18.465 PS5 due Tuesday, April 5, 2005

1. Let P put probability 1/3 each at the vertices of an equilateral triangle in the plane. Show that the spatial median of P is at the center (centroid) of the triangle. *Hint*: by Euclidean transformations we can assume that the triangle has vertices $V_1 = (-1, 0)$, $V_2 = (1, 0)$, and $V_3 = (0, \sqrt{3})$, so that the centroid is at $(0, 1/\sqrt{3})$. If a point (x, y) is a spatial median, show by symmetry that $(-x, y)$ is also one, and then by convexity that $(0, y)$ is, in fact by strict convexity x must equal 0, so the spatial median is on one of the perpendicular bisectors of the triangle.

2. Let P put mass 1/3 at each vertex V_i of an obtuse triangle, where the angle at the obtuse vertex V_2 is more than 120°. Then show that the spatial median equals V_2. *Hint*: By the proof in the handout it's enough to show that it gives a local minimum.

3. A function f defined for samples $Z_1, ..., Z_n$ of points in the plane $\mathbb{R}^2$ will be called an *affinely equivariant location estimator* if for any non-singular transformation (2×2 matrix) A and $v \in \mathbb{R}^2$ we have $f(AZ_1 + v, ..., AZ_n + V) = Af(Z_1, ..., Z_n) + v$. Show that the spatial median is not affinely equivariant. *Hint*: Consider the examples in problems 1 and 2, and triangles with vertices at $(-1, 0)$, $(0, y)$, and $(1, 0)$ as y varies.

4. Let $P = N(\mu, \sigma^2)$ and $Q = N(\nu, \sigma^2)$ be two normal distributions on the line with the same variance. Evaluate the Kullback-Leibler distance $I(P, Q)$ (defined just before Theorem 3.3.15) as a function of $|\mu - \nu|$.

5. Problem 1 of the "M-estimators and their consistency" handout.