

More 18.01 Final Practice Problems

Here are some further practice problems with solutions for the 18.01 Final Exam. Many of these problems are more difficult than problems on the exam.

Goal 1. Differentiation.

1.1 Find the equation of every tangent line to the curve $y = e^x$ containing the point $(-1, 0)$. This is not a point on the curve.

1.2 Let a and b be positive real numbers. Find the equation of every tangent line to the ellipse with implicit equation,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

containing the point $(2a, 2b)$. This is not a point on the ellipse.

1.3 Let a be a real number different from 0. Use the definition of the derivative as a limit of difference quotients to find the derivative to the following function,

$$f(x) = \frac{1}{x},$$

at the point $x = a$.

1.4 Use the definition of the derivative as a limit of difference quotients to find the derivative of the following function,

$$f(x) = \tan(x),$$

at the point $x = 0$. You may use without proof that the following limits exist and have the given values,

$$\lim_{x \rightarrow 0} \frac{\sin(x)}{x} = 1, \quad \lim_{x \rightarrow 0} \frac{1 - \cos(x)}{x} = 0.$$

1.5 For $x > 0$, let $f(x)$ be the function,

$$f(x) = e^{\sqrt{x}}.$$

Thus the inverse function,

$$y = f^{-1}(x),$$

satisfies the equations,

$$e^{\sqrt{y}} = x, \text{ and } \sqrt{y} = \ln(x).$$

Compute the derivative,

$$\frac{dy}{dx}.$$

Goal 2. Sketching graphs.

2.1 Sketch the graph of the function,

$$f(x) = \frac{1}{x-1} - \frac{2}{x} + \frac{1}{x+1}.$$

2.2 Sketch the implicit function,

$$y^2 - xy - x^2 = 1.$$

2.3 Sketch the graph of the function,

$$f(x) = \frac{x^2}{x-1} + \frac{x^2}{x+1}.$$

Goal 3. Applications of differentiation.

3.1 A sculpture has the form of a right triangle. The material used for the vertical leg has twice the cost of the material used for the horizontal leg. The length of the hypotenuse is fixed (thus its cost is irrelevant). What ratio of vertical leg to horizontal leg minimizes the total cost of the material?

3.2 A farmer has a fence running diagonally across her property at a 45 degree angle to the north-south and east-west lines. She decides to build a corral by adding a length $b - a$ of fence running north-south, a length $b - a$ of fence running east-west, and then connect the two corners with 2 length b of fence running north-south and east-west. Thus, the total new length of fence needed is $4b - 2a$, and the corral has the form of a square of length b , with a small isosceles triangle of leg length a removed from one corner (where the square corral meets the pre-existing diagonal fence). What ratio of a to b gives maximal area of the corral for a fixed length of new fence?

3.3 An icicle has the shape of a right circular cone whose ratio of length to base radius is 10. Assuming the icicle melts at a rate of 1 cubic centimeter per hour, how fast is the length of the icicle decreasing when it is 10 centimeters long?

3.4 A cube of ice rests on the ground. The cube of ice melts at a rate proportional to the surface area of the cube exposed to the air (thus, the area of the 5 sides other than the side sitting on the ground). Assuming it takes 5 hours before the volume of the melted cube equals 1/2 the initial volume, how much longer does it take for the cube to melt entirely?

Goal 4. Integration.

4.1 An integrable function $f(x)$ is defined on the interval $[0, 1]$. Give the formula for the Riemann sum of $f(x)$ on $[0, 1]$ with respect to the partition of $[0, 1]$ into n subintervals of equal length, evaluating the function $f(x)$ at the right endpoint of each subinterval.

4.2 Let r be a positive real number. Use your formula for the Riemann sum to reduce the following limit,

$$\lim_{n \rightarrow \infty} \frac{1}{n^{r+1}} \sum_{k=1}^n k^r,$$

to a Riemann integral. Then evaluate that Riemann integral (using whatever integration technique you like), and determine the limit.

4.3 Use your formula for the Riemann sum to reduce the following limit,

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{\sqrt{4k^2 + n^2}},$$

to a Riemann integral. Use an inverse hyperbolic substitution to evaluate the Riemann integral, and determine the limit. You are free to use the following formulas,

$$\cosh^2(t) - \sinh^2(t) = 1,$$

$$\frac{d}{dt} \sinh(t) = \cosh(t), \quad \frac{d}{dt} \cosh(t) = \sinh(t).$$

See also **Problem 6.1**.

4.4 Compute the following integral,

$$\int_0^2 e^{t^2} dt - 2 \int_0^1 e^{4u^2} du.$$

Hint: It is not possible to write down the antiderivatives of either of the separate integrands.

4.5 Compute the following integral,

$$\int_0^{\pi/3} \frac{1}{\sec(\theta) + \tan(\theta)} d\theta.$$

Goal 5. Applications of integration.

5.1 Denote by a the unique angle in the range $0 < a < \pi/2$ satisfying,

$$x - 2 \tan(x) = \frac{8 - \pi}{4}.$$

Compute the area bounded by the curve,

$$y = \tan(x), \quad -\pi/2 < x < \pi/2,$$

and the tangent line to the curve at $(x, y) = (-\pi/4, -1)$.

5.2 Find the unique, positive value of h making the area of the region bounded by the parabola,

$$y = x^2,$$

and the parabola,

$$y = h - x^2,$$

equal one half the area of the region bounded by the parabola,

$$y = x^2,$$

and the parabola,

$$y = 1 - x^2.$$

5.3 Let a and b be positive real numbers. Find the volume of the solid obtained by rotating about the y -axis the region in the first quadrant of the xy -plane bounded by the x -axis, the y -axis, the line $x = a\pi$ and the curve,

$$y = b \sin(x/a).$$

You may use either the washer method or the shell method.

5.4 Compute the arc length of the segment of the curve,

$$y = \ln(x),$$

bounded by $(x, y) = (1, 0)$ and $(x, y) = (e, 1)$. At some point, it will help to make an inverse substitution. You may make either an inverse trigonometric substitution or an inverse hyperbolic substitution (the resulting integrals are comparable). See also **Problem 6.2**.

5.5 Compute the area of the surface obtained by rotating about the y -axis the segment of the curve,

$$y = \ln(x),$$

bounded by $(x, y) = (1, 0)$ and $(x, y) = (e, 1)$. At some point, it will help to make an inverse substitution. It is best to make an inverse hyperbolic substitution. You are free to use the following formulas,

$$\cosh^2(t) = \frac{1}{2}(\cosh(2t) + 1),$$

$$\sinh^2(t) = \frac{1}{2}(\cosh(2t) - 1).$$

See also **Problem 6.3**.

Goal 6. Techniques of integration.

6.1 Do the inverse substitution part of **Problem 4.3**.

6.2 Do the inverse substitution part of **Problem 5.4**.

6.3 Do the inverse substitution part of **Problem 5.5**.

6.4 Compute the following antiderivative,

$$\int \sin^{-1}(x)\sqrt{1-x^2}dx.$$

Hint: Use the half-angle formula and integration by parts.

6.5 Compute the following antiderivative,

$$\int \sin^{-1}(x)dx.$$

6.6 Compute the following antiderivative,

$$\int \cos^{-1}(x)dx.$$

6.7 Compute the following antiderivative,

$$\int \tan^{-1}(x)dx.$$

Goal 7. L'Hospital's rule.

7.1 Compute the following limit,

$$\lim_{x \rightarrow 0^+} \frac{x+1}{x^2+1}.$$

7.2 Compute the following limit,

$$\lim_{x \rightarrow 0^+} \ln(x) \ln\left(\frac{x+1}{x^2+1}\right).$$

7.3 Try to find nonzero, analytic functions $f(x)$ and $g(x)$ defined near $x = 0$ such that $f(0) = g(0) = 0$ but

$$\lim_{x \rightarrow 0^+} f(x)^{g(x)},$$

does not equal 1. **Remark.** Don't try too hard! No problem on the final will ask anything this vague. Also, functions $f(x)$ and $g(x)$ satisfying the conditions do not exist :)

7.4 Denote,

$$f(x) = \begin{cases} e^{1/x^2}, & x \neq 0, \\ 0, & x = 0 \end{cases}$$

and denote,

$$g(x) = x^2.$$

Compute the limit,

$$\lim_{x \rightarrow 0^+} f(x)^{g(x)}.$$

7.5 Compute the limit,

$$\lim_{x \rightarrow 0^+} \ln(1 - \cos(x)) - \ln(\sin^2(x)).$$

Goal 8. Improper integrals.

8.1 Determine whether the following improper integral converges or diverges. If it converges, evaluate it.

$$\int_{0^+}^1 -\ln(x) dx.$$

8.2 Determine whether the following improper integral converges or diverges. If it converges, evaluate it.

$$\int_1^{\infty} \frac{1}{x\sqrt{1+x^2}} dx.$$

Goal 9. Taylor series.

9.1 Compute the Taylor series expansion about $x = 0$ of,

$$f(x) = \ln(1 + x).$$

9.2 Let $a > -1$ be a real number. Compute the Taylor series expansion about $x = a$ of,

$$f(x) = \ln(1 + x).$$

9.3 Let a be a real number. Compute the Taylor series expansion about $x = a$ of,

$$f(x) = \sin(x).$$

9.4 Let a be a real number. Compute the Taylor series expansion about $x = a$ of,

$$f(x) = \cos(x).$$

9.5 Compute the Taylor series expansion about $x = 0$ of,

$$f(x) = \tan^{-1}(x) = \arctan(x).$$

9.6 Compute the Taylor series expansion about $x = 0$ of,

$$f(x) = \int_0^x \frac{\sin(t)}{t} dt.$$

Solutions.

Solution to 1.1 Denote by (a, e^a) a point on the curve. The tangent line to the curve has slope,

$$\frac{dy}{dx}(a) = e^a.$$

Therefore the equation of the tangent line is,

$$y = e^a(x - a) + e^a.$$

The tangent line contains $(-1, 0)$ if and only if,

$$0 = e^a((-1) - a) + e^a.$$

Simplifying gives,

$$0 = -ae^a.$$

Since e^a is always nonzero, the equation holds if and only if a equals 0. Therefore, the equation of the unique tangent line to the curve containing $(-1, 0)$ is,

$$y = x + 1.$$

Solution to 1.2 Implicit differentiation with respect to x gives,

$$\frac{2x}{a^2} + \frac{2y}{b^2}y' = 0.$$

Solving gives,

$$y' = -\frac{b^2x}{a^2y}.$$

So for a point (x_0, y_0) on the ellipse, the equation of the tangent line to the ellipse is,

$$y = -\frac{b^2x_0}{a^2y_0}(x - x_0) + y_0.$$

Multiplying both sides by a^2y_0 and simplifying gives,

$$b^2x_0x + a^2y_0y = b^2x_0^2 + a^2y_0^2.$$

Because the point (x_0, y_0) lies on the ellipse, the right-hand side is,

$$a^2b^2 \left(\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} \right) = a^2b^2.$$

Therefore the equation of the tangent line is,

$$b^2x_0x + a^2y_0y = a^2b^2.$$

Plugging in $(x, y) = (2a, 2b)$, the tangent line contains $(2a, 2b)$ if and only if,

$$2ab^2x_0 + 2a^2by_0 = a^2b^2.$$

Solving for y_0 gives,

$$2ay_0 = -2bx_0 + ab.$$

Because (x_0, y_0) is contained in the ellipse, the pair satisfies the equation,

$$b^2x_0^2 + a^2y_0^2 = a^2b^2,$$

which is equivalent to,

$$4b^2x_0^2 + (2ay_0)^2 - a^2b^2 = 0.$$

Substituting the equation for $2ay_0$ gives the quadratic equation in x_0 ,

$$8b^2x_0^2 - 4ab^2x_0 - 3a^2b^2 = 0.$$

Simplifying, this becomes,

$$8x_0^2 - 4ax_0 - 3a^2 = 0.$$

By the quadratic formula, the solutions are,

$$x_0 = \frac{1 + \sqrt{7}}{4}a, \quad x_0 = \frac{1 - \sqrt{7}}{4}a.$$

The corresponding solutions for y_0 are,

$$y_0 = \frac{1 - \sqrt{7}}{4}b, \quad y_0 = \frac{1 + \sqrt{7}}{4}b.$$

The corresponding equations of the tangent lines are,

$$(\sqrt{7} + 1)bx - (\sqrt{7} - 1)ay = 4ab,$$

and,

$$(\sqrt{7} + 1)ay - (\sqrt{7} - 1)ax = 4ab.$$

Solution to 1.3 The difference quotient at $x = a$ is,

$$\frac{f(a+h) - f(a)}{h} = \frac{1}{h} \left(\frac{1}{a+h} - \frac{1}{a} \right).$$

Clearing denominators gives,

$$\frac{1}{h} \frac{1}{a(a+h)} (a - (a+h)).$$

Cancelling gives,

$$\frac{1}{h} \frac{1}{a(a+h)}(-h).$$

Simplifying gives,

$$\frac{f(a+h) - f(a)}{h} = -\frac{1}{a(a+h)}.$$

Taking the limit as $h \rightarrow 0$ gives,

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = -\frac{1}{a(a+0)} = -\frac{1}{a^2}.$$

Therefore the derivative function is,

$$f'(x) = -1/x^2.$$

Solution to 1.4 By definition, the difference quotient at $x = 0$ is,

$$\frac{f(h) - f(0)}{h} = \frac{1}{h}(\tan(h) - \tan(0)).$$

Since $\tan(0) = 0$, this simplifies to,

$$\frac{\tan(h)}{h} = \frac{1}{\cos(h)} \frac{\sin(h)}{h}.$$

Taking the limit $h \rightarrow 0$ gives,

$$f'(0) = \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} = \left(\lim_{h \rightarrow 0} \frac{1}{\cos(h)} \right) \left(\lim_{h \rightarrow 0} \frac{\sin(h)}{h} \right).$$

Evaluating gives,

$$\frac{1}{\cos(0)} \times 1 = \frac{1}{1} \times 1 = 1.$$

Therefore the derivative at 0 is,

$$f'(0) = 1.$$

Solution to 1.5 Directly,

$$y = [\ln(x)]^2.$$

Therefore, by the chain rule,

$$\frac{dy}{dx} = 2[\ln(x)] \frac{d}{dx} \ln(x) = \frac{2 \ln(x)}{x}.$$

Alternatively, implicitly differentiating,

$$e^{\sqrt{y}} = x,$$

gives,

$$e^{\sqrt{y}} \frac{1}{2\sqrt{y}} \frac{dy}{dx} = 1.$$

Plugging in $e^{\sqrt{y}} = x$ and $\sqrt{y} = \ln(x)$ gives,

$$x \frac{1}{2\ln(x)} \frac{dy}{dx} = 1.$$

Solving gives,

$$\frac{dy}{dx} = \frac{2\ln(x)}{x}.$$

Thus, both methods give the answer,

$$\frac{dy}{dx} = 2\ln(x)/x.$$

Solution to 2.1 From the form of the function, it is clear the function has vertical asymptotes at $x = -1$, $x = 0$ and $x = +1$. Also, the function has a horizontal asymptote $y = 0$. Observe that,

$$f(x) = \frac{1}{x-1} - \frac{2}{x} + \frac{1}{x+1} = \frac{2}{x(x^2-1)}.$$

Similarly,

$$f'(x) = -\frac{1}{(x-1)^2} + \frac{2}{x^2} - \frac{1}{(x+1)^2} = \frac{-2(3x^2-1)}{x^2(x^2-1)^2}.$$

Therefore the only critical points are,

$$x = -\frac{1}{\sqrt{3}}, \text{ and } x = +\frac{1}{\sqrt{3}}.$$

The corresponding points on the curve are,

$$(x, y) = \left(-\frac{1}{\sqrt{3}}, -\sqrt{3}\right), \quad (x, y) = \left(+\frac{\sqrt{3}}{3}, +\sqrt{3}\right).$$

The function $f(x)$ is decreasing for $x < -1/\sqrt{3}$, increasing for $-1/\sqrt{3} < x < 1/\sqrt{3}$, and decreasing for $x > 1/\sqrt{3}$ (except, of course, where $f(x)$ is undefined at $x = -1, 0, +1$). It is clear from the form of the derivative that $x = -1/\sqrt{3}$ is a local minimum and $x = +1/\sqrt{3}$ is a local maximum. The second derivative is,

$$f''(x) = \frac{2}{(x-1)^3} - \frac{4}{x^3} + \frac{2}{(x+1)^3} = \frac{4(6x^4 - 3x^2 + 1)}{x^3(x^2-1)^3}.$$

The numerator is a quadratic equation in x^2 . However, there are no real solutions of the quadratic equation. Therefore there are no inflection points. The function is concave down for $x < -1$,

concave up for $-1 < x < 0$, concave down for $0 < x < +1$, and concave up for $x > +1$. Finally, $f(x)$ is an odd function. This is more than enough to sketch the graph. A sketch is included on the course webpage.

Solution to 2.2 This is the equation of a hyperbola. The asymptotes of the hyperbola are the lines with equations,

$$2y - (\sqrt{5} + 1)x = 0,$$

and,

$$2y + (\sqrt{5} - 1)x = 0.$$

Implicit differentiation gives,

$$2yy' - y - xy' - 2x = 0,$$

or,

$$y' = \frac{2x + y}{2y - x}.$$

Thus the critical points are the intersection points of the hyperbola with the line,

$$y = -2x.$$

Solving gives a local minimum at,

$$(x, y) = \left(-\frac{\sqrt{5}}{5}, +\frac{2\sqrt{5}}{5} \right),$$

and a local maximum at,

$$(x, y) = \left(+\frac{\sqrt{5}}{5}, -\frac{2\sqrt{5}}{5} \right).$$

There are no inflection points. A sketch of the hyperbola is included on the course webpage.

Solution to 2.3 Clearly there are vertical asymptotes at $x = -1$ and $x = +1$. Simplifying,

$$f(x) = 2x + \frac{1}{x-1} + \frac{1}{x+1} = \frac{2x^3}{x^2-1}.$$

In particular, $y = f(x)$ is asymptotic to the line $y = 2x$ as x goes to $\pm\infty$. Also, $f(x)$ is an odd function that clearly has an inflection point at $(0, 0)$. The derivative is,

$$f'(x) = 2 - \frac{1}{(x-1)^2} - \frac{1}{(x+1)^2} = \frac{2x^2(x^2-3)}{(x^2-1)^2}.$$

Thus the critical points are $x = -\sqrt{3}$, $x = 0$ and $x = +\sqrt{3}$. The function is increasing for $x < -\sqrt{3}$, it is decreasing for $-\sqrt{3} < x < -1$, it is decreasing for $-1 < x < +1$, it is decreasing for

$+1 < x < +\sqrt{3}$, and it is increasing for $x > +\sqrt{3}$. Therefore $x = -\sqrt{3}$ is a local maximum, and $x = +\sqrt{3}$ is a local minimum. Finally, the second derivative is,

$$f''(x) = \frac{2}{(x-1)^3} + \frac{2}{(x+1)^3} = \frac{4x(x^2+3)}{(x^2-1)^3}.$$

In particular, the only inflection point is $x = 0$. The function is concave down for $x < -1$, it is concave up for $-1 < x < 0$, it is concave down for $0 < x < +1$, and it is concave up for $x > +1$. This is more than enough to sketch the graph. A sketch is included on the course webpage.

Solution to 3.1 Denote by x the length of the horizontal leg. Denote by y the length of the vertical leg. Denote by s the length of the hypotenuse. The constraint is that,

$$s^2 = x^2 + y^2,$$

is a constant. The cost function is,

$$C = x + 2y.$$

Take x as the independent variable. The endpoints are $x = 0$ and $x = s$. Implicitly differentiating with respect to x gives,

$$\frac{dC}{dx} = 1 + 2\frac{dy}{dx}.$$

Thus, at every critical point of C ,

$$\frac{dy}{dx} = -\frac{1}{2}.$$

Implicitly differentiating the constraint equation with respect to x gives,

$$2x + 2y\frac{dy}{dx} = 0.$$

Plugging in $dy/dx = -1/2$ at a critical point gives the linear equation,

$$2x - y = 0.$$

In other words, $y = 2x$. Substituting in to the constraint, the one critical point of C is,

$$x = \frac{\sqrt{5}s}{5}, \quad y = \frac{2\sqrt{5}s}{5}.$$

The cost at this point is,

$$C = \frac{4\sqrt{5}s}{5}.$$

In fact, this critical point is a local maximum. At the endpoint $x = 0$, y equals s and the cost is,

$$C = 2s.$$

At the endpoint $x = s$, y equals 0 and the cost is,

$$C = s.$$

Therefore, the ratio of vertical leg to horizontal leg minimizing the total cost is,

$$\frac{y}{x} = 0.$$

Solution to 3.2 The total length of fence is,

$$L = 4b - 2a.$$

The area of the corral is,

$$A = b^2 - \frac{1}{2}a^2.$$

Choose b as the independent variable. From the constraint equation,

$$a = 2b - \frac{L}{2}.$$

Since a must be between 0 and b , the endpoints of b are $b = L/4$ and $b = L/2$. Substituting $2b - \frac{L}{2}$ for a in the equation for A gives,

$$A(b) = b^2 - \frac{1}{2} \left(2b - \frac{L}{2} \right)^2 = -b^2 + Lb - \frac{L^2}{4}.$$

Differentiating,

$$\frac{dA}{db} = -2b + L.$$

Thus the only critical point is at $b = L/2$, which is also an endpoint. This gives a local maximum for A ,

$$A(L/2) = \frac{L^2}{8}.$$

The value of A at the other endpoint is,

$$A(L/4) = \frac{L^2}{16}.$$

Thus the global maximum is when $b = L/2$, and thus $a = L/2$. Therefore, the ratio of a to b giving the maximal area of the corral is,

$$\frac{a}{b} = 1.$$

Solution to 3.3 Denote the radius by r and the length by l . The volume of the icicle is,

$$\frac{1}{3}(\pi r^2)(l) = \frac{\pi r^2 l}{3}.$$

Using the constraint that $l = 10r$, the volume is,

$$V(r) = \frac{10\pi r^3}{3}.$$

Therefore the rate of change of V is,

$$\frac{dV}{dt} = 10\pi r^2 \frac{dr}{dt}.$$

By hypothesis, dV/dt equals -1 cubic centimeter per hour. Thus solving gives,

$$\frac{dr}{dt} = -\frac{1 \text{ cm}^3}{10\pi r^2 \text{ hr}}.$$

When the length is $l = 10\text{cm}$, the radius is $r = 1\text{cm}$. Plugging in, the radius of the icicle is decreasing at the rate,

$$\frac{dr}{dt} = -\frac{1 \text{ cm}}{10\pi \text{ hr}}.$$

Therefore the length is decreasing at the rate,

$$\frac{dl}{dt} = 10 \frac{dr}{dt} = -\frac{1 \text{ cm}}{\pi \text{ hr}}.$$

To summarize, the length of the icicle is decreasing at the rate,

$$\frac{dl}{dt} = \boxed{-(1/\pi) \text{ centimeter per hour.}}$$

Solution to 3.4 Denote by x the edge length of the cube. Then the volume is,

$$V(x) = x^3.$$

Also, the area of the 5 exposed sides is,

$$A(x) = 5x^2.$$

By hypothesis, the rate of change of the volume is proportional to the area of the exposed sides, i.e.,

$$\frac{dV}{dt} = CA,$$

for some constant C . Differentiating gives,

$$\frac{dV}{dt} = 3x^2 \frac{dx}{dt}.$$

Substituting gives,

$$\frac{dx}{dt} = \frac{5C}{3}.$$

But this is just another constant. Because dx/dt is a constant, $x(t)$ is a linear function,

$$x(t) = x_0 - at,$$

for some constant a . By hypothesis, $V(x(5)) = V(x_0)/2$. Solving gives,

$$x(5) = \frac{1}{\sqrt[3]{2}}x_0 = x_0 - 5a.$$

Solving for a gives,

$$a = \frac{\sqrt[3]{2} - 1}{5\sqrt[3]{2}}x_0.$$

Therefore the cube is completely melted at time,

$$t = \frac{5\sqrt[3]{2}}{\sqrt[3]{2} - 1}.$$

So the additional time from $t = 5$ before the ice cube melts is,

$$5/(\sqrt[3]{2} - 1) \text{ hours.}$$

Solution to 4.1 The endpoints of the k^{th} interval are $x_{k-1} = (k-1)/n$ and $x_k = k/n$. The length of each interval is $\Delta x_k = 1/n$. Thus the formula for the Riemann sum is,

$$\sum_{k=1}^n f(x_k)\Delta x_k = (1/n) \sum_{k=1}^n f(k/n).$$

Solution to 4.2 Plugging in $k = nx_k$ gives,

$$\sum_{k=1}^n \frac{1}{n^{r+1}} \sum_{k=1}^n k^r = \sum_{k=1}^n \frac{(nx_k)^r}{n^{r+1}} = \sum_{k=1}^n x_k^r \Delta x_k.$$

Therefore the function is $f(x) = x^r$. The Riemann integral is,

$$\int_0^1 f(x)dx = \int_0^1 x^r dx = \frac{1}{r+1} (x^{r+1})\Big|_0^1 = \frac{1}{r+1}.$$

Therefore the limit equals,

$$\lim_{n \rightarrow \infty} \frac{1}{n^{r+1}} \sum_{k=1}^n k^r = 1/(r+1).$$

Solution to 4.3 Plugging in $k = nx_k$ gives,

$$\sum_{k=1}^n \frac{1}{\sqrt{4k^2 + n^2}} = \sum_{k=1}^n \frac{1}{\sqrt{4x_k^2 + 1}} \frac{1}{n} = \sum_{k=1}^n \frac{1}{4x_k^2 + 1} \Delta x_k.$$

Thus the function is,

$$f(x) = \frac{1}{\sqrt{4x^2 + 1}}.$$

So the limit of the Riemann sums equals the Riemann integral,

$$\int_0^1 \frac{1}{\sqrt{4x^2 + 1}} dx.$$

To evaluate this integral, make the inverse hyperbolic substitution,

$$x = \frac{1}{2} \sinh(t), \quad dx = \frac{1}{2} \cosh(t) dt.$$

The new integral is,

$$\int_0^{\sinh^{-1}(2)} \frac{1}{2} dt = \frac{1}{2} \sinh^{-1}(2).$$

In fact,

$$\sinh^{-1}(2) = \ln(2 + \sqrt{5}).$$

Therefore, the limit is,

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{\sqrt{4k^2 + n^2}} = \ln(2 + \sqrt{5})/2.$$

Solution to 4.4 To simplify the second integral, make the substitution,

$$v = 2u, \quad dv = 2du.$$

The new integral is,

$$2 \int_0^2 e^{4u^2} dx = \int_0^2 e^{v^2} dv.$$

But this equals the first integral. Therefore the difference is,

$$\int_0^2 e^{t^2} dt - \int_0^2 e^{v^2} dv = 0.$$

Solution to 4.5 To simplify the integral, multiply numerator and denominator by $\cos(\theta)$,

$$\int_0^{\pi/3} \frac{\cos(\theta)}{1 + \sin(\theta)} d\theta.$$

Substituting,

$$u = 1 + \sin(\theta), \quad du = \cos(\theta) d\theta, \quad u(0) = 1, \quad u(\pi/3) = \frac{\sqrt{3} + 2}{2},$$

gives the new integral,

$$\int_1^{(\sqrt{3}+2)/2} \frac{dv}{v} = (\ln(v)) \Big|_1^{(\sqrt{3}+2)/2}.$$

Thus the integral equals,

$$\int_0^{\pi/3} \frac{1}{\sec(\theta) + \tan(\theta)} d\theta = \ln(\sqrt{3} + 2) - \ln(2).$$

Solution to 5.1 The derivative of $\tan(x)$ is,

$$\frac{dy}{dx} = \sec^2(x).$$

In particular, the slope of the tangent line to the curve at $x = -\pi/4$ is,

$$\sec^2(-\pi/4) = \frac{1}{2}.$$

So the equation of the tangent line is,

$$y = \frac{1}{2} \left(x + \frac{\pi}{4} \right) - 1.$$

Therefore the other intersection point of the line with the curve $y = \tan(x)$ occurs at the unique point x such that,

$$\tan(x) = \frac{1}{2} \left(x + \frac{\pi}{4} \right) - 1.$$

This is precisely the equation defining a . The area of the region is thus,

$$\text{Area} = \int_{-\pi/4}^a \left[\frac{1}{2} \left(x + \frac{\pi}{4} \right) - 1 \right] - [\tan(x)] dx = \int_{-\pi/4}^a \frac{1}{2} x dx - \int_{-\pi/4}^a \frac{8 - \pi}{8} dx + \int_{-\pi/4}^a \frac{-\sin(x)}{\cos(x)} dx.$$

The first 2 integrals give,

$$\int_{-\pi/4}^a \frac{1}{2} x dx = \frac{16a^2 - \pi^2}{64},$$

and,

$$\int_{-\pi/4}^a \frac{8 - \pi}{8} dx = \frac{(8 - \pi)(4a + \pi)}{32}.$$

For the third integral, substitute,

$$u = \cos(x), \quad du = -\sin(x) dx,$$

to get,

$$\int \frac{-\sin(x)}{\cos(x)} dx = \int \frac{du}{u} = \ln(u) + C = \ln(\cos(x)) + C.$$

Therefore,

$$\int_{-\pi/4}^a \frac{-\sin(x)}{\cos(x)} dx = \ln(\cos(a)) + \frac{1}{2} \ln(2).$$

Altogether, the area is,

$$\text{Area} = \frac{1}{4}a^2 - \frac{8-\pi}{8}a + \ln(\cos(a)) + \frac{\pi^2}{64} - \frac{\pi}{4} + \frac{1}{2} \ln(2).$$

Solution to 5.2 The parabola $y = x^2$ intersects the parabola $y = h - x^2$ at (x, y) only if,

$$x^2 = y = h - x^2.$$

Equivalently,

$$2x^2 = h,$$

or,

$$x = \pm \frac{\sqrt{2h}}{2}.$$

In particular, when $h = 1$, the intersection points are,

$$x = -\frac{\sqrt{2}}{2}, \text{ and } x = +\frac{\sqrt{2}}{2}.$$

Therefore the area of the bigger region is,

$$\text{Area} = \int_{-\sqrt{2}/2}^{+\sqrt{2}/2} [1 - x^2] - [x^2] dx = 2 \int_0^{\sqrt{2}/2} 1 - 2x^2 dx.$$

Evaluating gives,

$$2 \left(x - \frac{2x^3}{3} \Big|_0^{\sqrt{2}/2} = \frac{2\sqrt{2}}{3}.$$

Thus the goal is to find the unique value of h so that,

$$\text{Area} = \int_{-\sqrt{2h}/2}^{+\sqrt{2h}/2} [h - x^2] - [x^2] dx,$$

equals $\sqrt{2}/3$. Simplifying gives,

$$\text{Area} = 2 \int_0^{\sqrt{2h}/2} h - 2x^2 dx = 2 \left(hx - \frac{2x^3}{3} \Big|_0^{\sqrt{2h}/2}.$$

Evaluating gives,

$$\text{Area} = \frac{2h\sqrt{2h}}{3}.$$

So the equation is,

$$\frac{2h\sqrt{2h}}{3} = \frac{\sqrt{2}}{3}.$$

Simplifying and squaring gives,

$$8h^3 = 2.$$

Thus the solution is,

$$h = \sqrt[3]{2}/2.$$

Solution to 5.3 Using the shell method, the infinitesimal volume of a shell with inner radius x , outer radius $x + dx$, and height $y = b \sin(x/a)$ is,

$$dV = 2\pi(\text{radius})(\text{height})(\text{width}) = 2\pi xy dx = 2b\pi x \sin(x/a) dx.$$

Therefore the total volume equals,

$$V = \int dV = \int_{x=0}^{x=a\pi} 2b\pi x \sin(x/a) dx.$$

To compute this integral, use integration by parts,

$$\begin{array}{l} u = 2b\pi x \quad | \quad dv = \sin(x/a) dx \\ du = 2b\pi dx \quad | \quad v = -a \cos(x/a) \end{array}$$

This gives,

$$\begin{aligned} \int u dv &= uv - \int v du, \\ \int_{x=0}^{x=a\pi} 2b\pi x \sin(x/a) dx &= (-2ab\pi x \cos(x/a)) \Big|_0^{a\pi} + 2ab\pi \int_0^{a\pi} \cos(x/a) dx. \end{aligned}$$

For the second integral, notice that $\cos(x/a)$ is positive for $0 < x < a\pi/2$, and then for $a\pi/2 < x < a\pi$, $\cos(x/a)$ is the negative of $\cos(x/a)$ on the first interval. Therefore the integral from 0 to $a\pi/2$ cancels exactly the integral from $a\pi/2$ to $a\pi$. Therefore,

$$\text{Volume} = 2\pi a^2 b.$$

Observe that the volume of the cylinder with base radius $a\pi$ and height b is $\pi^3 a^2 b$. Thus the fraction of the volume of the cylinder taken up by the solid is $2/\pi^2$.

Solution to 5.4 The infinitesimal element of arc length, ds , satisfies the equation,

$$(ds)^2 = (dx)^2 + (dy)^2 = (1 + (y')^2)(dx)^2.$$

In this case,

$$y' = \frac{1}{x}, \quad 1 + (y')^2 = \frac{x^2 + 1}{x^2}.$$

Therefore, the infinitesimal element of arc length equals,

$$ds = \sqrt{1 + (y')^2} dx = \frac{\sqrt{x^2 + 1}}{x} dx.$$

So the arc length of the curve is,

$$s = \int ds = \int_{x=1}^{x=e} \frac{\sqrt{x^2 + 1}}{x} dx.$$

To solve this integral, use an inverse substitution. The goal is that $x^2 + 1$ equals a square. This occurs for,

$$x = \tan(\theta), \quad dx = \sec^2(\theta) d\theta.$$

Inverse substitution gives,

$$\int \frac{\sqrt{x^2 + 1}}{x} dx = \int \frac{\sec^3(\theta)}{\tan(\theta)} d\theta.$$

To simplify this, write it in terms of $\sin(\theta)$ and $\cos(\theta)$ as,

$$\int \frac{1}{\sin(\theta) \cos^2(\theta)} d\theta = \int \frac{1}{\sin^2(\theta) \cos^2(\theta)} \sin(\theta) d\theta = \int \frac{1}{(1 - \cos^2(\theta)) \cos^2(\theta)} \sin(\theta) d\theta.$$

This simplifies by making the direct substitution,

$$u = \cos(\theta), \quad du = -\sin(\theta) d\theta.$$

Drawing a right triangle gives,

$$u(x) = \cos(\tan^{-1}(x)) = \frac{1}{\sqrt{1 + x^2}}.$$

Thus the new limits of integration are,

$$u(1) = \frac{1}{\sqrt{2}}, \quad u(e) = \frac{1}{\sqrt{1 + e^2}}.$$

Substitution gives,

$$\int_{1/\sqrt{2}}^{1/\sqrt{1+e^2}} \frac{1}{(u^2 - 1)u^2} du.$$

The partial fraction decomposition of the integrand has the form,

$$\frac{1}{(u^2 - 1)u^2} = \frac{1}{(u - 1)(u + 1)u^2} = \frac{A}{u - 1} + \frac{B}{u + 1} + \frac{C}{u^2} + \frac{D}{u}.$$

Heaviside's cover-up method determines the first 3 coefficients,

$$A = \frac{1}{((1) + 1)(1)^2} = \frac{1}{2},$$

$$B = \frac{1}{((-1) - 1)(-1)^2} = \frac{-1}{2},$$
$$C = \frac{1}{((0) - 1)((0) + 1)} = -1.$$

For the last coefficient, observe that $1/(u^2(u^2 - 1))$ is an even function. Therefore the coefficient D must be 0. So the partial fraction decomposition is,

$$\frac{1}{(u^2 - 1)u^2} = \frac{1}{2} \frac{1}{u - 1} - \frac{1}{2} \frac{1}{u + 1} - \frac{1}{u^2}.$$

So the integral is,

$$\int_{1/\sqrt{2}}^{1/\sqrt{1+e^2}} \frac{1}{2} \frac{1}{u - 1} - \frac{1}{2} \frac{1}{u + 1} - \frac{1}{u^2} du = \left[\frac{1}{2} \ln \left(\left| \frac{u - 1}{u + 1} \right| \right) + \frac{1}{u} \right]_{1/\sqrt{2}}^{1/\sqrt{1+e^2}}.$$

Therefore the total arclength is,

$$s = \sqrt{1 + e^2} - \sqrt{2} + \frac{1}{2} \ln \left(\frac{(\sqrt{2}-1)(\sqrt{1+e^2}+1)}{(\sqrt{2}+1)(\sqrt{1+e^2}-1)} \right).$$

Solution to 5.5 By the **Solution to 5.4**,

$$ds = \frac{\sqrt{x^2 + 1}}{x} dx.$$

The infinitesimal surface area is,

$$dA = 2\pi(\text{radius})(\text{slant height}) = 2\pi x ds = 2\pi \sqrt{x^2 + 1} dx.$$

Therefore the total area is,

$$A = \int dA = 2\pi \int_{x=1}^{x=e} \sqrt{x^2 + 1} dx.$$

To compute this, it is best to make an inverse hyperbolic substitution. Since the goal is that $x^2 + 1$ is a square, it is best to use $x = \sinh(t)$, so that $x^2 + 1 = \cosh^2(t)$. The substitution is,

$$x = \sinh(t), \quad dx = \cosh(t) dt.$$

Then the new integral is,

$$\int \sqrt{x^2 + 1} dx = \int \cosh^2(t) dt.$$

To simplify this, use the half-angle formula for hyperbolic functions,

$$\cosh^2(t) = \frac{1}{2}(\cosh(2t) + 1).$$

Substituting gives,

$$\frac{1}{2} \int \cosh(2t) + 1 dt = \frac{1}{2} \left(\frac{1}{2} \sinh(2t) + t \right) + C.$$

Using the double-angle formula,

$$\sinh(2t) = 2 \sinh(t) \cosh(t),$$

the antiderivative is,

$$\int \cosh^2(t) dt = \frac{1}{2} (\sinh(t) \cosh(t) + t) + C.$$

Back-substituting gives,

$$\int \sqrt{x^2 + 1} dx = \frac{1}{2} \left(x\sqrt{x^2 + 1} + \sinh^{-1}(x) \right) + C.$$

Therefore the total area is,

$$2\pi \int_1^e \sqrt{x^2 + 1} dx = \pi \left(x\sqrt{x^2 + 1} + \sinh^{-1}(x) \right) \Big|_1^e.$$

Evaluating gives,

$$\text{Area} = \pi(e\sqrt{e^2 + 1} + \ln(e + \sqrt{1 + e^2}) - \sqrt{2} - \ln(1 + \sqrt{2})).$$

Solution to 6.1 See **Solution to 4.3**.

Solution to 6.2 See **Solution to 5.4**.

Solution to 6.3 See **Solution to 5.5**.

Solution to 6.4 First make the inverse substitution,

$$x = \sin(\theta), \quad dx = \cos(\theta)d\theta.$$

The new integral is,

$$\int \theta \cos(\theta)(\cos(\theta)d\theta) = \int \theta \cos^2(\theta)d\theta.$$

To simplify this, use the half-angle formula,

$$\cos^2(\theta) = \frac{1}{2}(1 + \cos(2\theta)).$$

Substituting gives,

$$\frac{1}{2} \int \theta d\theta + \frac{1}{2} \int \theta \cos(2\theta)d\theta.$$

Of course the first antiderivative is simply,

$$\frac{1}{2} \int \theta d\theta = \frac{1}{4} \theta^2 + C.$$

For the second antiderivative, use integration by parts,

$$\begin{array}{l|l} u = \theta & dv = \cos(2\theta)d\theta, \\ du = d\theta & v = \sin(2\theta)/2. \end{array}$$

Thus,

$$\begin{aligned} \int u dv &= uv - \int v du, \\ \frac{1}{2} \int \theta \cos(2\theta) d\theta &= \frac{1}{4} \theta \sin(2\theta) - \frac{1}{4} \int \sin(2\theta) d\theta. \end{aligned}$$

The last integral is straightforward, giving,

$$\frac{1}{2} \int \theta \cos(2\theta) d\theta = \frac{1}{4} \theta \sin(2\theta) + \frac{1}{8} \cos(2\theta) + C.$$

In total, the integral is,

$$\int \theta \cos^2(\theta) d\theta = \frac{1}{4} \theta^2 + \frac{1}{2} \theta \sin(\theta) \cos(\theta) + \frac{1}{8} (1 - 2 \sin^2(\theta)) + C.$$

Back-substituting, the original integral is,

$$\int \sin^{-1}(x) \sqrt{1-x^2} dx = (1/4)[\sin^{-1}(x)]^2 + (1/2)x\sqrt{1-x^2} \sin^{-1}(x) + (1/8)(1-2x^2) + C.$$

Solution to 6.5 First make the inverse substitution,

$$x = \sin(\theta), \quad dx = \cos(\theta)d\theta.$$

The new integral is,

$$\int \theta(\cos(\theta)d\theta) = \int \theta \cos(\theta) d\theta.$$

This can be computed using integration by parts.

$$\begin{array}{l|l} u = \theta & dv = \cos(\theta)d\theta, \\ du = d\theta & v = \sin(\theta). \end{array}$$

Thus,

$$\int u dv = uv - \int v du,$$

$$\int \theta \cos(\theta) d\theta = \theta \sin(\theta) - \int \sin(\theta) d\theta = \theta \sin(\theta) + \cos(\theta) + C.$$

Back-substituting gives,

$$\int \sin^{-1}(x) dx = x \sin^{-1}(x) + \sqrt{1-x^2} + C.$$

Solution to 6.6 This is very similar to the last case. The answer is,

$$\int \cos^{-1}(x) dx = x \cos^{-1}(x) - \sqrt{1-x^2} + C.$$

Solution to 6.7 Make the inverse substitution,

$$x = \tan(\theta), \quad dx = \sec^2(\theta) d\theta.$$

The new integral equals,

$$\int \theta (\sec^2(\theta) d\theta) = \int \theta \sec^2(\theta) d\theta.$$

Next use integration by parts,

$$\begin{array}{l|l} u = \theta & dv = \sec^2(\theta) d\theta, \\ du = d\theta & v = \tan(\theta) \end{array}$$

Therefore,

$$\begin{aligned} \int u dv &= uv - \int v du, \\ \int \theta \sec^2(\theta) d\theta &= \theta \tan(\theta) - \int \tan(\theta) d\theta. \end{aligned}$$

For the second integral, rewrite it as,

$$- \int \tan(\theta) d\theta = \int \frac{1}{\cos(\theta)} (-\sin(\theta) d\theta).$$

Making the substitution,

$$u = \cos(\theta), \quad du = -\sin(\theta) d\theta,$$

the integral becomes,

$$\int \frac{1}{u} du = \ln(|u|) + C = \ln(|\cos(\theta)|) + C.$$

Therefore the total integral is,

$$\int \theta \sec^2(\theta) d\theta = \theta \tan(\theta) + \ln(|\cos(\theta)|) + C.$$

Using right triangles,

$$\cos(\theta) = \cos(\tan^{-1}(x)) = \frac{1}{\sqrt{1+x^2}}.$$

Back-substituting for x gives,

$$\int \tan^{-1}(x) dx = x \tan^{-1}(x) - \ln(\sqrt{1+x^2}) + C.$$

Solution to 7.1 The limit is simply,

$$\lim_{x \rightarrow 0^+} \frac{x+1}{x^2+1} = \frac{(0)+1}{(0)^2+1} = 1.$$

Solution to 7.2 Here it is best to first compute the linearization of the second factor,

$$\frac{1+x}{1+x^2} = (1+x)(1-x^2+x^4-x^6+\dots) = 1+x - x^2 - x^3 + x^4 + x^5 - x^6 - x^7 + \dots$$

Using the linearization,

$$\ln(1+u) = u - \frac{u^2}{2} + \frac{u^3}{3} - \frac{u^4}{4} + \dots,$$

gives the linearization,

$$\ln\left(\frac{1+x}{1+x^2}\right) = x - \frac{3}{2}x^2 - \frac{8}{3}x^3 + \dots$$

Therefore, the limit is,

$$\lim_{x \rightarrow 0^+} \ln(x)x \left(1 - \frac{3}{2}x - \frac{8}{3}x^2 + \dots\right) = \left(\lim_{x \rightarrow 0^+} \ln(x)x\right) \left(1 - \frac{3}{2}0 - \frac{8}{3}0^2 + \dots\right) = \lim_{x \rightarrow 0^+} \frac{\ln(x)}{1/x}.$$

Applying L'Hospital's rule,

$$\lim_{x \rightarrow 0^+} \frac{\ln(x)}{1/x} = \lim_{x \rightarrow 0^+} \frac{(1/x)}{-(1/x^2)} = - \lim_{x \rightarrow 0^+} x = 0.$$

Therefore the limit is,

$$\lim_{x \rightarrow 0^+} \ln(x) \ln\left(\frac{1+x}{1+x^2}\right) = 0.$$

Solution to 7.3 It is impossible. Write $f(x) = x^m(a_m + a_{m+1}x + a_{m+2}x^2 + \dots)$, with $a_m \neq 0$. Write $g(x) = x^n(b_n + b_{n+1}x + b_{n+2}x^2 + \dots)$, with $b_n \neq 0$. By hypothesis, both m and n are positive integers. The limit to be computed is,

$$f(x)^{g(x)} = e^{g(x) \ln(f(x))}.$$

Therefore the real limit needed is,

$$\lim_{x \rightarrow 0^+} g(x) \ln(f(x)) = b_n \lim_{x \rightarrow 0^+} x^n \ln(x^m(a_m + \dots)) = b_n \lim_{x \rightarrow 0^+} x^n \ln(x^m) + b_n \lim_{x \rightarrow 0^+} x^n \ln(a_m + \dots).$$

Simplifying the second limit gives,

$$b_n(0)^n \ln(a_m) = 0.$$

This leaves only the first limit,

$$mb_n \lim_{x \rightarrow 0^+} x^n \ln(x) = mb_n \lim_{x \rightarrow 0^+} \frac{\ln(x)}{x^{-n}}.$$

Applying L'Hospital's rule,

$$\lim_{x \rightarrow 0^+} \frac{\ln(x)}{x^{-n}} = \lim_{x \rightarrow 0^+} \frac{x^{-1}}{-nx^{-n-1}} = \frac{-1}{n} \lim_{x \rightarrow 0^+} x^n = 0.$$

Thus, altogether,

$$\lim_{x \rightarrow 0^+} g(x) \ln(f(x)) = 0.$$

Therefore,

$$\lim_{x \rightarrow 0^+} f(x)^{g(x)} = e^0 = \boxed{1}.$$

Solution to 7.4 By definition,

$$f(x)^{g(x)} = (e^{1/x^2})^{x^2} = e^{(1/x^2) \cdot x^2} = e^1,$$

for $x \neq 0$. Since this is a constant function,

$$\lim_{x \rightarrow 0^+} f(x)^{g(x)} = \lim_{x \rightarrow 0^+} e = \boxed{e}.$$

Solution to 7.5 Write the limit as,

$$\ln(1 - \cos(x)) - \ln(\sin^2(x)) = \ln\left(\frac{1 - \cos(x)}{\sin^2(x)}\right).$$

Using the Taylor series expansion,

$$1 - \cos(x) = 1 - (1 - x^2/2 + x^4/24 + \dots) = x^2/2 - x^4/24 + \dots,$$

and,

$$(\sin(x))^2 = (x - x^3/6 + \dots)^2 = x^2 - x^4/3 + \dots,$$

the Taylor series expansion of the ratio is,

$$\frac{1 - \cos(x)}{(\sin(x))^2} = \frac{x^2/2 - x^4/24 + \dots}{x^2 - x^4/3 + \dots} = \frac{1}{2} + \frac{1}{8}x^2 + \dots$$

Therefore,

$$\lim_{x \rightarrow 0^+} \ln \left(\frac{1 - \cos(x)}{(\sin(x))^2} \right) = \lim_{x \rightarrow 0^+} \ln \left(\frac{1}{2} + \dots \right) = \ln(1/2).$$

So the limit is,

$$\lim_{x \rightarrow 0^+} \ln(1 - \cos(x)) - \ln((\sin(x))^2) = -\ln(2).$$

Solution to 8.1 Using integration by parts,

$$\int -\ln(x) dx = -x \ln(x) + x + C.$$

Therefore the improper integral is,

$$\lim_{h \rightarrow 0^+} \int_h^1 -\ln(x) dx = \lim_{h \rightarrow 0^+} (-x \ln(x) + x) \Big|_h^1 = \lim_{h \rightarrow 0^+} 1 - h + h \ln(h).$$

As computed above, using L'Hospital's rule,

$$\lim_{h \rightarrow 0^+} h \ln(h) = 0.$$

Therefore the limit is $1 - 0 + 0 = 1$. So the improper integral converges to,

$$\int_{0^+}^1 -\ln(x) dx = 1.$$

Solution to 8.2 By comparison to $1/x^2$, the improper integral converges. To evaluate it, make the inverse trigonometric substitution,

$$x = \tan(\theta), \quad dx = \sec^2(\theta) d\theta.$$

The new integral is,

$$\int \frac{1}{\sec(\theta) \tan(\theta)} \sec^2(\theta) d\theta = \int \frac{1}{\sin(\theta)} d\theta = \frac{1}{\cos^2(\theta) - 1} (-\sin(\theta) d\theta).$$

Now make the substitution,

$$u = \cos(\theta), \quad du = -\sin(\theta) d\theta.$$

Using right triangles,

$$u(x) = \cos(\tan^{-1}(x)) = \frac{1}{\sqrt{1+x^2}}.$$

So the new limits of integration are,

$$u(1) = \frac{1}{\sqrt{2}}, \quad \lim_{R \rightarrow \infty} u(R) = 0^+.$$

So the integral is,

$$\int_{1/\sqrt{2}}^{0^+} \frac{1}{u^2 - 1} du = \int_{0^+}^{1/\sqrt{2}} \frac{1}{1 - u^2} du.$$

The partial fractions decomposition of $1/(1 - u^2)$ has the form,

$$\frac{1}{1 - u^2} = \frac{-1}{(u + 1)(u - 1)} = \frac{A}{u - 1} + \frac{B}{u + 1}.$$

Using the Heaviside cover-up method,

$$A = \frac{-1}{(1) + 1} = -\frac{1}{2},$$

and,

$$B = \frac{-1}{(-1) - 1} = \frac{1}{2}.$$

Thus the integral is,

$$\int_{0^+}^{1/\sqrt{2}} \frac{1}{2} \frac{1}{u + 1} - \frac{1}{2} \frac{1}{u - 1} du = \left(\frac{1}{2} \ln \left(\left| \frac{u + 1}{u - 1} \right| \right) \right) \Big|_{0^+}^{1/\sqrt{2}}.$$

Evaluating gives,

$$\int_1^{\infty} \frac{1}{x\sqrt{1+x^2}} dx = (1/2) \ln((\sqrt{2} + 1)/(\sqrt{2} - 1)).$$

Solution to 9.1 The Taylor series expansion is,

$$\ln(1 + x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n.$$

The simplest way to see this is to observe,

$$\ln(1 + x) = \int_0^x \frac{1}{1 + t} dt = \int_0^x \frac{1}{1 - (-t)} dt.$$

Using the Taylor series expansion,

$$\frac{1 - u}{1 + u} = \sum_{k=0}^{\infty} u^k,$$

with $u = (-t)$ gives,

$$\ln(1 + x) = \int_0^x \sum_{k=0}^{\infty} (-1)^k t^k dt.$$

Integrating term-by-term,

$$\ln(1 + x) = \sum_{k=0}^{\infty} (-1)^k \int_0^x t^k dt = \sum_{k=0}^{\infty} (-1)^k \frac{t^{k+1}}{k+1}.$$

Finally, substituting $n = k + 1$ gives the result.

Solution to 9.2 The Taylor series expansion is,

$$\ln(1+x) = \ln(1+a) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n(1+a)^n} (x-a)^n.$$

The easiest way to see this is to write,

$$\ln(1+x) = \ln(1+a+(x-a)) = \ln((1+a)(1+(x-a)/(1+a))).$$

Because $\ln(AB) = \ln(A) + \ln(B)$, this gives,

$$\ln(1+x) = \ln(1+a) + \ln\left(1 + \frac{x-a}{1+a}\right).$$

Using the expansion,

$$\ln(1+u) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} u^n,$$

with $u = \frac{x-a}{1+a}$ gives the Taylor series expansion.

Solution to 9.3 The Taylor series expansion is,

$$\sin(x) = \sum_{m=0}^{\infty} \frac{(-1)^m \cos(a)}{(2m+1)!} (x-a)^{2m+1} + \sum_{m=0}^{\infty} \frac{(-1)^m \sin(a)}{(2m)!} (x-a)^{2m}.$$

The easiest way to see this is to use the angle-addition formulas,

$$\sin(x) = \sin(a+(x-a)) = \cos(a) \sin(x-a) + \sin(a) \cos(x-a),$$

together with the Taylor series expansions,

$$\sin(u) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} u^{2m+1},$$

$$\cos(u) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} u^{2m},$$

with the substitution $u = x - a$.

Solution to 9.4 The Taylor series expansion is,

$$\cos(x) = \sum_{m=0}^{\infty} \frac{(-1)^{m+1} \sin(a)}{(2m+1)!} (x-a)^{2m+1} + \sum_{m=0}^{\infty} \frac{(-1)^m \cos(a)}{(2m)!} (x-a)^{2m}.$$

This follows similarly to **Solution to 9.3**.

Solution to 9.5 The Taylor series expansion is,

$$\tan^{-1}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1} x^{2m+1}.$$

The easiest way to see this is to write,

$$\tan^{-1}(x) = \int_0^x \frac{1}{1+t^2} dt = \int_0^x \sum_{k=0}^{\infty} (-1)^k t^{2k} dt = \sum_{k=0}^{\infty} (-1)^k \int t^{2k} dt.$$

Solution to 9.6 The Taylor series expansion is,

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2(2n)!} x^{2n+1}.$$

This is similar to **Solution to 9.5**.