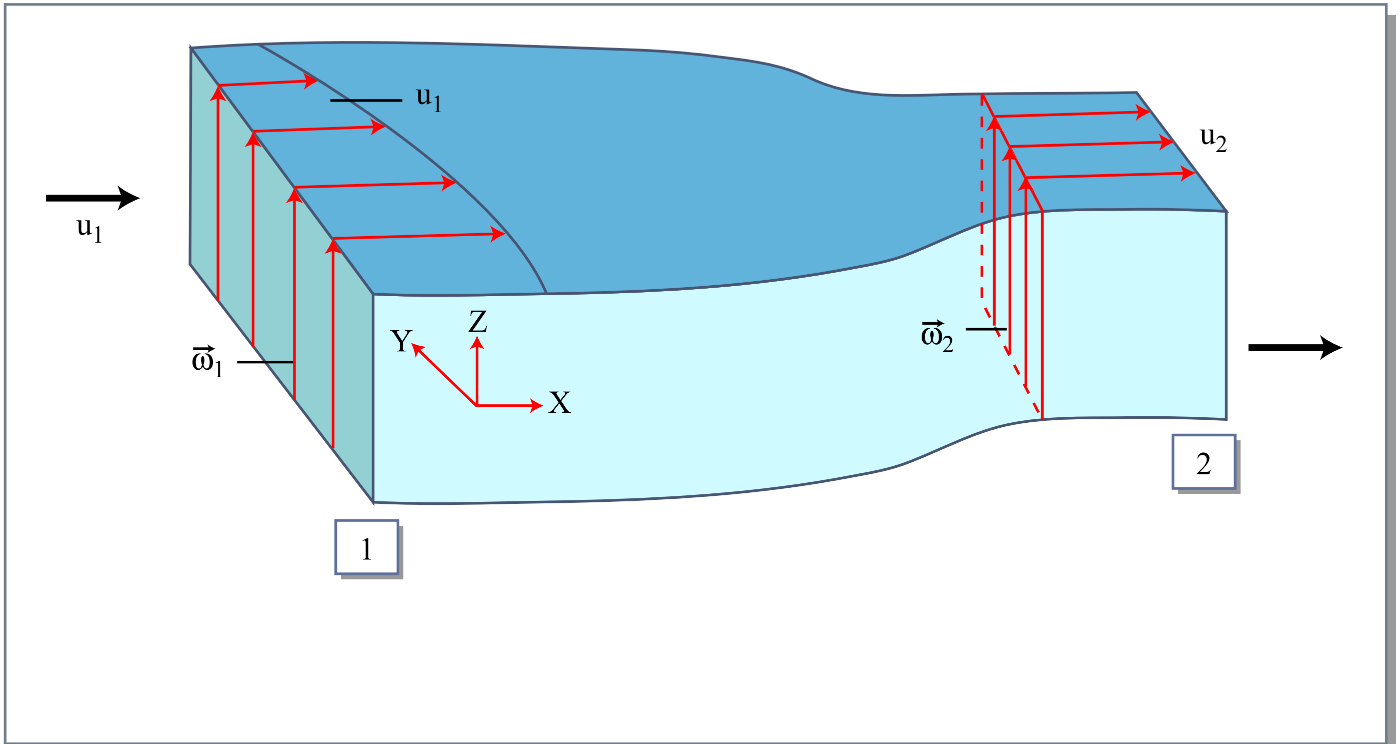
- Thus:

$$\frac{\alpha_2}{\alpha_1} \sim \frac{r_2}{r_1} \sim \sqrt{\frac{u_1}{u_2}} = \sqrt{\text{Area ratio}}\,; \quad \text{Area ratio} = \frac{A_2}{A_1}$$

- Nozzles increase flow uniformity with regard to swirl angularity

- Diffusers worsen it

EFFECT OF NORMAL VORTICITY



Transverse (normal) vorticity in a nozzle, $\vec{\varpi} \sim \omega_z \hat{e}_z$

Figure by MIT OCW.

VELOCITY NON-UNIFORMITY DUE TO NORMAL VORTICITY

- **Vorticity normal to flow: Non-uniformity in streamwise velocity**
- **2-D nozzle $\Rightarrow$ length of vortex lines is constant**
- ω_z **= constant along a mean streamline**
- **Parallel flow at inlet and exit**

$$\omega_1 = \frac{du_1}{dy} = \omega_2$$

- **Channel width decreases,** $\omega_1 \rightarrow \omega_2$
- **Local velocity gradient remains same**

EFFECT OF NOZZLE ON VELOCITY NON-UNIFORMITY

$$\frac{\Delta u_{x_2}}{\Delta u_{x_1}} \approx \frac{\omega_2}{\omega_1} = \text{Area ratio}$$

- Look at relative velocity non-uniformity $\frac{\Delta u_x}{U}$

$$\frac{\Delta u_{x_2}}{U_2} \bigg/ \frac{\Delta u_{x_1}}{U_1} = (\text{Area ratio})^2$$

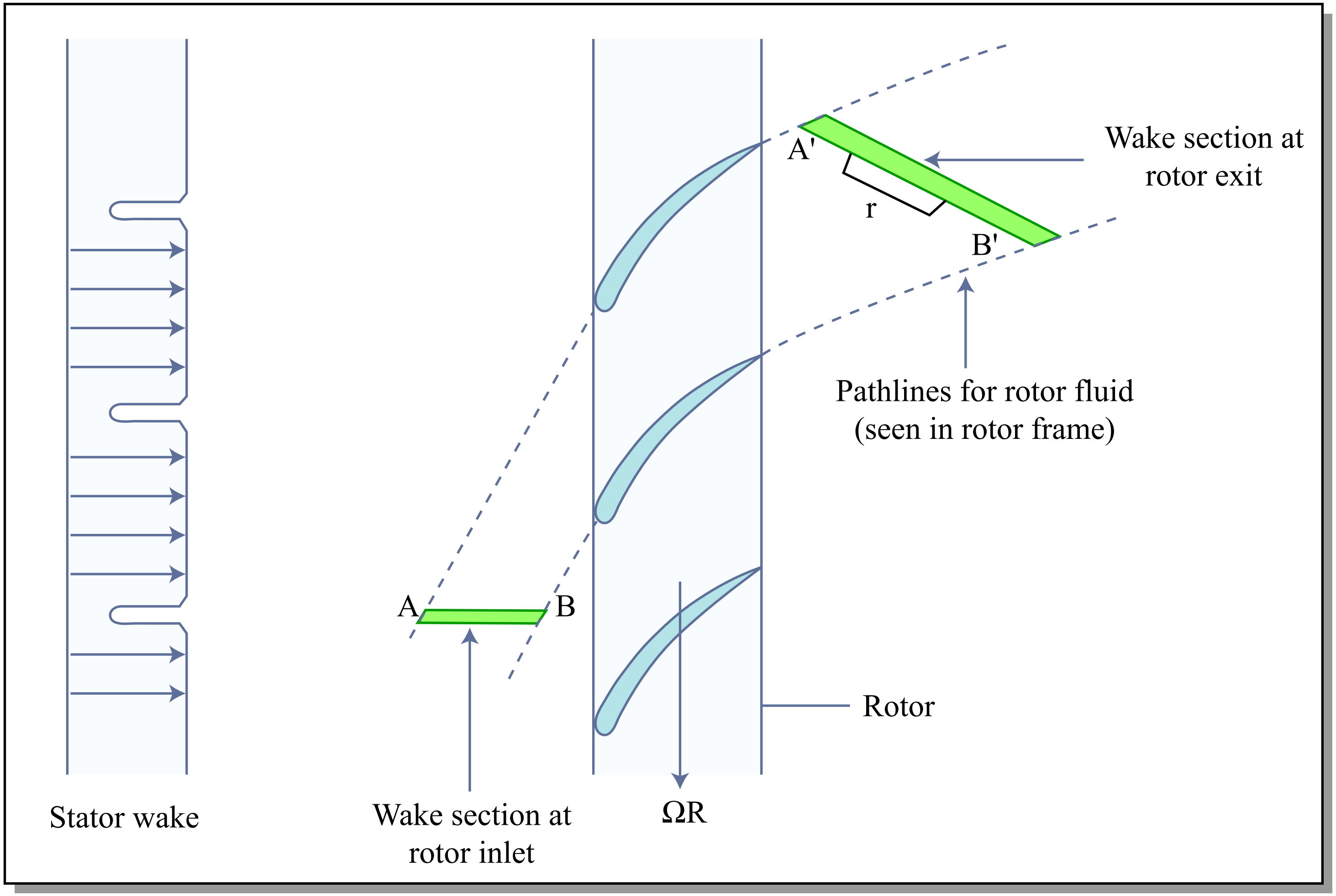
- Nozzles suppress velocity non-uniformities
- Diffusers worsen them
- Suppose vorticity is in y-direction
  - Then “width” is constant
- Is velocity non-uniformity altered?

PASSAGE OF TURBOMACHINE WAKE THROUGH SUCCEEDING BLADE ROW (COMPRESSOR)

- **View wake as 2-D, inviscid, convected by "mean" flow**
- **Compare length of wake segment at inlet and at exit**
- **Length increases because**
  - **1) Width of mean streamtube increases**
  - **2) Net circulation around blades (A,B)**
- Γ **~ wake length x velocity difference freestream – wake**
- Γ **= constant, length** $\uparrow \Rightarrow$ **ΔV freestream – wake** $\downarrow$
- **Wake gets attenuated in compressor**

PASSAGE OF STATOR WAKE THROUGH ROTOR

[Argument due to L. H. Smith]



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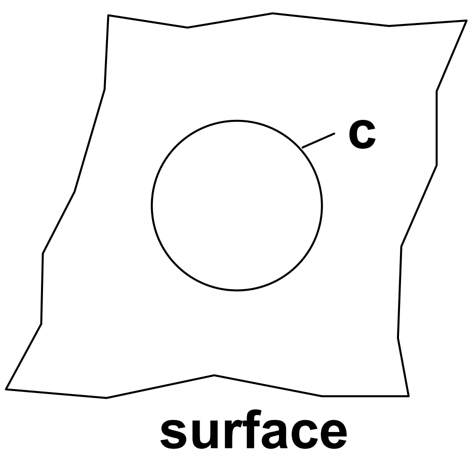
BEHAVIOR OF VORTICITY AT SOLID SURFACES

- **Viscous flow, no slip condition,** $\vec{u} = \vec{u}_{\substack{solid \\ surface}}$ **on the surface**
- **Stationary surface,** $\vec{u} = 0$ **at surface**
- **What is circulation on surface (any contour)**



$$\Gamma_c = \oint \vec{u} \cdot d\vec{l}$$

but $\vec{u} = 0$ **on surface**

$$\Gamma_c = 0 \text{ for any contour}$$

$$\Gamma = \iint_A \vec{\omega} \cdot \hat{n} dA = \iint_A \left[\omega_{\substack{\text{normal} \\ \text{to surface}}} \right] dA$$

Result: on <u>stationary</u> surface $\omega_{normal} = 0$ vorticity (vortex lines) are tangent to surface - cannot end in fluid

BEHAVIOR OF VORTEX LINES AT A SOLID SURFACE

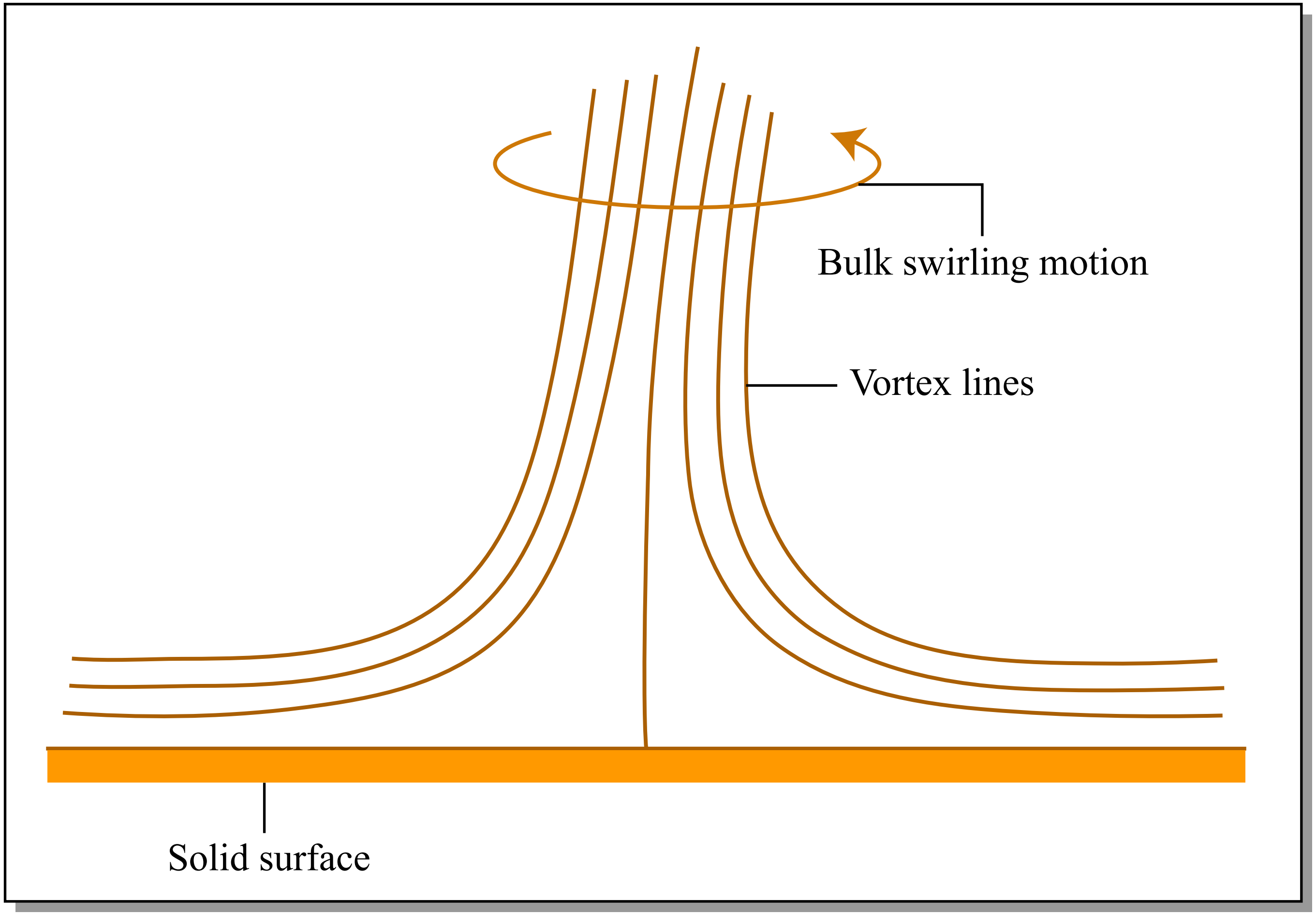


Figure by MIT OCW.

- **What about rotating surface: can vortex lines end on these?**
- **To see generation of vorticity on solid surfaces start with**

momentum equation (viscous, const. ρ)

On surface: $\vec{u} = 0$ on stationary surface

$$\left[\frac{1}{\rho}\nabla p = \nu\nabla^2 u\right]_{\text{surface}}$$

look at 2-D case - surface is plane $y = 0$

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = \nu\frac{\partial^2 u_x}{\partial y^2}$$

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = -\nu\frac{\partial \omega}{\partial y}$$

Whenever there is a pressure gradient along the solid boundary there is a gradient of tangential vorticity at the surface - a diffusion of vorticity into fluid

Analogy with heat transfer

$$-k\,\frac{\partial T}{\partial y} = \text{Heat flux} \qquad\qquad \nu\frac{\partial \omega}{\partial y}$$

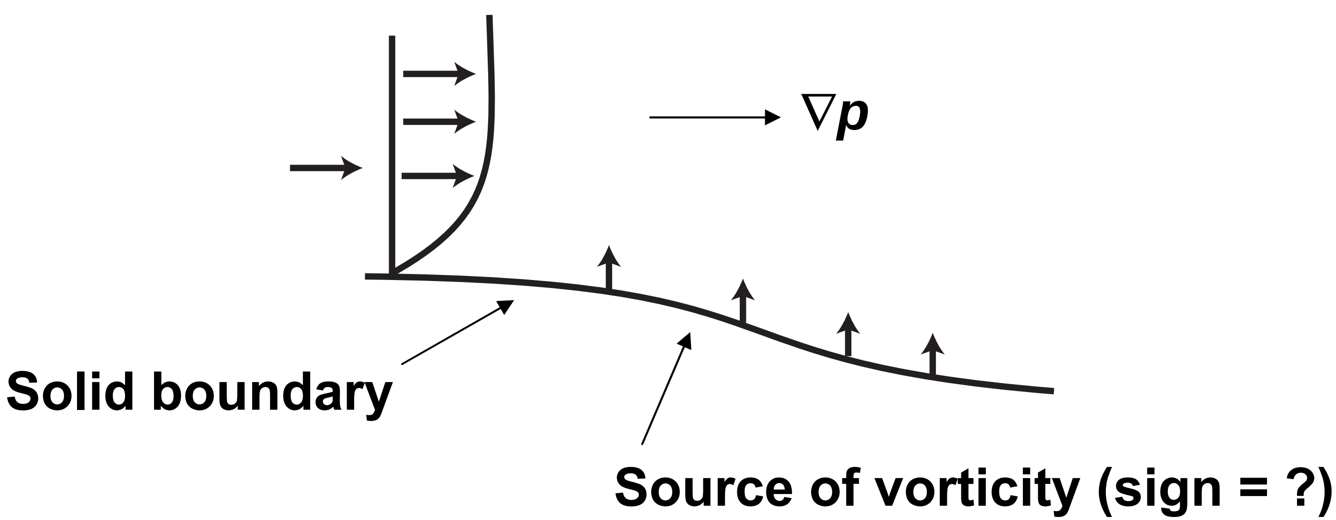
temperature gradient

gradient of vorticity

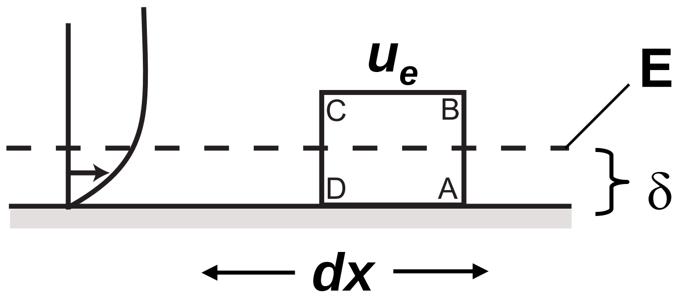
Boundary layer flow

dp **set by changes in free stream velocity,** u_e

$$\frac{\partial u_e}{\partial t} + u_e\,\frac{\partial u_e}{\partial x} = \nu\left[\frac{\partial \omega}{\partial y}\right]_{\text{surface}}$$



NET CIRCULATION/UNIT LENGTH IN BOUNDARY LAYER



Look at contour

$\int \vec{u} \cdot dl = 0$ on DA

on CD $\approx u_y \delta$

on AB $\approx u_y \delta + \frac{d}{dx}\left(u_y \delta\right) dx$

on BC $\approx -u_e dx$

$$\Gamma = \left[\frac{d}{dx}\left(u_y \delta\right) - u_e\right] dx$$

Length in *X* direction

But $\frac{d}{dx}\left(u_y \delta\right) \sim \frac{\delta}{L}$

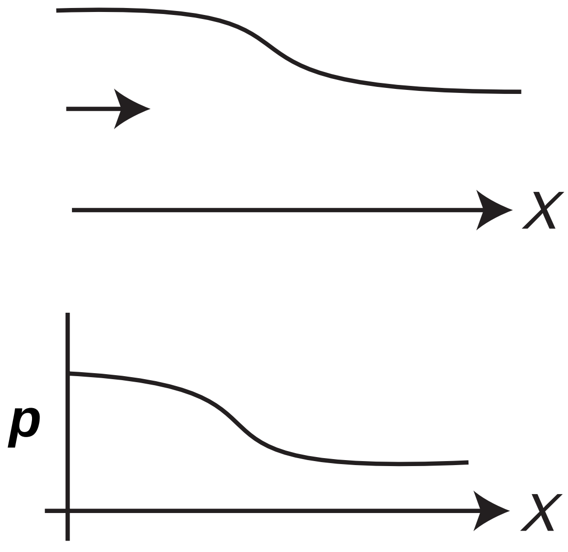
ratio: $\frac{u_y \delta}{LU} \sim \frac{u_y}{U} \frac{\delta}{L}$ $\qquad \frac{\partial u_y}{\partial y} + \frac{\partial u_x}{\partial x} = 0$

$$\frac{u_y}{u_x} \sim \frac{u_y}{U} \sim \frac{\delta}{L}$$

ratio $\sim (\delta/L)^2 << 1$

Boundary layer circulation/unit length = - u_e

FLOW IN A CONTRACTION



Velocity increases $\Rightarrow$ *p* decreases
Vorticity diffused into flow

$$\frac{\partial p}{\partial X} < 0$$ **so vorticity is same sign as existing vorticity**

New vorticity has short time to diffuse (be spread by viscosity) away from wall - is concentrated near wall

Velocity gradient large near wall profile is "fuller"

CONTOUR USED FOR EVALUATION OF CIRCULATION IN BOUNDARY LAYER; $\Gamma_{ABCD} = -u_e$

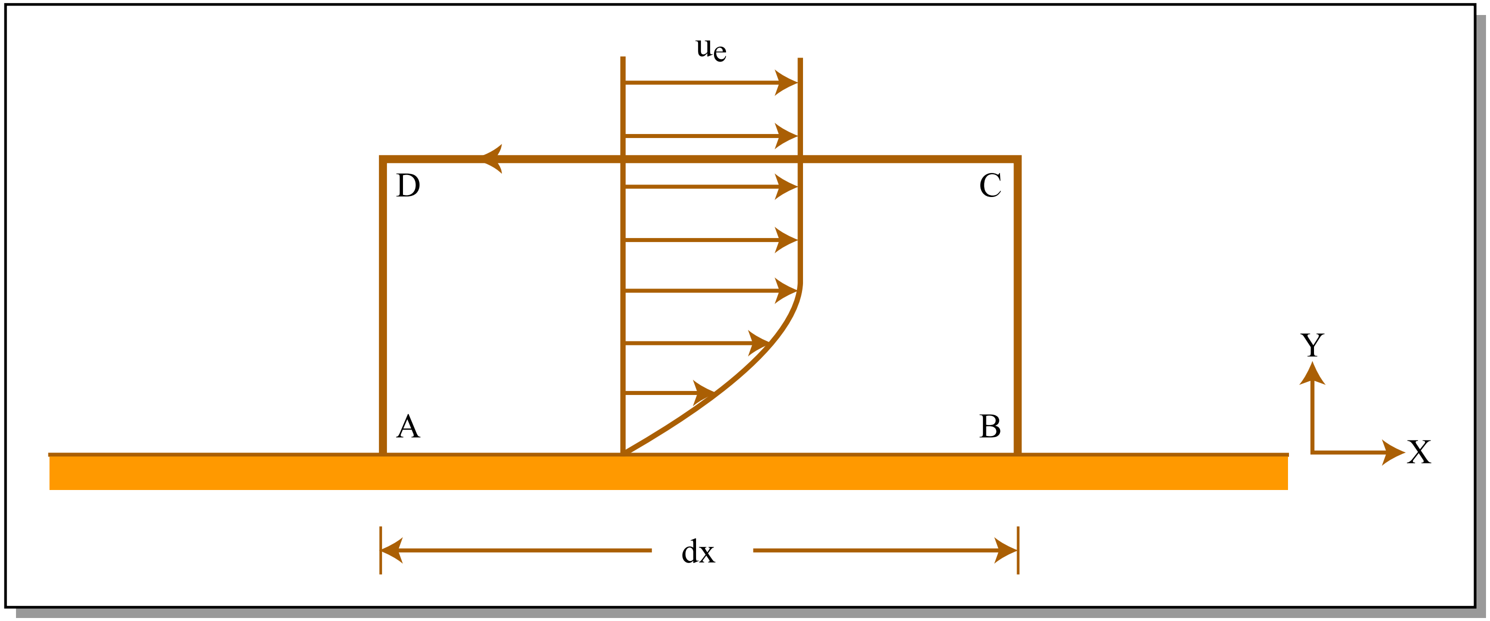
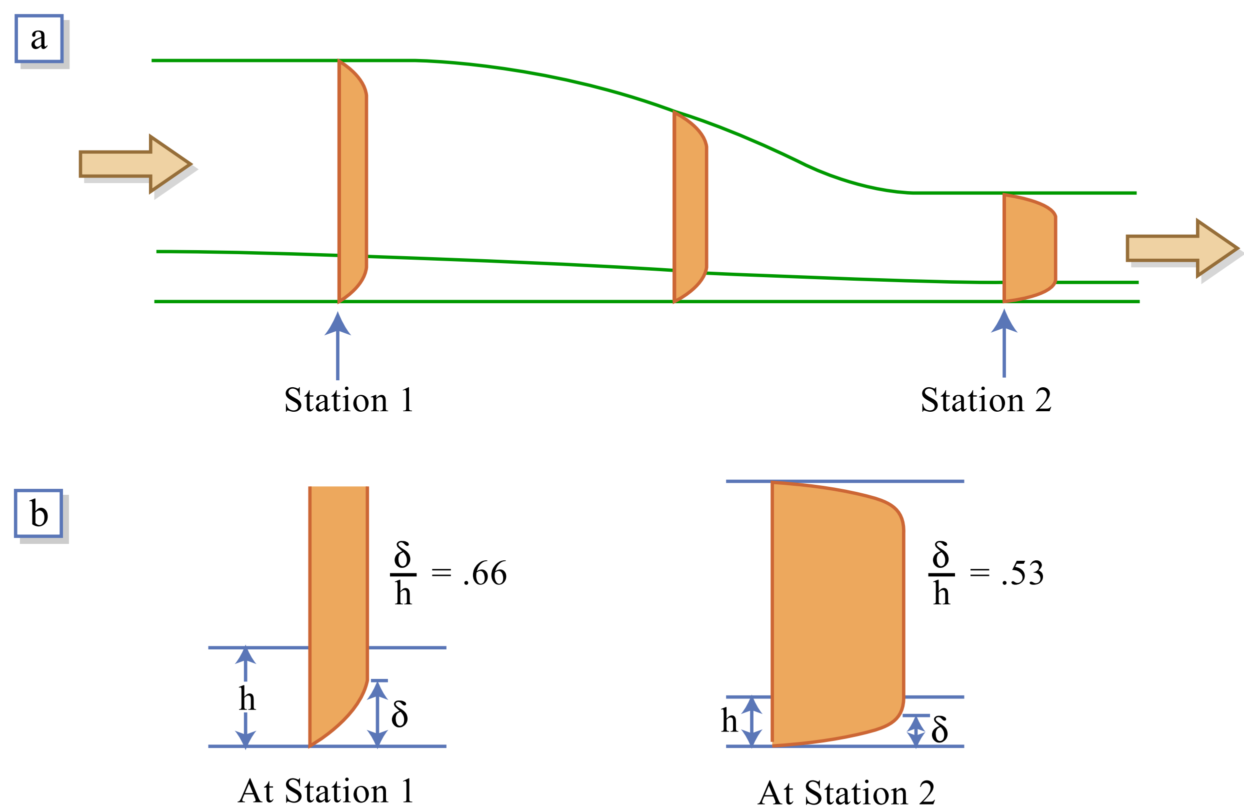


Figure by MIT OCW.

FLOW IN A 2:1 CONTRACTION; (A) OVERALL VELOCITY PROFILES; (B) BLOWUP OF (A) AT STATIONS 1 AND 2. HYDROGEN BUBBLE FLOW VISUALIZATION



Flow in a 2:1 contraction;
(a) Overall velocity profiles; (b) Blowup of (a) at station 1 and 2. Hydrogen bubble flow visualization.

Figure by MIT OCW.

Free stream velocity increases ⇒ implies that more vorticity has entered flow

Flat plate boundary layer Γ = - u_e = constant!
No vorticity put in anywhere except at leading edge.

Horseshoe vortex
Consider contour as shown
Vortex lines from upstream keep coming into contour. Does circulation continually increase?

Convection and diffusion of vorticity into contour ABCD on plane of symmetry upstream of a strut

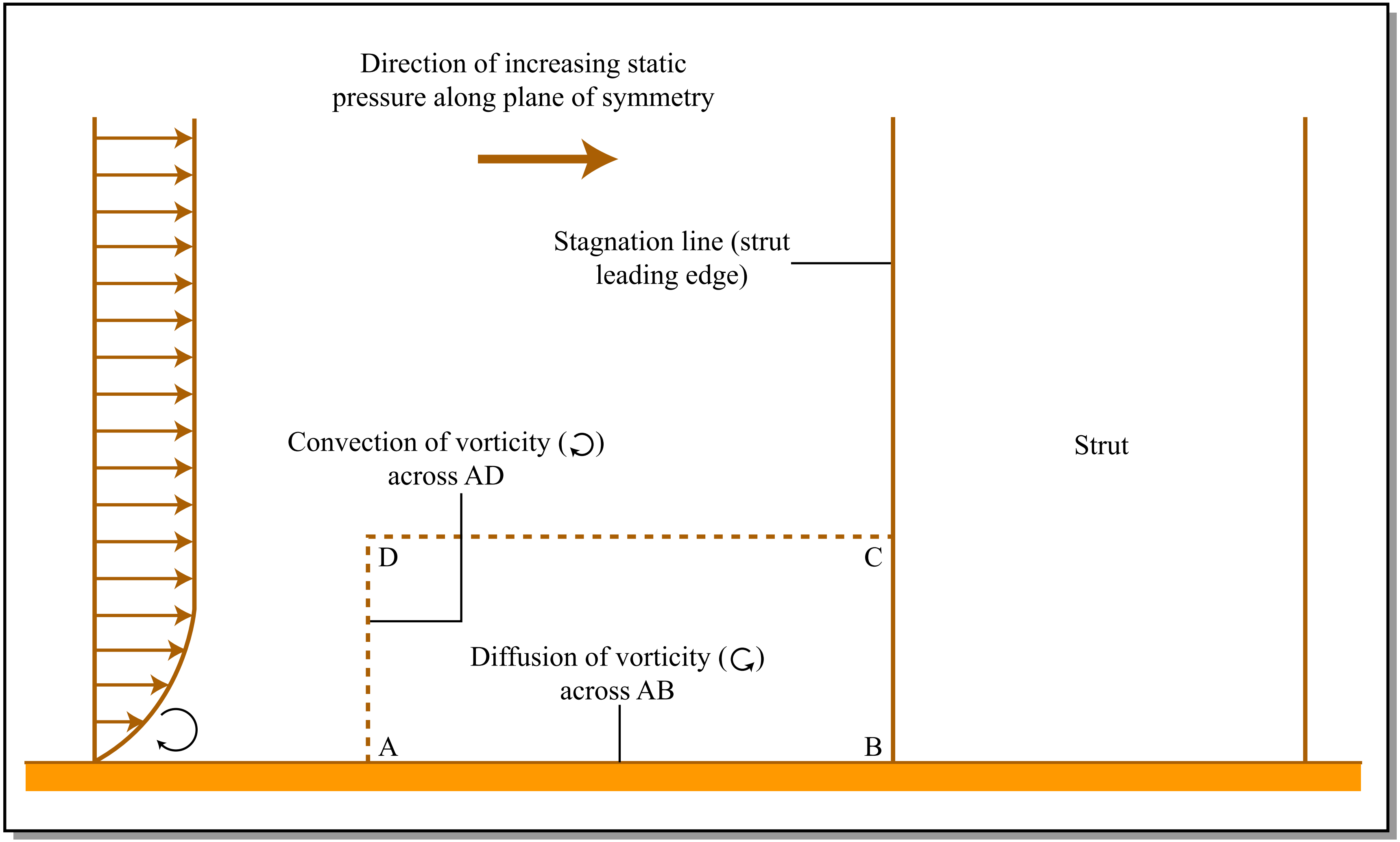


Figure by MIT OCW.

For horseshoe vortex have a balance between convection of ↻ and diffusion in of ↺ . Net vorticity in contour (net circulation) remains constant

Note also in vortex - balance between stretching, diffusion sets scale of vortex (radius of vortex)

We have been working in 2-D, but arguments can be extended to 3-D. Pressure field (gradients) not one-dimensional so two components of vorticity can be diffused into flow from vorticity sources at wall

RELATION BETWEEN KINEMATIC AND THERMODYNAMIC QUANTITIES

- These relate vorticity and ∇p_t, ∇T_t, ds
- Most useful for "inviscid" flows
- Momentum equation

$$\nabla\left(\frac{u^2}{2}\right) - \vec{u} \times \vec{\omega} = -\frac{\nabla p}{\rho} - \nabla \Psi$$

F_{body}

$$T\nabla s = \nabla h - \frac{1}{\rho} - \nabla p$$

$$-\vec{u} \times \vec{\omega} = T\nabla s - \nabla h - \nabla\left(\frac{u^2}{2}\right) - \nabla \Psi$$

$$\vec{u} \times \vec{\omega} = \nabla\left[h + \frac{u^2}{2} + \Psi\right] - T\nabla s$$

STAGNATION QUANTITIES

In internal flow situations often work with stagnation quantities

1) Convient to measure, 2) Relate directly to loss

<u>Stagnation temperature defined</u>

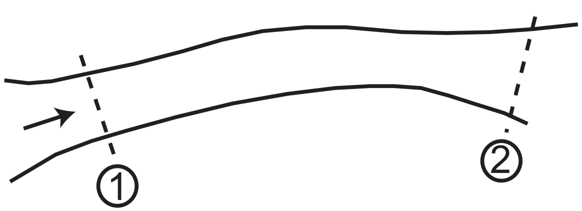
Adiabatic process, no work - bring stream to rest

First law (steady flow energy equation)

Along a streamtube:

$$\dot{m}_1 h_{t1} = \dot{m}_2 h_{t2}$$

but $\dot{m}_1 = \dot{m}_2 \Rightarrow h_{t1} = h_{t2}$



If station 2 has velocity = 0

$$h_t = C_p T_t = C_p T + \frac{u^2}{2}$$

$$T_t = T + \frac{u^2}{2C_p} = T\left[1 + \frac{u^2}{2C_pT}\right]$$

$$= T\left[1 + \frac{\gamma - 1}{2}\frac{u^2}{\gamma RT}\right]$$

Stagnation temperature →

$$T_t = T\left[1 + \frac{\gamma - 1}{2}M^2\right]$$

Note: Nothing yet about "frictionless"

Now: If frictionless

$$p/p_{initial} = (T/T_{initial})^{\gamma/\gamma-1}$$

Any two states

Stagnation pressure →

$$p_t = p\left[1 + \frac{\gamma - 1}{2}M^2\right]^{\gamma/\gamma-1}$$

Low speed flow,

$$p_t = p\left[1 + \frac{\gamma - 1}{2}M^2 \cdot \frac{\gamma}{\gamma - 1} + \cdots\right]$$

$$\cong p + \frac{\gamma}{2}\frac{u^2 p}{\gamma RT} = p + \frac{\rho u^2}{2}$$

p_t for low speed "incompressible" flow

- **If no body forces**

$$\boxed{\vec{u} \times \vec{\omega} = \nabla h_t - T\nabla s}$$

"Crocco's Theorem"

- **Consequences of Crocco's Theorem**

1) If a steady flow has constant entropy and stagnation enthalpy, $\omega = 0$ or vorticity is parallel to velocity

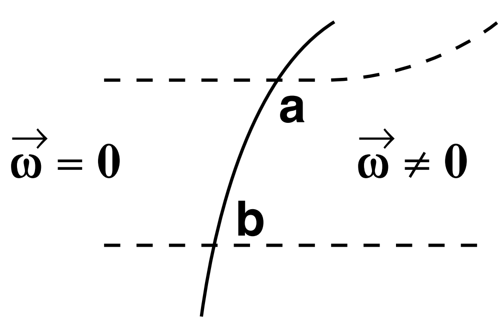
2) Vorticity can be produced by phenomena which generate gradients of entropy or stagnation enthalpy

3) In an irrotational flow with uniform entropy, h_t can vary only if the flow is <u>unsteady</u>

$$\frac{-\partial \vec{u}}{\partial t} + \underbrace{\vec{u} \times \vec{\omega}}_{0} = \nabla h_t - \underbrace{T\nabla s}_{0}$$

EXAMPLES

1) Flow downstream of a curved shock
   - h_t is constant across shock
   - $\nabla s_a < \nabla s_b$
   - $\vec{\omega} \neq 0$ downstream of shock



2) Flow downstream of an ideal inlet guide vane row
   - $\nabla h_t = \nabla s = 0$
   - $\vec{u} \times \vec{\omega} = 0$ so $\vec{u}, \vec{\omega}$ parallel

     (trailing vorticity as on a finite wing)

FLOW DOWNSTREAM OF A CURVED SHOCK

- **Geometry - M=2 flow round an airfoil**

- **Static and stagnation pressure**

- **Axial velocity profiles for**

different static pressure rise

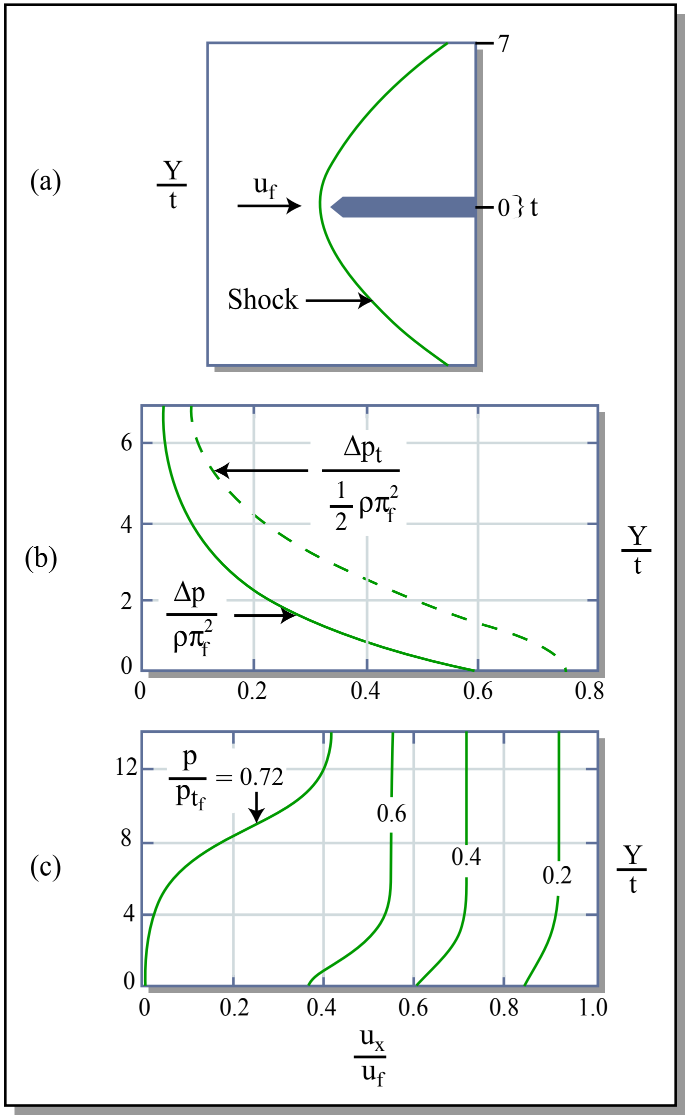
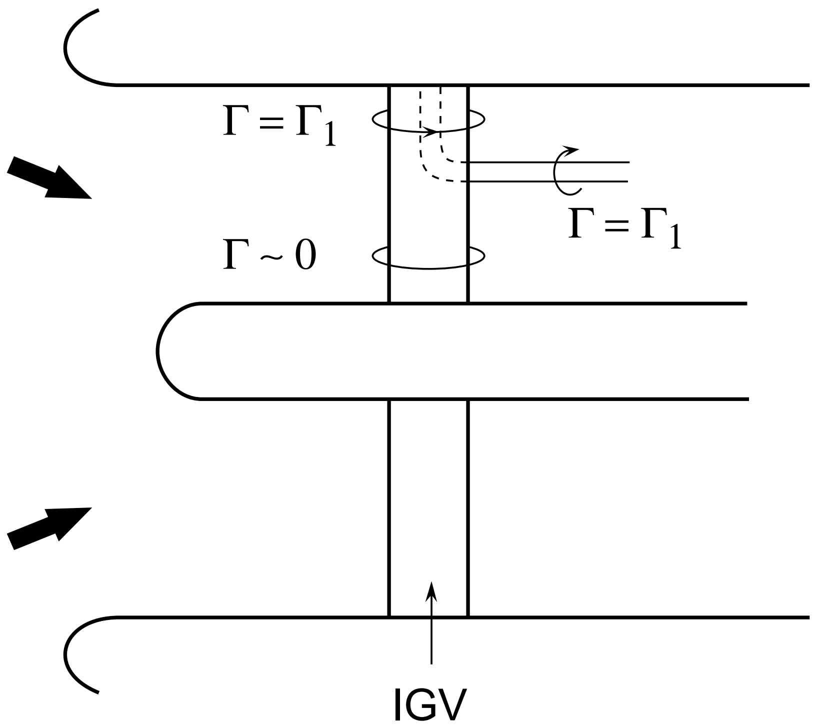


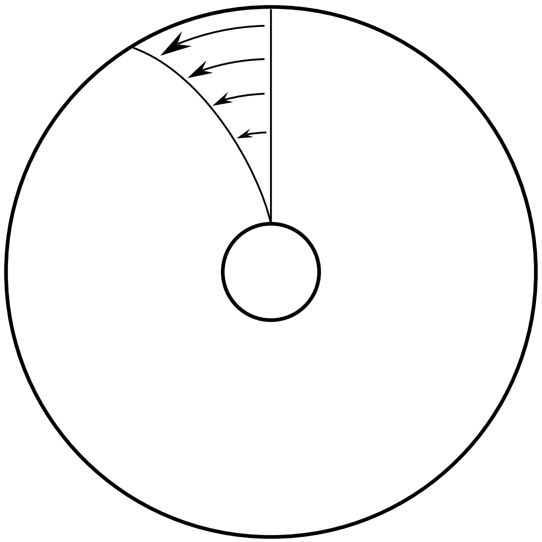
Figure by MIT OCW.

ROTATIONAL FLOW DOWNSTREAM OF IGV

What approximations are made in showing this figure?



Turbomachine Annulus and Inlet Guide Vane (IGV); Uniform Entropy and Stagnation Enthalpy



Rotational Swirl Flow Distribution Downstream of IGV

INCOMPRESSIBLE FLOW FORM

- **Incompressible, inviscid flow, ρ uniform**

$$\frac{\partial \vec{\mathbf{u}}}{\partial \mathbf{t}} - \vec{\mathbf{u}} \times \vec{\omega} = -\frac{\nabla \mathbf{p_t}}{\rho}$$

- **Steady flow**

$$\boxed{\vec{\mathbf{u}} \times \vec{\omega} = \frac{\nabla \mathbf{p_t}}{\rho}}$$

Incompressible form of Crocco's equation

PERSPECTIVE ON INTERPRETATION AND INSIGHT

- **Concepts of vorticity and circulation are useful in understanding fluid motions – most notably those with SWIRL, and/or UNSTEADINESS, and/or THREE-DIMENSIONALITY**

- **Focus on vortex line structure often provides clues to overall flow field behavior**

- **Focus on vorticity can give insight for complex motions**

- **Strongly complementary partner to pressure-acceleration approach**