Let $x \in \mathcal{X}^n$. Suppose $A_1, A_2 \subseteq \mathcal{X}^n$. We want to define $d(A_1, A_2, x)$.

Definition 29.1.

$$d(A_1, A_2, x) = \inf\{\mathit{card}\ \{i \le n : x_i \ne y_i^1 \text{ and } x_i \ne y_i^2\}, y^1 \in A_1, y^2 \in A_2\}$$

Theorem 29.1.

$$\mathbb{E}2^{d(A_1,A_2,x)} = \int 2^{d(A_1,A_2,x)} dP^n(x) \le \frac{1}{P^n(A_1)P^n(A_2)}$$

and

$$\mathbb{P}\left(d(A_1, A_2, x) \ge t\right) \le \frac{1}{P^n(A_1)P^n(A_2)} \cdot 2^{-t}$$

We first prove the following lemma:

Lemma 29.1. *Let* $0 \le g_1, g_2 \le 1$, $g_i : \mathcal{X} \mapsto [0,1]$. *Then*

$$\int \min\left(2, \frac{1}{g_1(x)}, \frac{1}{g_2(x)}\right) dP(x) \cdot \int g_1(x) dP(x) \cdot \int g_2(x) dP(x) \le 1$$

Proof. Notice that $\log x \le x - 1$.

So enough to show

$$\int \min\left(2, \frac{1}{g_1}, \frac{1}{g_2}\right) dP + \int g_1 dP + \int g_2 dP \le 3$$

which is the same as

$$\int \left[\min\left(2, \frac{1}{g_1}, \frac{1}{g_2}\right) + g_1 + g_2\right] dP \le 3$$

It's enough to show

$$\min\left(2, \frac{1}{g_1}, \frac{1}{g_2}\right) + g_1 + g_2 \le 3.$$

If min is equal to 2, then $g_1, g_2 \le \frac{1}{2}$ and the sum is less than 3.

If min is equal to $\frac{1}{g_1}$, then $g_1 \ge \frac{1}{2}$ and $g_1 \ge g_2$, so $\min + g_1 + g_2 \le \frac{1}{g_1} + 2g_1 \le 3$. □

We now prove the Theorem:

Proof. Proof by induction on n.

$\mathbf{n = 1}$:

$$d(A_1, A_2, x) = 0 \text{ if } x \in A_1 \cup A_2 \text{ and } d(A_1, A_2, x) = 1 \text{ otherwise}$$

$$\begin{aligned}\int 2^{d(A_1,A_2,x)} dP(x) &= \int \min\left(2, \frac{1}{I(x\in A_1)}, \frac{1}{I(x\in A_2)}\right) dP(x) \\ &\le \frac{1}{\int I(x\in A_1)dP(x)\cdot\int I(x\in A_2)dP(x)} \\ &= \frac{1}{P(A_1)P(A_2)}\end{aligned}$$

$\mathbf{n \to n+1}$:

Let $x \in \mathcal{X}^{n+1}$, $A_1, A_2 \subseteq \mathcal{X}^{n+1}$. Denote $x = (x_1, \ldots, x_n, x_{n+1}) = (z, x_{n+1})$.

Define

$$A_1(x_{n+1}) = \{z \in \mathcal{X}^n : (z, x_{n+1}) \in A_1\}$$

$$A_2(x_{n+1}) = \{z \in \mathcal{X}^n : (z, x_{n+1}) \in A_2\}$$

and

$$B_1 = \bigcup_{x_{n+1}} A_1(x_{n+1}), \quad B_2 = \bigcup_{x_{n+1}} A_2(x_{n+1})$$

Then

$$d(A_1, A_2, x) = d(A_1, A_2, (z, x_{n+1})) \le 1 + d(B_1, B_2, z),$$

$$d(A_1, A_2, (z, x_{n+1})) \le d(A_1(x_{n+1}), B_2, z),$$

and

$$d(A_1, A_2, (z, x_{n+1})) \le d(B_1, A_2(x_{n+1}), z).$$

Now,

$$\int 2^{d(A_1,A_2,x)} dP^{n+1}(z, x_{n+1}) = \int \underbrace{\int 2^{d(A_1,A_2,(z,x_{n+1}))} dP^n(z)}_{I(x_{n+1})} dP(x_{n+1})$$

The inner integral ca ne bounded by induction as follows

$$\begin{aligned}I(x_{n+1}) &\le \int 2^{1+d(B_1,B_2,z)} dP^n(z) \\ &= 2\int 2^{d(B_1,B_2,z)} dP^n(z) \\ &\le 2\cdot\frac{1}{P^n(B_1)P^n(B_2)}\end{aligned}$$

Moreover, by induction,

$$I(x_{n+1}) \leq \int 2^{d(A_1(x_{n+1}),B_2,z)} dP^n(z) \leq \frac{1}{P^n(A_1(x_{n+1}))P^n(B_2)}$$

and

$$I(x_{n+1}) \leq \int 2^{d(B_1,A_2(x_{n+1}),z)} dP^n(z) \leq \frac{1}{P^n(B_1)P^n(A_2(x_{n+1}))}$$

Hence,

$$\begin{aligned} I(x_{n+1}) &\leq \min\left(\frac{2}{P^n(B_1)P^n(B_2)}, \frac{1}{P^n(A_1(x_{n+1}))P^n(B_2)}, \frac{1}{P^n(B_1)P^n(A_2(x_{n+1}))}\right) \\ &= \frac{1}{P^n(B_1)P^n(B_2)} \min\left(2, \underbrace{\frac{1}{P^n(A_1(x_{n+1})/P^n(B_1)}}_{1/g_1(x_{n+1})}, \underbrace{\frac{1}{P^n(A_2(x_{n+1})/P^n(B_2)}}_{1/g_2(x_{n+1})}\right) \end{aligned}$$

So,

$$\begin{aligned} \int I(x_{n+1}) dP(x_{n+1}) &\leq \frac{1}{P^n(B_1)P^n(B_2)} \int \min\left(2, \frac{1}{g_1}, \frac{1}{g_2}\right) \; dP \\ &\leq \frac{1}{P^n(B_1)P^n(B_2)} \cdot \frac{1}{\int g_1 dP \cdot \int g_2 dP} \\ &= \frac{1}{P^n(B_1)P^n(B_2)} \cdot \frac{1}{P^{n+1}(A_1)/P^n(B_1) \cdot P^{n+1}(A_2)/P^n(B_2)} \\ &= \frac{1}{P^{n+1}(A_1)P^{n+1}(A_2)} \end{aligned}$$

because $\int P^n(A_1(x_{n+1}))dP(x_{n+1}) = P^{n+1}(A_1)$. □