Frank
Wagner's book. Simple Theories.
+ various articles.
Nice Paper: From Stability to Simplicity / Kim & Pillay JSL?

Stability = theory of independence + multiplicity.
simplicity
types over models have unique extensions.

Framework of F.O. model theory.

Organisation:
1. Basic Definitions & Framework
2. Simplicity & independence
3. Canonical Bases.
} Basic stuff everyone should know.
4. "Generic Constructions" - adding random predicate etc.
5. "Simple" groups.
6. Lovely Pairs.

Analogue of Monster Model in CAT.

Defn: Let $\mathcal{L}$ be a signature, $\Delta \subseteq \mathcal{L}_{\omega,\omega}$. (set of f.o. formulas).

Assume that Δ is closed under positive boolean combinations. $\vee \wedge$.
Then Δ is a positive fragment of $\mathcal{L}$.

We fix a positive fragment Δ. A "formula" always means a formula from Δ.
* Work in purely relational language or Δ is closed under substitution of terms for variable.
Definition: An $\mathcal{L}$-structure $\mathcal{U}$ is a κ-universal domain (κ is a cardinal) if it satisfies:
1. Homogeneity: If $A \subseteq \mathcal{U}$ & $|A| < \kappa$ and $f: A \to \mathcal{U}$ is a mapping s.t. $\forall \varphi(\bar{x}) \in \Delta$ $\forall \bar{a} \in A$, if $\mathcal{U} \models \varphi(\bar{a}) \Rightarrow \mathcal{U} \models \varphi(f(\bar{a}))$.
[We say that f is a partial Δ-endomorphism of $\mathcal{U}$.]
Then $\exists \tilde{f} \in \mathrm{Aut}(\mathcal{U})$ which extends f.

② Compactness: If $\Sigma(\bar{x})$ is a set of Δ-formulas where $\bar{x}$ is a possibly infinite tuple of variables, then either $\Sigma(\bar{x})$ is realised in $\mathcal{U}$ (ie $\exists \bar{a} \in \mathcal{U}$ st $\mathcal{U} \models \Sigma(\bar{a})$) or there is $\Sigma_0 \subseteq \Sigma$ finite which is not realised.

Important

Fact: Assume $\mathcal{U}$ is a universal domain. $\bar{a} \in \mathcal{U}$. $\varphi(\bar{x},\bar{y}) \in \Delta$ and $\mathcal{U} \not\models \exists \bar{y}\, \varphi(\bar{a},\bar{y})$.

Then there is a formula $\psi(\bar{x}) \in \Delta$ st
① $\mathcal{U} \models \psi(\bar{a})$.
② $\mathcal{U} \models \forall \bar{x}(\psi(\bar{x}) \to \neg \exists y\, \varphi(\bar{x},\bar{y}))$.

Proof Set $p(\bar{x}) = tp(\bar{a}) := \{ \chi(\bar{x}) \in \Delta : \mathcal{U} \models \chi(\bar{a}) \}$
Then $p(\bar{x}) \cup \varphi(\bar{x},\bar{y})$ is not realised in $\mathcal{U}$. by homogeneity.

(otherwise we'd get $\bar{c},\bar{d} \in \mathcal{U}$ st. $p(\bar{c}) \wedge \varphi(\bar{c},\bar{d})$.
so $f : \bar{a} \mapsto \bar{c}$ is a partial endomorphism extending to an automorphism $\tilde{f}$ and then $\mathcal{U} \models \varphi(\bar{a}, \tilde{f}^{-1}(\bar{d}))$.)

by compactness: $\exists\, \chi(\bar{x}) \in p(\bar{x})$ st $\chi(\bar{x}) \wedge \varphi(\bar{x},\bar{y})$ is not realised in $\mathcal{U}$. □

Remark: If $\mathcal{U}$ is a universal domain wrt Δ and $\Delta' = \exists\Delta := \{ \exists \bar{y}\, \varphi(\bar{x},\bar{y}) : \varphi \in \Delta \}$.
Then $\mathcal{U}$ is a universal domain wrt Δ'.

Proof Homogeneity becomes easier ✓
Compactness: by replacing each $\exists \bar{y}\, \varphi(\bar{x},\bar{y})$ with $\varphi(\bar{x},\bar{y}_\varphi)$ □
new variables.

Therefore we allow ourselves the following additional assumption:

Convention: For every $\varphi(\bar{x},\bar{y}) \in \Delta$, the formula $\exists \bar{y}\, \varphi(\bar{x},\bar{y})$ is equivalent in $\mathcal{U}$ to a partial Δ-type.

We say that $\mathcal{U}$ is a universal domain for $T = Th(\mathcal{U})$
where $Th(\mathcal{U}) = \{ \neg \exists \bar{x} \varphi(\bar{x}) : \varphi \in \Delta, \mathcal{U} \models \ldots \}$.

~~Definition~~ Lemma: $S_n(T) = \{ tp(\bar{a}) : \bar{a} \in \mathcal{U}^n \} = \{$ all maximal sets of Δ-formulas in $(x_0 \cdots x_{n-1}) = \bar{x}$ consistent with $T \}$

Proof ① Let $p(\bar{x}) = tp(\bar{a})$ where $\bar{a} \in \mathcal{U}^n$.
Then $p(\bar{x})$ is consistent with T since $\mathcal{U} \models p(\bar{a}) \cup T$.
~~[illegible]~~
If $\varphi(\bar{x}) \notin p(\bar{x})$ (but $\varphi \in \Delta$). then $\mathcal{U} \not\models \varphi(\bar{a})$.
so by the fact, there is $\psi(\bar{x}) \in p(\bar{x})$ s.t.
$\mathcal{U} \models \underbrace{\neg \exists \bar{x}\, \varphi(\bar{x}) \wedge \psi(\bar{x})}_{\in T}$. (by the fact).

so $p(\bar{x}) \cup \{\varphi(\bar{x})\} \cup T$ is inconsistent.
$\Rightarrow$ p is maximal.

② Assume that $p(\bar{x})$ is maximal consistent with T.
Since it is consistent with T, it is realised in $\mathcal{U}$ (by compactness)

Say by $\bar{a}$: $\mathcal{U} \models p(\bar{a})$.

Then $p(\bar{x}) \subseteq tp(\bar{a})$. and $tp(\bar{a})$ is consistent with T.
So by maximality $p = tp(\bar{a})$. □

From homogeneity, two tuples are the same type $\Leftrightarrow$ they correspond by an automorphism of $\mathcal{U}$.

Therefore $S_\alpha(T) \cong \mathcal{U}^\alpha / Aut(\mathcal{U})$. orbits of action of Aut $\mathcal{U}$ on $\mathcal{U}^\alpha$.
(α: possibly infinite)

The logic topology on $S_\alpha(T)$: If $\varphi(x) \in \Delta$, then $\langle \varphi \rangle \subseteq S_\alpha(T) ::= \{ p(\bar{x}) : \varphi(\bar{x}) \in p \}$

The ~~[illegible]~~ closed sets are generated by the sets of this form.

2/4
Simplicity
Theory
(4)

This is a compact topology.