MEASURE AND INTEGRATION: LECTURE 19

Product spaces in $\mathbb{R}^n$.

Proposition 0.1. *Let $\mathbb{R}^n = \mathbb{R}^\ell \times \mathbb{R}^m$. Let $X \subset \mathbb{R}^\ell$ be $\mathcal{L}^\ell$-measurable and $Y \subset \mathbb{R}^m$ be $\mathcal{L}^m$-measurable. Then $X \times Y \subset \mathbb{R}^n$ is $\mathcal{L}^n$-measurable, and $\lambda(X \times Y) = \lambda(X)\lambda(Y)$.*

Proof. If $X \times Y \in \mathcal{L}^n$, then by Fubini I,

$$\begin{aligned}\lambda(X \times Y) &= \int_{\mathbb{R}^n} \chi_{(X\times Y)}\ dz = \int_{\mathbb{R}^n} \chi_X \chi_Y\ dz \\ &= \int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^\ell} \chi_X \chi_Y dx \right) dy = \lambda(X)\lambda(Y).\end{aligned}$$

Just NTS $X \times Y \in \mathcal{L}^n$.

We may assume X and Y have finite measure. Let $X = \cup_{k=1}^\infty X_k$ and $Y = \cup_{k=1}^\infty Y_k$, where $X_k = X \cap B(0,k)$ and $Y_k = Y \cap B(0,k)$. Then

$$X \times Y = \bigcup_{j,k}^{\infty} X_k \times Y_k.$$

So if $X_k \times Y_k \in \mathcal{L}^n$, since L^n is a σ-algebra, then $X \times Y \in \mathcal{L}^n$.

Now, given $\epsilon > 0$, there exists $K_1 \subset X \subset G_1$ and $K_2 \subset Y \subset G_2$, with $K_1 \subset \mathbb{R}^\ell$ and $K_2 \subset \mathbb{R}^m$ compact, $G_1 \subset \mathbb{R}^\ell$ and $G_2 \subset \mathbb{R}^m$ open, such that $\lambda^\ell(G_1 \setminus K_1) < \epsilon$ and $\lambda^m(G_2 \setminus K_2) < \epsilon$. We have $K_1 \times K_2 \subset \mathbb{R}^n$ is compact, $G_1 \times G_2 \subset \mathbb{R}^n$ is open, and $K_1 \times K_2 \subset X \times Y \subset G_1 \times G_2$.

Now

$$\begin{aligned}G_1 \times G_2 \setminus K_1 \times K_2 &= ((G_1 \setminus K_1) \times G_2) \cup (K_1 \times (G_2 \setminus K_2)) \\ &\subset ((G_1 \setminus K_1) \times G_2) \cup (G_1 \times (G_2 \setminus K_2)).\end{aligned}$$

Thus,

$$\begin{aligned}\lambda(G_1 \times G_2 \setminus K_1 \times K_2) &= \lambda(G_1 \setminus K_1)\lambda(G_2) + \lambda(G_1)\lambda(G_2 \setminus K_2) \\ &\leq \epsilon\lambda(G_2) + \epsilon\lambda(G_1) \\ &< \epsilon(\lambda(K_2) + \epsilon) + \epsilon(\lambda(K_1) + \epsilon) \\ &\leq \epsilon(\lambda(X) + \lambda(Y) + 2\epsilon).\end{aligned}$$

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Hence $\lambda(G_1 \times G_2 \setminus K_1 \times K_2)$ can be made arbitrarily small. By the approximation theorem, $X \times Y$ is $\mathcal{L}^n$-measurable. $\square$

General product spaces. Let $(X, \mathcal{M}_X, \mu_X)$ and $(Y, \mathcal{M}_Y, \mu_Y)$ be measure spaces. What is a measure on $X \times Y$? Define $\mathcal{M}_X \times \mathcal{M}_Y$ to be the smallest σ-algebra containing measurable rectangles (i.e., $A \times B$ with $A \in \mathcal{M}_X$ and $B \in \mathcal{M}_Y$).

Proposition 0.2. *If $E \in \mathcal{M}_X \times \mathcal{M}_Y$, then E_y (the x-section of E at y) is in $\mathcal{M}_X$ for all $y \in Y$.*

Proof. Let Ω be the class of all $E \in \mathcal{M}_X \times \mathcal{M}_Y$ such that $E_y \in \mathcal{M}_X$ for every $y \in Y$. If $E = A \times B$, then clearly $E \in \Omega$. Then Ω is a σ-algebra: (a) $X \times Y \in \Omega$, (b) $E \in \Omega$, then $(E^c)_y = (E_y)^c \in \mathcal{M}_X$ since $\mathcal{M}_X$ is a σ-algebra. (c) If $E_i \in \Omega$, then $(\cup_{i=1}^{\infty} E_i)_y = \cup_{i=1}^{\infty} (E_i)_y \in \mathcal{M}_X$ since M_X is a σ-algebra. $\square$

For $\mathbb{R}^n = \mathbb{R}^\ell \times \mathbb{R}^m$, it is not true that $\mathcal{L}^n$ is the product measure (but it is the completion of the product measure). How do we define $\mu_{X\times Y}$?

Proposition 0.3. *If $E \in \mathcal{M}_X \times \mathcal{M}_Y$, then $E_y \in \mathcal{M}_X$ for all y and $\lambda(E_y)$ is a measurable function on Y.*

Define $\lambda_{X\times Y}(E) = \int_Y \lambda(E_y)\, d\mu_Y$. If X and Y are σ-finite (countable unions of sets with finite measure), then this also equals $\int_X \lambda(E_X)\, d\mu_X$.

Fubini's theorem. Let $(X, \mathcal{M}_X, \mu_X)$ and $(Y, \mathcal{M}_Y, \mu_Y)$ be σ-finite measure spaces and $f = \mathcal{M}_X \times \mathcal{M}_Y$ measurable. Then, for each $y \in Y$, f_y is $\mathcal{M}_X$-measurable, and for each $x \in X$, f_x is $\mathcal{M}_Y$-measurable.

(a) Let $0 \le f \le \infty$,

$$\varphi(x) = \int_Y f_x\, d\mu_Y, \quad \psi(y) = \int_X f_y\, d\mu_X.$$

Then φ is $\mathcal{M}_X$-measurable and ψ is M_Y-measurable, and

$$\int_X \varphi\, d\mu_X = \int_{X\times Y} f\, d(\mu_X \times \mu_Y) = \int_Y \psi\, d\mu_Y.$$

(b) Let $f\colon X \times Y \to \mathbb{C}$. If $f \in L^1(\mu_X \times \mu_Y)$, then $f_X \in L^1(\mu_Y)$ for a.e. $x \in X$ and $f_Y \in L^1(\mu_X)$ for a.e. $y \in Y$, and the above holds ($\varphi \in L^1(\mu_X)$ and $\psi \in L^1(\mu_Y)$).

If μ_X and μ_Y are complete and use $\overline{\mu_X \times \mu_Y}$ (the completion of $\mu_X \times \mu_Y$, then the only change is f_Y is $\mathcal{M}_X$-measurable for a.e. y and f_X is $\mathcal{M}_Y$-measurable for a.e. x.

Proposition 0.4. *Let $f\colon X \times Y \to \mathbb{C}$ is $\mathcal{M}_X \times \mathcal{M}_Y$-measurable. Then*

(a) *for every* $x \in X$, $f_x \colon Y \to \mathbb{C}$ *is* $\mathcal{M}_Y$*-measurable,*
(b) *for every* $y \in Y$, $f_y \colon X \to \mathbb{C}$ *is* $\mathcal{M}_X$*-measurable.*

Proof. If V is open, let $Q = f^{-1}(V)$, $Q \in \mathcal{M}_X \times \mathcal{M}_Y$. We have $Q_x = \{y \mid f_x(y) \in V\} = f_x^{-1}(V) \in \mathcal{M}_Y$ from earlier. □

Theorem 0.5. *Let* $\mathbb{R}^n = \mathbb{R}^\ell \times \mathbb{R}^m$. *Then* $\mathcal{L}^n$ *is the completion of* $L^\ell \times \mathcal{L}^m$.