Recitation 5

Summary of Plate Bending:

	Out-of-plane response	In-plane response
Geometry:	$\kappa_{\alpha\beta} = -w_{,\alpha\beta}$	$\varepsilon^o_{\alpha\beta} = \frac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha})$
Equilibrium:	$M_{\alpha\beta,\alpha\beta} + p = 0$	$N_{\alpha\beta,\beta} = 0$
Constitutive Law:	$M_{\alpha\beta} = D[(1-\nu)\kappa_{\alpha\beta} + \nu\kappa_{jj}\delta_{\alpha\beta}]$	$N_{\alpha\beta} = C[(1-\nu)\varepsilon^o_{\alpha\beta} + \nu\varepsilon^o_{jj}\delta_{\alpha\beta}]$
Governing PDE:	$Dw_{,\alpha\alpha\beta\beta} = p$ $D\nabla^4 w = p$	$(1-\nu)u_{\alpha,\beta\beta} + (1-\nu)u_{\beta,\alpha\beta} = 0$ (2 eqns)

For moderately large deflections: $D\nabla^4 w + N_{\alpha\beta} w_{,\alpha\beta} = p$

D = plate bending regidity $= \dfrac{Eh^3}{12(1-\nu^2)}$

Rectangular coordinates:

$$\nabla^2 w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \tag{5.1}$$

$$\nabla^4 w = \frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \tag{5.2}$$

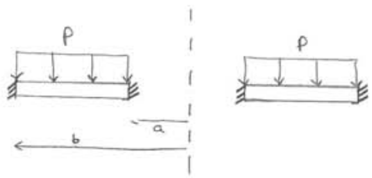
Polar coordinates:

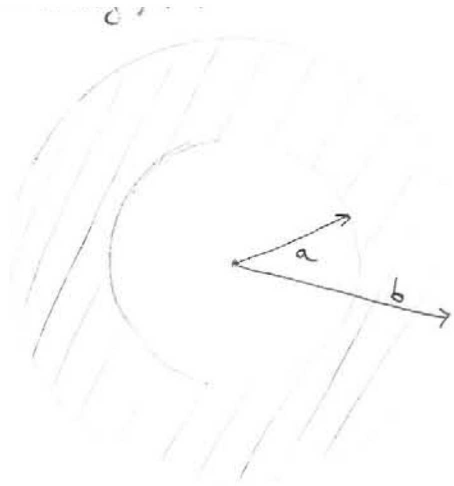
$$\nabla^2 w = \frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} \tag{5.3}$$

$$\nabla^4 w = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)\right)\right) + \frac{2}{r^2}\frac{\partial^4 w}{\partial r^2 \partial \theta^2} + \frac{1}{r^4}\frac{\partial^4 w}{\partial \theta^4} - \frac{2}{r^3}\frac{\partial^3 w}{\partial r \partial \theta^2} + \frac{4}{r^4}\frac{\partial^2 w}{\partial \theta^2} \tag{5.4}$$

Example: Find the displacement, $w(r)$, for an annular plate with uniform pressure loading, P.

Both inner and outer edges are clamped.





General Solution: $w(r) = C_1 \ln r + C_2 r^2 + C_3 r^2 \ln r + C_4 + \frac{Pr^4}{64D}$ (eqn 7.19)

BC's: at $r = a$: $w = w' = 0$
at $r = b$: $w = w' = 0$

$$w' = \frac{\partial w}{\partial r} = \frac{C_1}{r} + 2C_2 r + C_3(2r \ln r + \frac{r^2}{r} + \frac{4Pr^3}{64D})$$

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① $w(a) = 0$: $C_1 \ln a + C_2 a^2 + C_3 a^2 \ln a + C_4 + \frac{Pa^4}{64D} = 0$

② $w'(a) = 0$: $\frac{C_1}{a} + 2C_2 a + C_3(2a \ln a + a) + \frac{Pa^3}{16D} = 0$

③ $w(b) = 0$: $C_1 \ln b + C_2 b^2 + C_3 b^2 \ln b + C_4 + \frac{Pb^4}{64D} = 0$

④ $w'(b) = 0$: $\frac{C_1}{b} + 2C_2 b + C_3(2b \ln b + b) + \frac{Pb^3}{16D} = 0$

4 eons, 4 unknowns $\rightarrow$ solve with MATLAB

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10/17/12 11:14 AM   MATLAB Command Window                              1 of 1

>> syms c1 c2 c3 c4 a b P D
>> F=[log(a) a^2 a^2*log(a) 1;1/a 2*a 2*a*log(a)+a 0;log(b) b^2 b^2*log(b) 1;1/b 2*b
2*b*log(b)+b 0]

F =

[ log(a), a^2,       a^2*log(a), 1]
[    1/a, 2*a, a + 2*a*log(a), 0]
[ log(b), b^2,       b^2*log(b), 1]
[    1/b, 2*b, b + 2*b*log(b), 0]

>> B=[-P*a^4/64/D;-P*a^3/16/D;-P*b^4/64/D;-P*b^3/16/D]

B =

 -(P*a^4)/(64*D)
 -(P*a^3)/(16*D)
 -(P*b^4)/(64*D)
 -(P*b^3)/(16*D)

>> C=F\B

C =

C1
(P*a^2*b^6 - 2*P*a^4*b^4 + P*a^6*b^2 + P*a^2*b^6*log(a) - P*a^6*b^2*log(a) -
P*a^2*b^6*log(b) + P*a^6*b^2*log(b))/(16*D*(a^4 - 4*a^2*b^2*log(a)^2 + 8*a^2*b^2*log(a)
*log(b) - 4*a^2*b^2*log(b)^2 - 2*a^2*b^2 + b^4))

C2
(P*(2*a^6*log(a) - a^6 - 8*a^4*b^2*log(a)*log(b) - 4*a^4*b^2*log(a) + 8*a^4*b^2*log(b)
^2 + 2*a^4*b^2*log(b) + a^4*b^2 + 8*a^2*b^4*log(a)^2 - 8*a^2*b^4*log(a)*log(b) +
2*a^2*b^4*log(a) - 4*a^2*b^4*log(b) + a^2*b^4 + 2*b^6*log(b) - b^6))/(64*D*(a^4 -
4*a^2*b^2*log(a)^2 + 8*a^2*b^2*log(a)*log(b) - 4*a^2*b^2*log(b)^2 - 2*a^2*b^2 + b^4))

C3
-(P*(a^6 + b^6 - a^2*b^4 - a^4*b^2 + 4*a^2*b^4*log(a) - 4*a^4*b^2*log(a) - 4*a^2*b^4*log
(b) + 4*a^4*b^2*log(b)))/(32*D*(a^4 - 4*a^2*b^2*log(a)^2 + 8*a^2*b^2*log(a)*log(b) -
4*a^2*b^2*log(b)^2 - 2*a^2*b^2 + b^4))

C4
-(P*a^2*b*(- 4*a^4*b*log(a)*log(b) + 2*a^4*b*log(a) + 4*a^4*b*log(b)^2 +
2*a^4*b*log(b) - a^4*b - 4*a^2*b^3*log(a) - 4*a^2*b^3*log(b) - 2*a^2*b^3 + 4*b^5*log(a)
^2 - 4*b^5*log(a)*log(b) + 2*b^5*log(a) + 2*b^5*log(b) - b^5))/(64*D*(a^4 -
4*a^2*b^2*log(a)^2 + 8*a^2*b^2*log(a)*log(b) - 4*a^2*b^2*log(b)^2 - 2*a^2*b^2 + b^4))
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To demonstrate the *messy* result...

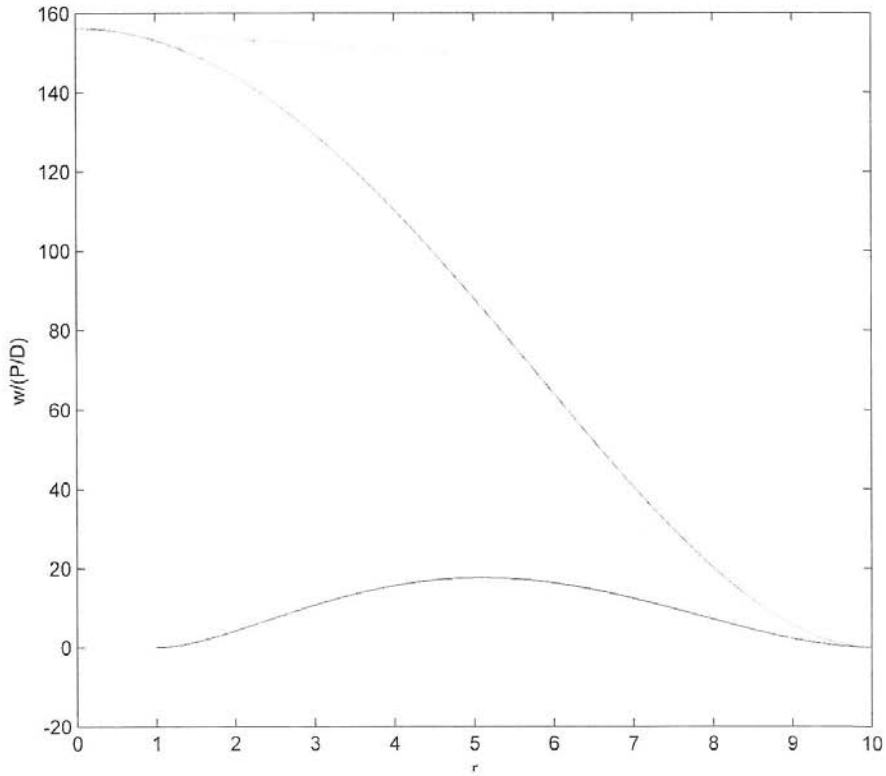
— For illustration, let $a = 1$ and $b = 10$:

$$\begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \frac{P}{D} \begin{bmatrix} -10.8 \\ 7.2 \\ -3.7 \\ -7.2 \end{bmatrix}$$

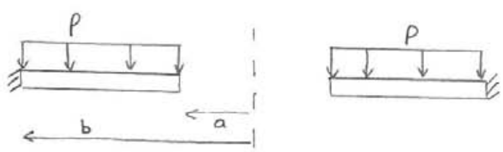
then $w(r) = \dfrac{P}{D}\left(-10.8 \ln r + 7.2r^2 - 3.7r^2 \ln r - 7.2 + \dfrac{r^4}{64}\right)$ (for $1 \leqslant r \leqslant 10$)

— Compare to solid plate with clamped edges, $R = 10$:

$$w(r) = \frac{P}{D}\frac{(10)^4}{64}\left[1 - \left(\frac{r}{10}\right)^2\right]^2 \quad \text{(eqn. 7.24)} \tag{5.5}$$



Suppose the inner edge is Free instead of clamped:



<u>BC's</u>: at $r = b$: $w = w' = 0$
at $r = a$: $M_r = 0$, $V = 0$

$$M_r = D\left[\frac{\partial^2 w}{\partial r^2} + \nu \frac{1}{r}\frac{\partial w}{\partial r}\right] \quad \text{eqn. 7.25} \tag{5.6}$$

$$M_\theta = D\left[\frac{1}{r}\frac{\partial w}{\partial r} + \nu \frac{\partial^2 w}{\partial r^2}\right] \tag{5.7}$$

$$\begin{aligned} V &= \frac{1}{r}\left[\frac{\partial}{\partial r}(rM_r) - M_\theta\right] \\ &= \frac{1}{r}\left(m_r + r\frac{\partial M_r}{\partial r} - M_\theta\right) \\ &= \frac{\partial M_r}{\partial r} + \frac{M_r - M_\theta}{r} \end{aligned} \tag{5.8}$$

Substitute for M_r and M_θ:

$$V = D\left[\frac{\partial^3 w}{\partial r^3} + \nu\left(-\frac{1}{r^2}\right)\frac{\partial w}{\partial r} + \nu\frac{1}{r}\frac{\partial^2 w}{\partial r^2}\right] + \frac{D\left[\frac{\partial^2 w}{\partial r^2} + \nu\frac{1}{r}\frac{\partial w}{\partial r} - \frac{1}{r}\frac{\partial w}{\partial r} - \nu\frac{\partial^2 w}{\partial r^2}\right]}{r} \tag{5.9}$$

$$V = D\left[\frac{\partial^3 w}{\partial r^3} + \frac{1}{r}\frac{\partial^2 w}{\partial r^2} - \frac{1}{r^2}\frac{\partial w}{\partial r}\right] \tag{5.10}$$

General Solution:

$$w(r) = C_1 \ln r + C_2 r^2 + C_3 r^2 \ln r + C_4 + \frac{Pr^4}{64D} \tag{5.11}$$

$$\frac{\partial w}{\partial r} = \frac{C_1}{r} + 2C_2 r + C_3(2r\ln r + r) + \frac{Pr^3}{16D} \tag{5.12}$$

$$\frac{\partial^2 w}{\partial r^2} = -\frac{C_1}{r^2} + 2C_2 + C_3(2\ln r + 2 + 1) + \frac{3Pr^2}{16D} \tag{5.13}$$

$$\frac{\partial^3 w}{\partial r^3} = \frac{2C_1}{r^3} + \frac{2C_3}{r} + \frac{6Pr}{16D} \tag{5.14}$$

Then

$$\begin{aligned} M_r &= D\left[-\frac{C_1}{r^2} + 2C_2 + C_3(2\ln r + 3) + \frac{3Pr^2}{16D} + \frac{\nu}{r}\left(\frac{C_1}{r} + 2C_2 r + C_3(2r\ln r + r) + \frac{Pr^3}{16D}\right)\right] \\ &= D\left[\frac{C_1}{r^2}(\nu - 1) + 2C_2(\nu + 1) + C_3(2\ln r + 3 + 2\nu\ln r + \nu) + \frac{Pr^2}{16D}(3+\nu)\right] \end{aligned} \tag{5.15}$$

and

$$\begin{aligned} V &= D\left[\frac{2C_1}{r^2} + \frac{2C_3}{r} + \frac{3Pr}{8D} - \frac{C_1}{r^3} + \frac{2C_2}{r} + \frac{C_3}{r}(2\ln r + 3) + \frac{3Pr}{16D} - \frac{C_1}{r^3} - \frac{2C_2}{r} - \frac{C_3}{r}(2\ln r + 1) + \frac{Pr}{16d}\right] \\ &= D\left[\frac{4C_3}{r} + \frac{5Pr}{8D}\right] \end{aligned} \tag{5.16}$$

BC's:

① $V(a) = 0 \rightarrow D\left[4\frac{C_3}{a} + \frac{5Pa}{8D}\right] = 0 \rightarrow \boxed{C_3 = -\frac{5Pa^2}{32D}}$

② $M_r(a) = 0$ (sub. $\nu = 0.3$) $\rightarrow D\left[\frac{C_1}{a^2}(.3-1) + 2C_2(.3+1) + C_3(2\ln a + 3 + 2(.3)\ln a + .3) + \frac{Pa^2}{160}(3+.3)\right] = 0$

③ $w(b) = 0 \rightarrow C_1 \ln b + C_2 b^2 + C_3 b^2 \ln b + C_4 + \frac{Pb^4}{64D} = 0$

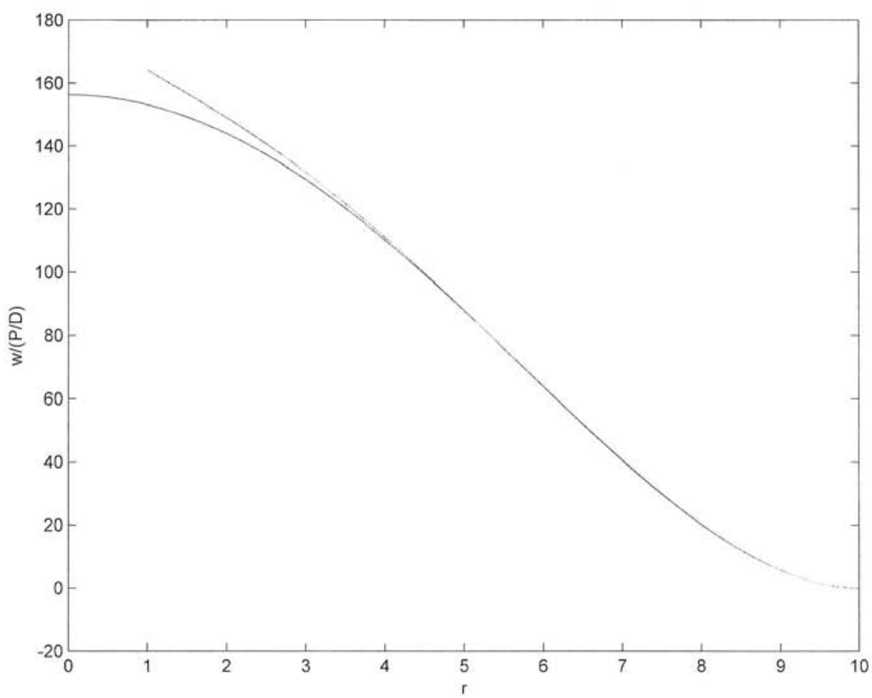
④ $w'(b) = 0 \rightarrow \frac{C_1}{b} + 2C_2 b + C_3(2b\ln b + b) + \frac{Pb^3}{16D} = 0$

$\boxed{\text{4 eqns, 4 C's}}$

For $a = 1$ and $b = 10$:

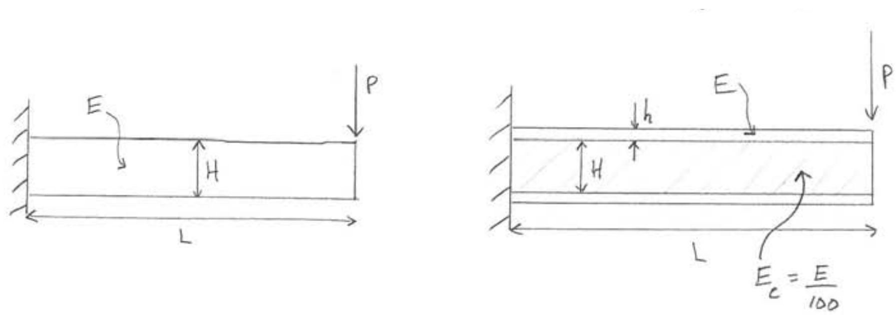
$$\begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \frac{P}{D}\begin{bmatrix} -10.2 \\ -2.6 \\ -0.2 \\ 166.9 \end{bmatrix} \tag{5.17}$$

So $w(r) = \frac{P}{D}\left(-10.2\ln r - 2.6r^2 - 0.2r^2\ln r + 166.9 + \frac{r^4}{64}\right)$



Example: Sandwich Beam

Consider the two cantilever beam configurations shown:



Find h so that the 2 beams have the same stiffness.

For cantilever with end load, $w(L) = \dfrac{PL^3}{3EI}$

Monolithic beam: $I = \dfrac{bH^3}{12} \rightarrow \dfrac{w(L)}{P} = \dfrac{L^3}{3E(\frac{bH^3}{12})} = \dfrac{4L^3}{EbH^3}$

Sandwich beam: $I_c = \dfrac{bH^3}{12}$ (core)

$$I_f = 2\Big[\frac{bh^3}{12} + \underbrace{bh\left(\frac{H}{2}\right)^2}_{\text{parallel axis theorem}}\Big] \text{ (2 face plates)}$$

We expect $h \ll H \rightarrow \dfrac{bh^3}{12} \ll bh\left(\dfrac{H}{2}\right)^2$

So $I_f \simeq \dfrac{bhH^2}{2}$

If $h = CH$, then $I_f = \dfrac{bCH^3}{2}$

Sandwich beam:

$$\begin{aligned} \frac{w(L)}{P} &= \frac{L^3}{3(EI_f + E_cI_c)} \quad (E_c = \frac{E}{100}) \\ &= \frac{L^3}{3E(\frac{bCH^3}{2} + \frac{1}{100}\cdot\frac{bH^3}{12})} \end{aligned} \tag{5.18}$$

For equivalent stiffness,

$$\underbrace{\frac{L^3}{3EbH^3\left(\frac{C}{2}+\frac{1}{1200}\right)}}_{\text{sandwich}} = \underbrace{\frac{4L^3}{EbH^3}}_{\text{monolithic}} \tag{5.19}$$

Solve for C:

$$\frac{1}{3\left(\frac{C}{2}+\frac{1}{1200}\right)} = 4 \tag{5.20}$$

$$\frac{C}{2}+\frac{1}{1200} = \frac{1}{12} \tag{5.21}$$

$$C = 0.165 \tag{5.22}$$

$$\text{So} \quad \boxed{h = 0.165H} \tag{5.23}$$

Verify our assumption that $\frac{bh^3}{12} \ll bh\left(\frac{H}{2}\right)^2$:

$$\frac{\frac{bh^3}{12}}{bh\left(\frac{H}{2}\right)^2} = \frac{\frac{(.165H)^2}{12}}{\frac{H^2}{4}} = 0.009 \simeq \frac{1}{100} \quad \checkmark \tag{5.24}$$

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