CHAPTER III

A. Solvable and Nilpotent Lie Algebras

A.1. Consider the derived algebra and observe, as a consequence of Cor. 6.3, Chapter II that a semisimple Lie algebra equals its derived algebra.

A.2. A direct computation shows $[\mathfrak{t}(n), \mathfrak{t}(n)] = \mathfrak{n}(n)$ and also that center $(\mathfrak{t}(n)) = \boldsymbol{R}(E_{11} + \dots + E_{nn})$ and center $(\mathfrak{n}(n)) = \boldsymbol{R}E_{1n}$. By Theorem 2.4 (i), $\mathfrak{n}(n)$ is nilpotent, thus by Cor. 2.6 solvable, whence $\mathfrak{t}(n)$ is solvable. Thus (i) and (ii) are proved. For (iii) we have

$$B(X, [Y, Z]) = \mathrm{Tr}(\mathrm{ad}\, X\, \mathrm{ad}\, Y\, \mathrm{ad}\, Z - \mathrm{ad}\, X\, \mathrm{ad}\, Z\, \mathrm{ad}\, Y) = 0$$

because ad X, ad Y, and ad Z can, on the complexification, be expressed by upper triangular matrices and thereby the two matrix products on the right have the same diagonal elements.

A.3. We indicate a proof of this except for the second implication $\Leftarrow$, for which see e.g. Bourbaki [2], I, §5. If $\mathfrak{D}\mathfrak{g}$ is nilpotent, then it is solvable (Cor. 2.6); so, by definition, $\mathfrak{g}$ is solvable. Conversely, if $\mathfrak{g}$ is solvable, For a fixed j the summand above equals

$$\epsilon_{ij}(-1)^{i-1}\omega(\widetilde{X}_1, \ldots, \widetilde{X}_{i-1}, [\widetilde{X}_i, \widetilde{X}_j], \widetilde{X}_{i+1}, \ldots, \widehat{X}_j, \ldots, \widetilde{X}_{p+1}).$$

Here $\epsilon_{ij}(-1)^{i-1}$ is independent of i so summation over i gives O because of equation (2) for $X = X_j$ (Chevalley-Eilenberg [1]).

Page 147, line 13 from below: Here we give some details of general interest which also yield the construction of G on the basis of the Levi decomposition $\mathfrak{g} = \mathfrak{r} + \mathfrak{s}$.

Theorem 8.1 *Let G and H be Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively. Assume G simply connected. If $\sigma : \mathfrak{g} \to \mathfrak{h}$ is a homomorphism there exists a homomorphism $S : G \to H$ such that $dS = \sigma$.*

For the proof (cf. Chevalley [2], Ch. IV) consider the product $G \times H$ and the two projections $p_1 : G \times H \to G$, $p_2 : G \times H \to H$ given by $p_1(g, h) = g$, $p_2(g, h) = h$, with differentials $dp_1(X, Y) = X$, $dp_2(X, Y) = Y$. Let $\mathfrak{k}$ denote the graph $\{(X, \sigma X) : X \in \mathfrak{g}\}$ and K the corresponding analytic subgroup of $G \times H$. The restriction $\varphi = p_1|K$ has differential $d\varphi = dp_1|\mathfrak{k}$ which is the isomorphism $(X, \sigma X) \to X$ of $\mathfrak{k}$ onto $\mathfrak{g}$. Thus $\varphi : K \to G$ is surjective with a discrete kernel so is a covering. Hence φ is an isomorphism and the homomorphism $S = p_2 \circ \varphi^{-1}$ of G into H has differential σ.

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$, G^* its universal covering group so $G = G^*/C$, where C is a discrete central subgroup. The group Aut (G^*) of analytic automorphisms of G^* is naturally identified with Aut $(\mathfrak{g})$ and is thus a Lie group. Then Aut (G) is identified with the closed subgroup preserving C so is also a Lie group. In fact, if $\sigma \in$ Aut (G) there exists an automorphism θ of G^* such that $d\theta = d\pi^{-1} \circ d\sigma \circ d\pi$, where π denotes the covering map of G^* onto G. Then $\pi\theta = \sigma\pi$ so θ maps C into itself.

If A and B are abstract groups and $b \to \sigma_b$ a homomorphism of B into Aut (A) the semi-direct product $A \times_\sigma B$ is the group defined by the product

$$(a, b)(a', b') = (a\sigma_b(a'), bb')$$

on $A \times B$. This is indeed a group containing A as a normal subgroup.

Proposition 8.2 *Suppose A and B are connected Lie groups, σ an analytic homomorphism of B into* Aut (A). *Let $\mathfrak{a}$ and $\mathfrak{b}$ denote their respective Lie algebras. Then the group $G = A \times_\sigma B$ has Lie algebra*

$$\mathfrak{g} = \mathfrak{a} + \mathfrak{b}$$

with the bracket relation

$$[X + Y, X' + Y'] = [X, X'] + d\psi(Y)(X') - d\psi(Y')(X) + [Y, Y'],$$

where $X, X' \in \mathfrak{a}$, $Y, Y' \in \mathfrak{b}$ and ψ is the map $b \to d\sigma_b$ of B into Aut $(\mathfrak{a})$.

Proof Since $\mathfrak{a}$ and $\mathfrak{b}$ are subalgebras of $\mathfrak{g}$ it remains to prove

$$[X,Y] = -d\psi(Y)(X), \quad X \in \mathfrak{a}, Y \in \mathfrak{b}.$$

The differential $d\psi$ is a homomorphism of $\mathfrak{b}$ into $\partial(\mathfrak{a})$, the Lie algebra of derivations of $\mathfrak{a}$. We have

$$d\sigma_{\exp tY} = \psi(\exp tY) = e^{t\,d\psi(Y)}.$$

Hence by the multiplication in $A \times_\sigma B$,

$$\begin{aligned}\exp(-tX)\exp(-tY)\exp(tX)\exp(tY) &= \exp(-tX)\sigma_{\exp(-tY)}(\exp(tX)) \\ = \exp(-tX)\exp(t\,d\sigma_{\exp(-tY)}(X)) &= \exp(-tX)\exp(te^{-t\,d\psi(Y)}).\end{aligned}$$

Expanding this in powers of t we deduce from Lemma 1.8, $[X,Y] = -d\psi(Y)(X)$ as desired.

Lemma 8.3 *If $\mathfrak{g}$ is a solvable Lie algebra then there exists a Lie group G with Lie algebra $\mathfrak{g}$.*

Proof. This is proved by induction on $\dim\mathfrak{g}$. If $\dim\mathfrak{g} = 1$ we take $G = \mathbf{R}$. If $\dim\mathfrak{g} > 1$ then $\mathfrak{D}\mathfrak{g} \neq \mathfrak{g}$ so there exists a subspace $\mathfrak{h}$ such that $\mathfrak{D}\mathfrak{g} \subset \mathfrak{h}$ and $\dim\mathfrak{h} = \dim\mathfrak{g} - 1$. Let $X \in \mathfrak{g}$, $X \notin \mathfrak{h}$. Then $[\mathfrak{h}, \mathfrak{g}] \subset \mathfrak{D}\mathfrak{g} \subset \mathfrak{h}$ so $\mathfrak{h}$ is an ideal in $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{h} + \mathbf{R}X$. Let by induction H be a simply connected Lie group with Lie algebra $\mathfrak{h}$ and A a Lie group with Lie algebra $\mathbf{R}X$. The derivation $Y \to [X,Y]$ of $\mathfrak{h}$ extends to a homomorphism $\sigma : A \to \mathrm{Aut}(\mathfrak{h})$ so by Proposition 8.2, $H \times_\sigma A$ serves as the desired G.

Now let $\mathfrak{g}$ be an arbitrary Lie algebra over $\mathbf{R}$. Assuming the Levi decomposition $\mathfrak{g} = \mathfrak{r} + \mathfrak{s}$, we deduce from the Lemma 8.3, Proposition 8.2 and Corollary 6.5 that $\mathfrak{g}$ is the Lie algebra of a Lie group.

Page 173, line 5 from below: By definition $\alpha(H) = \alpha'(\varphi(H))$ so

$$B(H_\alpha, H_\beta) = \alpha(H_\beta) = \alpha'(\varphi(H_\beta)) = B'(H_{\alpha'}, \varphi(H_\beta)).$$

Combining with $(2')$ we deduce $\varphi(H_\beta) = H_{\beta'}$. Thus $(2')$ becomes $B(H_\alpha, H_\beta) = B'(\varphi(H_\alpha), \varphi(H_\beta))$ and the isometry of φ follows by linearity.

Page 200, line 5 from below: Here it is useful to clarify some simple features of affine transformations. Let M and M' be manifolds with affine connections ∇ and ∇', respectively. A diffeomorphism $\psi : M \to M'$ is said to be an *affine transformation* if $\nabla'_{\psi X} = \psi \circ \nabla_X \circ \psi^{-1}$ on $\mathfrak{D}^1(M')$ for each $X = \mathfrak{D}^1(M)$.

Let $p \in M$ and suppose $X_1, \ldots, X_m$ is a basis (over $C^\infty(N_p)$) of $\mathfrak{D}^1(N_p)$, N_p being a neighborhood of p. Then $X_i' = \psi X_i (1 \leq i \leq m)$ is a basis