For $f \in F \subseteq [-1,1]^n$, define $R(f) = \frac{1}{n}\sum_{i=1}^n \varepsilon_i f_i$. Let $d(f,g) := \left(\frac{1}{n}\sum_{i=1}^n (f_i - g_i)^2\right)^{1/2}$.

Theorem 14.1.

$$\mathbb{P}\left(\forall f \in F, R(f) \le \frac{2^{9/2}}{\sqrt{n}} \int_0^{d(0,f)} \log^{1/2} \mathcal{D}(F,\varepsilon,d) d\varepsilon + 2^{7/2} d(0,f)\sqrt{\frac{u}{n}}\right) \ge 1 - e^{-u}$$

for anyu > 0.

Proof. Without loss of generality, assume $0 \in F$.

*Kolmogorov's chaining technique*define a sequence of subsets

$$\{0\} = F_0 \subseteq F_1 \ldots \subseteq F_j \subseteq \ldots \subseteq F$$

where F_j is defined such that

(1) $\forall f, g \in F_j, d(f,g) > 2^{-j}$

(2) $\forall f \in F$, we can find $g \in F_j$ such that $d(f,g) \le 2^{-j}$

How to construct F_{j+1} if we have F_j:

- $F_{j+1} := F_j$
- Find $f \in F$, $d(f,g) > 2^{-(j+1)}$ for all $g \in F_{j+1}$
- Repeat until you cannot find such f

Define projection $\pi_j : F \mapsto F_j$ as follows: for $f \in F$ find $g \in F_j$ with $d(f,g) \le 2^{-j}$ and set $\pi_j(f) = g$.

For any $f \in F$,

$$\begin{aligned} f &= \pi_0(f) + (\pi_1(f) - \pi_0(f)) + (\pi_2(f) - \pi_1(f)) \ldots \\ &= \sum_{j=1}^{\infty} (\pi_j(f) - \pi_{j-1}(f)) \end{aligned}$$

Moreover,

$$\begin{aligned} d(\pi_{j-1}(f), \pi_j(f)) &\le d(\pi_{j-1}(f), f) + d(f, \pi_j(f)) \\ &\le 2^{-(j-1)} + 2^{-j} = 3 \cdot 2^{-j} \le 2^{-j+2} \end{aligned}$$

Define the links

$$L_{j-1,j} = \{f - g : f \in F_j, g \in F_{j-1}, d(f,g) \le 2^{-j+2}\}.$$

Since R is linear, $R(f) = \sum_{j=1}^{\infty} R(\pi_j(f) - \pi_{j-1}(f))$. We first show how to control R on the links. Assume $\ell \in L_{j-1,j}$. Then by Hoeffding's inequality

$$\begin{aligned}\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n}\varepsilon_i \ell_i \geq t\right) &\leq \exp\left(-\frac{t^2}{2\sum \frac{1}{n^2}\ell_i^2}\right)\\ &= \exp\left(-\frac{nt^2}{2\frac{1}{n}\sum_{i=1}^{n}\ell_i^2}\right)\\ &\leq \exp\left(-\frac{nt^2}{2\cdot 2^{-2j+4}}\right)\end{aligned}$$

Note that

$$\mathrm{card} L_{j-1,j} \leq \mathrm{card} F_{j-1} \cdot \mathrm{card} F_j \leq (\mathrm{card} F_j)^2.$$

$$\begin{aligned}\mathbb{P}\left(\forall \ell \in L_{j-1,j}, R(\ell) = \frac{1}{n}\sum_{i=1}^{n}\varepsilon_i \ell_i \leq t\right) &\geq 1 - (\mathrm{card} F_j)^2 e^{-\frac{nt^2}{2\cdot 2^{-2j+5}}}\\ &= 1 - \frac{1}{(\mathrm{card} F_j)^2} e^{-u}\end{aligned}$$

after changing the variable such that

$$t = \sqrt{\frac{2^{-2j+5}}{n}\left(4\log(\mathrm{card} F_j) + u\right)} \leq \sqrt{\frac{2^{-2j+5}}{n} 4\log(\mathrm{card} F_j)} + \sqrt{\frac{2^{-2j+5}}{n} u}.$$

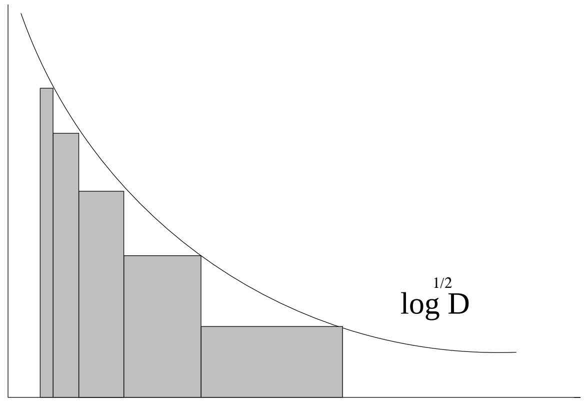
Hence,

$$\mathbb{P}\left(\forall \ell \in L_{j-1,j}, R(\ell) \leq \frac{2^{7/2}2^{-j}}{\sqrt{n}}\log^{1/2}(\mathrm{card} F_j) + 2^{5/2}2^{-j}\sqrt{\frac{u}{n}}\right) \geq 1 - \frac{1}{(\mathrm{card} F_j)^2} e^{-u}.$$

If $F_{j-1} = F_j$ then by definition $\pi_{j-1}(f) = \pi_f$ and $L_{j-1,j} = \{0\}$.

By union bound for all steps,

$$\begin{aligned}&\mathbb{P}\left(\forall j \geq 1, \forall \ell \in L_{j-1,j}, R(\ell) \leq \frac{2^{7/2}2^{-j}}{\sqrt{n}}\log^{1/2}(\mathrm{card} F_j) + 2^{5/2}2^{-j}\sqrt{\frac{u}{n}}\right)\\ &\geq 1 - \sum_{j=1}^{\infty}\frac{1}{(\mathrm{card} F_j)^2} e^{-u}\\ &\geq 1 - \left(\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2}\right) e^{-u}\\ &= 1 - (\pi^2/6 - 1)e^{-u} \geq 1 - e^{-u}\end{aligned}$$



Recall that $R(f) = \sum_{j=1}^{\infty} R(\pi_j(f) - \pi_{j-1}(f))$. If f is close to 0, $-2^{k+1} < d(0,f) \le 2^{-k}$. Find such a k. Then $\pi_0(f) = \ldots = \pi_k(f) = 0$ and so

$$\begin{aligned}
R(f) &= \sum_{j=k+1}^{\infty} R(\pi_j(f) - \pi_{j-1}(f)) \\
&\le \sum_{j=k+1}^{\infty} \left(\frac{2^{7/2}}{\sqrt{n}} 2^{-j} \log^{1/2}(\text{card} F_j) + 2^{5/2} 2^{-j} \sqrt{\frac{u}{n}} \right) \\
&\le \sum_{j=k+1}^{\infty} \left(\frac{2^{7/2}}{\sqrt{n}} 2^{-j} \log^{1/2} \mathcal{D}(F, 2^{-j}, d) \right) + 2^{5/2} 2^{-k} \sqrt{\frac{u}{n}}
\end{aligned}$$

Note that $2^{-k} < 2d(f,0)$, so

$$2^{5/2} 2^{-k} < 2^{7/2} d(f,0).$$

Furthermore,

$$\begin{aligned}
\frac{2^{9/2}}{\sqrt{n}} \sum_{j=k+1}^{\infty} \left(2^{-(j+1)} \log^{1/2} \mathcal{D}(F, 2^{-j}, d) \right) &\le \frac{2^{9/2}}{\sqrt{n}} \int_0^{2^{-(k+1)}} \log^{1/2} \mathcal{D}(F, \varepsilon, d) d\varepsilon \\
&\le \frac{2^{9/2}}{\sqrt{n}} \underbrace{\int_0^{d(0,f)} \log^{1/2} \mathcal{D}(F, \varepsilon, d) d\varepsilon}_{\text{Dudley's entropy integral}}
\end{aligned}$$

since $2^{-(k+1)} < d(0,f)$.

□