

2.171 Quiz 2 Solutions

Problem 1:

$$H(s) = \frac{a^2}{s^2 - a^2}$$

a) Calculate the zero order hold equivalent $H_{eq}(z)$.

$$H_{eq}(z) = \frac{z-1}{z} Z\left\{\frac{G(s)}{s}\right\}$$

$$Z\left\{\frac{G(s)}{s}\right\} = Z\left\{\frac{a^2}{s(s^2 - a^2)}\right\}$$

$$Z\left\{\frac{G(s)}{s}\right\} = Z\left\{\frac{A}{s(s+a)} + \frac{B}{s(s-a)}\right\}$$

$$Z\left\{\frac{G(s)}{s}\right\} = Z\left\{-\frac{1}{2} \frac{a}{s(s+a)} + \frac{1}{2} \frac{a}{s(s-a)}\right\}$$

Using Transform 12 in Table B.2 of FPW, we have:

$$Z\left\{\frac{G(s)}{s}\right\} = -\frac{z(1 - e^{-aT})}{(z-1)(z - e^{-aT})} - \frac{z(1 - e^{aT})}{(z-1)(z - e^{aT})}$$

$$Z\left\{\frac{G(s)}{s}\right\} = -\frac{z(1 - e^{-aT})(z - e^{aT}) + z(1 - e^{aT})(z - e^{-aT})}{(z-1)(z - e^{-aT})(z - e^{aT})}$$

$$H_{eq}(z) = -\frac{z-1}{z} \frac{z(1 - e^{-aT})(z - e^{aT}) + z(1 - e^{aT})(z - e^{-aT})}{(z-1)(z - e^{-aT})(z - e^{aT})}$$

$$H_{eq}(z) = -\frac{(1 - e^{-aT})(z - e^{aT}) + (1 - e^{aT})(z - e^{-aT})}{(z - e^{-aT})(z - e^{aT})}$$

$$H_{eq}(z) = -\frac{(2 - e^{aT} - e^{-aT})z - e^{aT} - e^{-aT}}{(z - e^{-aT})(z - e^{aT})}$$

$$H_{eq}(z) = \frac{(e^{aT} + e^{-aT} - 2)z + e^{aT} + e^{-aT}}{(z - e^{-aT})(z - e^{aT})}$$

b) Show that for

$$a = 40, T = 0.001 \text{ sec}, H_{eq}(z) = \frac{8 \times 10^{-4}(z + 1)}{z^2 - 2.0016z + 1}$$

$$H_{eq}(z) = \frac{(e^{0.04} + e^{-0.04} - 2)z + e^{0.04} + e^{-0.04}}{(z - e^{-0.04})(z - e^{0.04})}$$

$$H_{eq}(z) = \frac{8 \times 10^{-4}(z + 1)}{z^2 - 2.0016z + 1}$$

The answer matches, as expected.

The poles map closely to $z = e^{sT}$ due to the fact that the sampling frequency is significantly higher than the frequency of the poles.

$$e_1^{sT} = 1.0408 = p_{1z}$$

$$e_2^{sT} = 0.9608 = p_{2z}$$

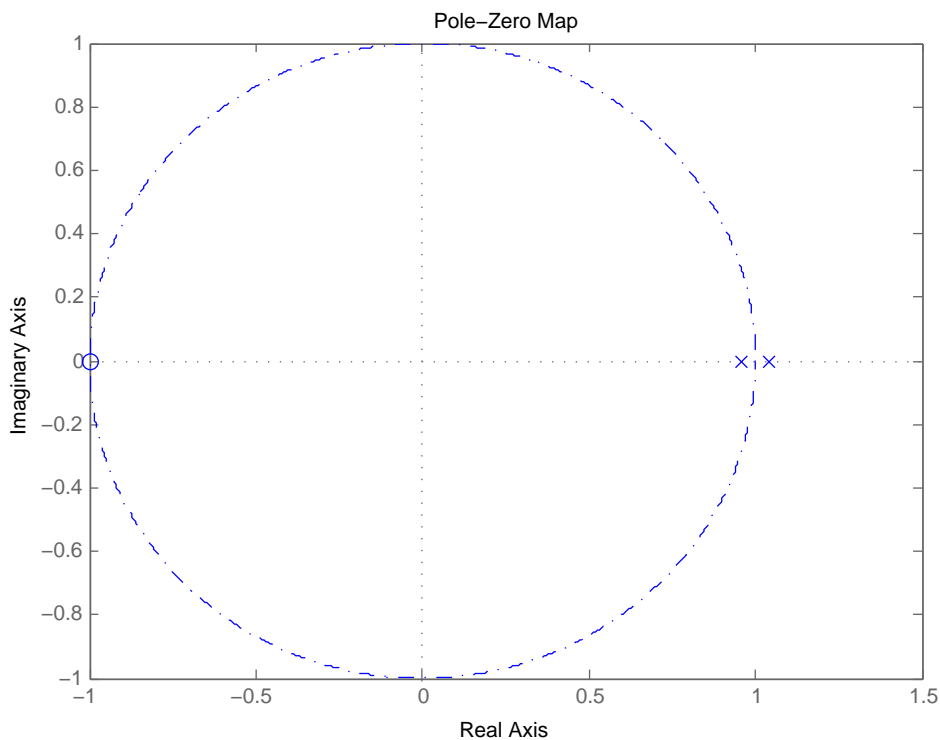


Figure 1: Pole Zero Map of Plant

- c) The DC value of the transfer function is -1, so the magnitude is 1 with a phase of -180. This result can be found by plugging in $z = 1$. This discrete DC gain agrees with the continuous time DC gain, as both are -1. The high frequency gain is 0, as if $z = -1$, then the numerator goes to 0.

The poles near $z=1$ can both be treated as if they were at $z=0.9608$ for the purposes of the magnitude plot. Because of this, the magnitude plot starts off with a flat slope, and the slope turns to -2 at the pole frequency. The value of the magnitude at the breakpoint is -3db. The plot continues with this -2 slope until the zero is approached, at which the magnitude decreases sharply towards 0.

The two poles wind up contributing -180° of phase to the system for low frequencies and -360° of phase at high frequencies. The zero is approached at an angle nearly perpendicular to the real axis, so it contributes 90° at high frequencies. Other interesting points such as the $\frac{\pi}{2}$ frequency point can be observed. At $\frac{\pi}{2}$ rad/sample, each pole contributes -135° of phase while the zero contributes 45° . The magnitude at that point is $\frac{1}{\sqrt{2}} \times 8 \times 10^{-4}$. Connecting these points, we can form a reasonable Bode plot.

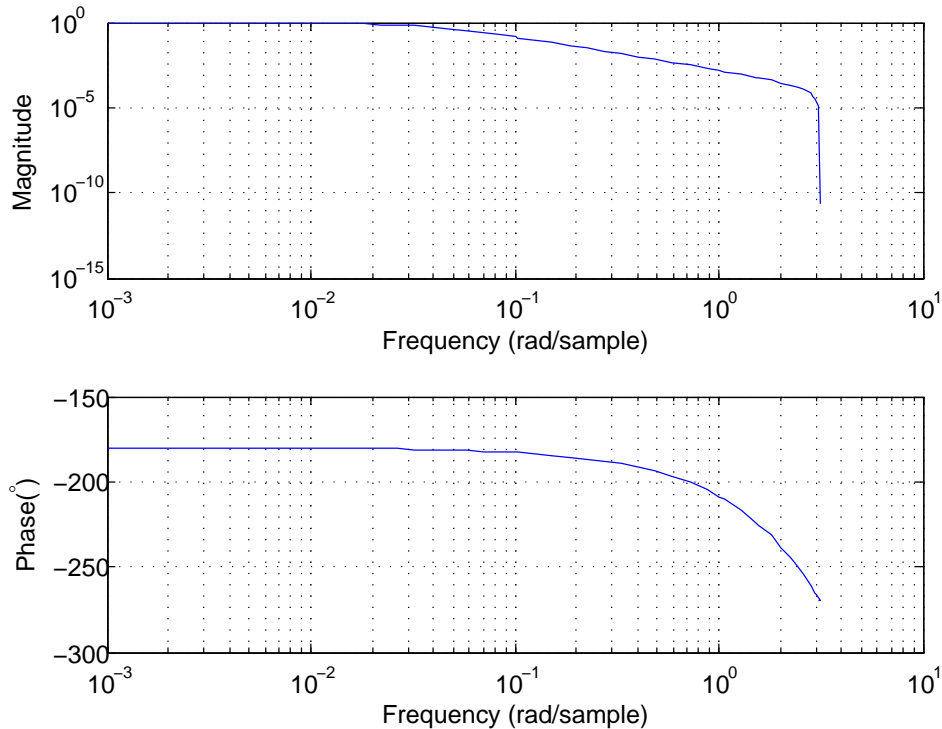


Figure 2: Bode Plot of Plant

- d) There are no values of G_0 that can stabilize the system. The poles break away from the real axis outside of the unit circle, and proceed to circle around the zero in a circular arch of radius 2. They break back into the real axis, but even at high gains, one pole continues to remain unstable.

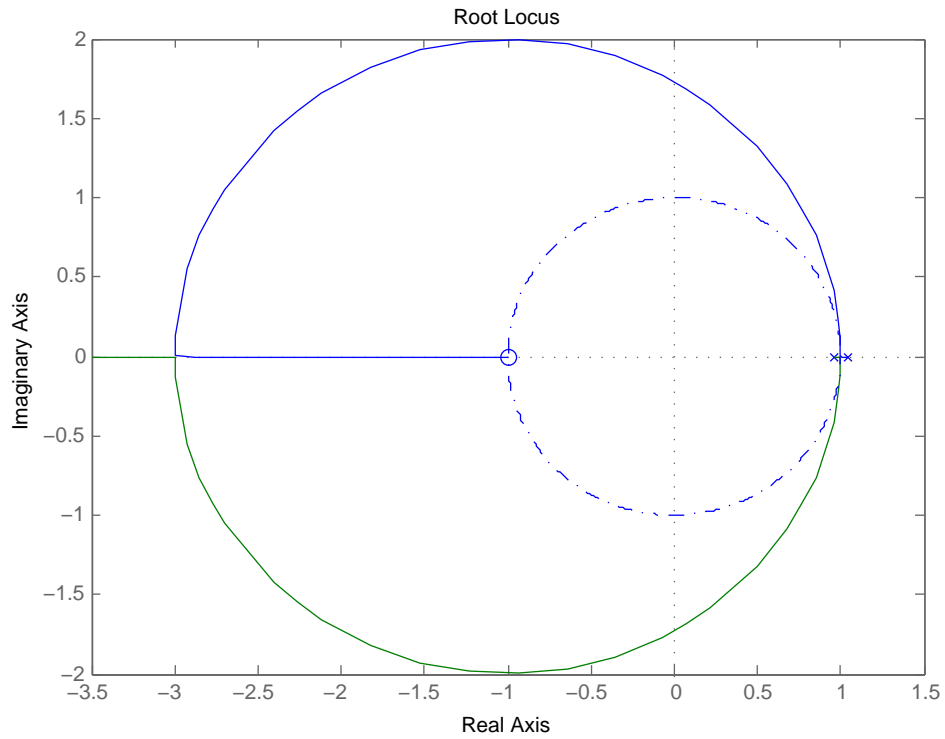


Figure 3: Root Locus of Proportional Controller + Plant

e) The Nyquist contour can be developed using the unit circle, and thus the same data points as described in the Bode plot construction.

In Figure 4, three points are indicated. These exist at frequencies of 0 , $\frac{\pi}{2}$, and π . The Nyquist contour will first be drawn for $G_0 = 1$ and generalized afterwards.

At DC, the magnitude was already found to be 1 with a phase of -180° . At $\frac{\pi}{2}$, the magnitude was found to be $\frac{1}{\sqrt{2}} \times 8 \times 10^{-4}$ with a phase of -225° . Finally, at π , the magnitude is 0 due to the presence of the zero, and the phase is $-180^\circ - 180^\circ + 90^\circ = -270^\circ$.

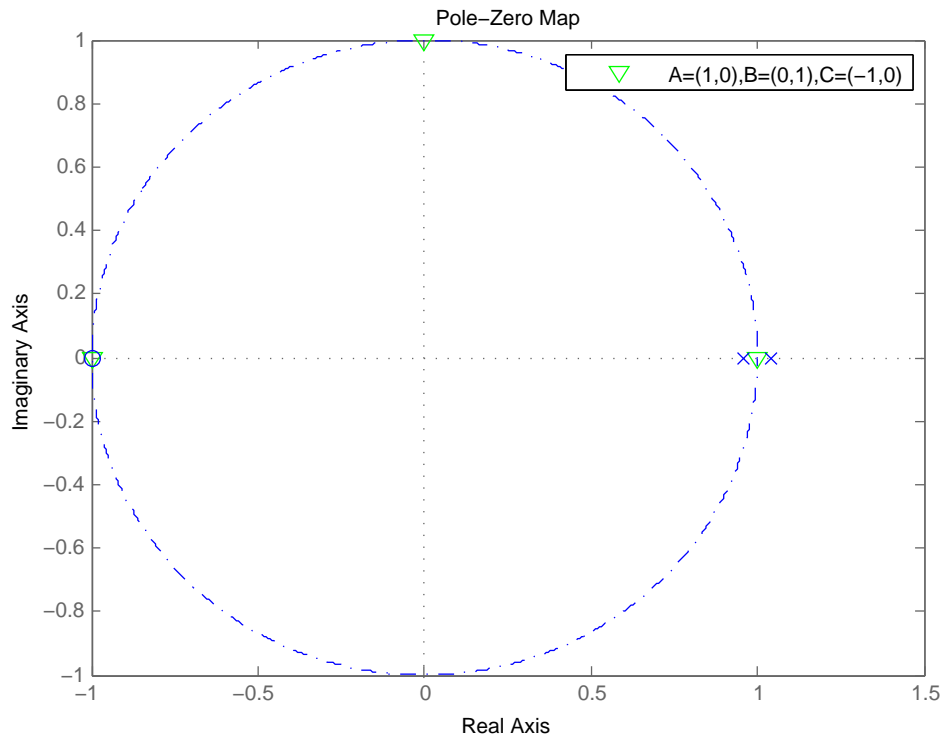


Figure 4: Pole Zero Map of Plant and Useful Points for Nyquist

Using the presented points, a Nyquist plot can be constructed.

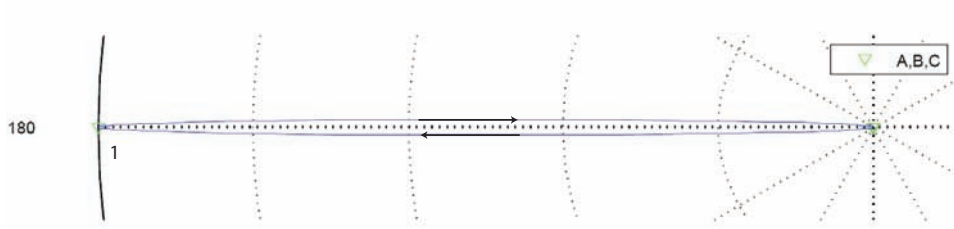


Figure 5: Nyquist Plot

The loop that is created is a clockwise loop. We can normalize the system by observing the encirclements of the $\frac{1}{G_0}$ point. Because of the unstable loop-transmission pole, our goal is to produce a counterclockwise encirclement of the $\frac{1}{G_0}$ point. However, this is never achieved. For small G_0 , there is no encirclement at all, so $Z=N+P=0+1=1$. For large G_0 , there is a clockwise encirclement, and $Z=N+P=1+1=2$.

f) Design a lead compensator for the system of the form

$$G(z) = \frac{K(z - \alpha)}{z}$$

The lead pole has already been set at the origin. The location of the zero has to be determined. One possibility is to place the zero near the pole at $z=0.9608$. This will greatly offset the phase contribution of that pole.

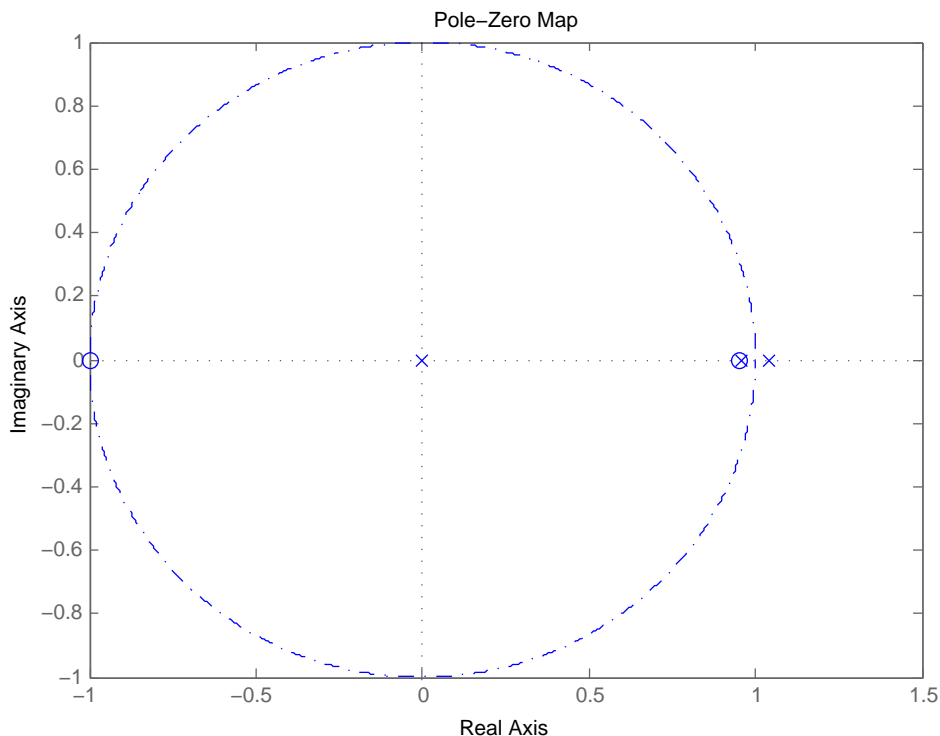


Figure 6: PZ Map with Lead with Zero at $z=0.95$

A suitable location for the crossover can be estimated from the pole-zero plots. For both cases, at very small frequencies ($\Omega \ll 1$), the pole at the origin and the zero at -1 have very little effect. For our approximation, the 2 poles and zero near $z = 1$ appear as a single pole near $z = 1$. Let's test some approximate points. At $\Omega = \pi/2$, the phase is $-135^\circ - 90^\circ + 45^\circ = -180^\circ$. There is not enough phase margin at this point. At $\Omega = \pi/4$, the phase is $\sim -110^\circ - 45^\circ + 22.5^\circ = \sim -135^\circ$. The phase margin is approximately 45° , so that frequency will suffice. All that is left to do is solve for the gain necessary at that frequency. The gain can quickly be approximated. The gain introduced by the lead compensator is $K(1 - \alpha)$. Putting aside the K term, the lead compensator has already contributed $1 - \alpha = .04$ to the gain. With the design of the lead compensator, the zero nearly cancels the pole near the origin, so it appears as if there were only one pole near the origin in total. The pole outside of the unit circle can be treated as if it were at $z=0.96$ for magnitude calculations. Thus there is a breakpoint at $\Omega = 0.04$, and the following asymptote has a slope of -1 . At the desired crossover frequency, $\Omega = 0.8 \frac{\text{rad}}{\text{sample}}$,

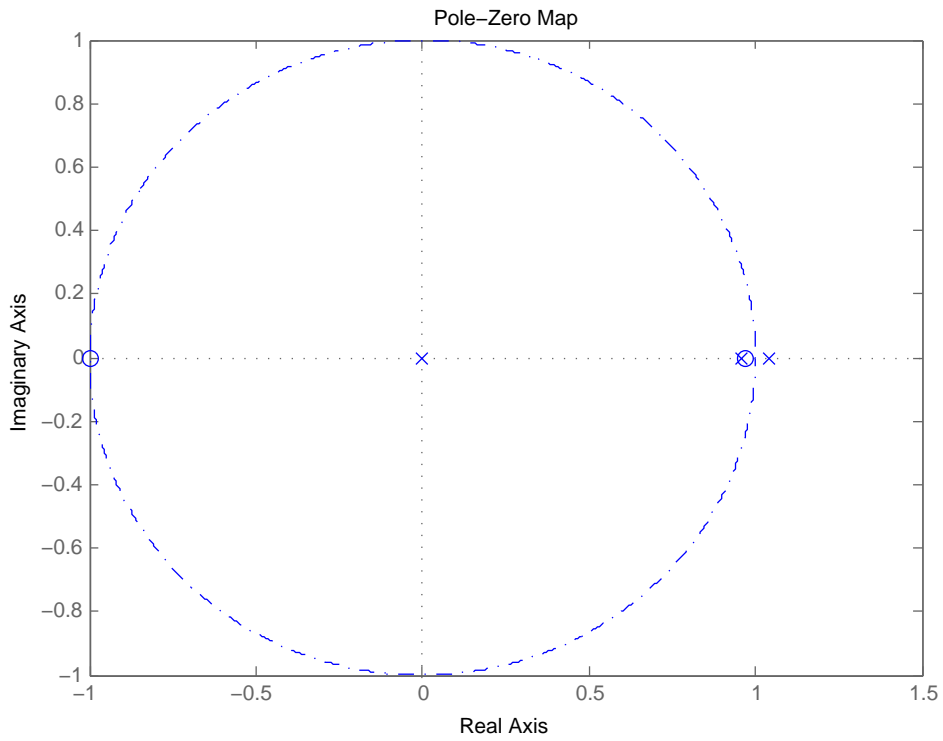


Figure 7: PZ Map with Lead with Zero at $z=0.97$

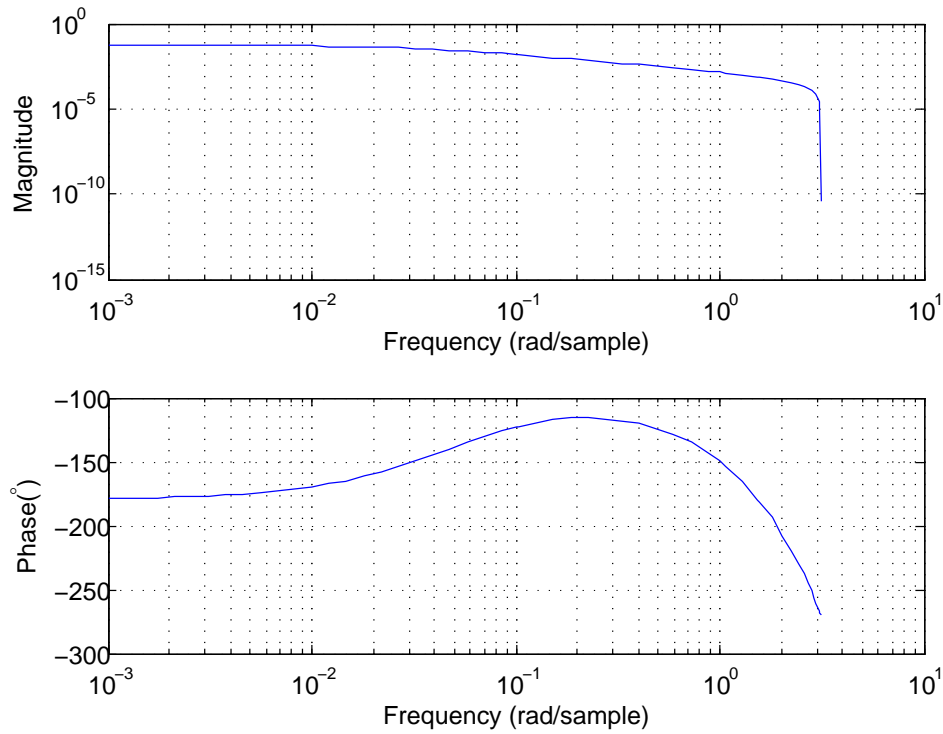


Figure 8: Bode Plot of Compensated System, $K=1$

the -1 slope has resulted in a gain of $\frac{1}{20}$. These two gains result in a gain of $\frac{1}{25} \times \frac{1}{20} = \frac{1}{500}$. $K = 500$.

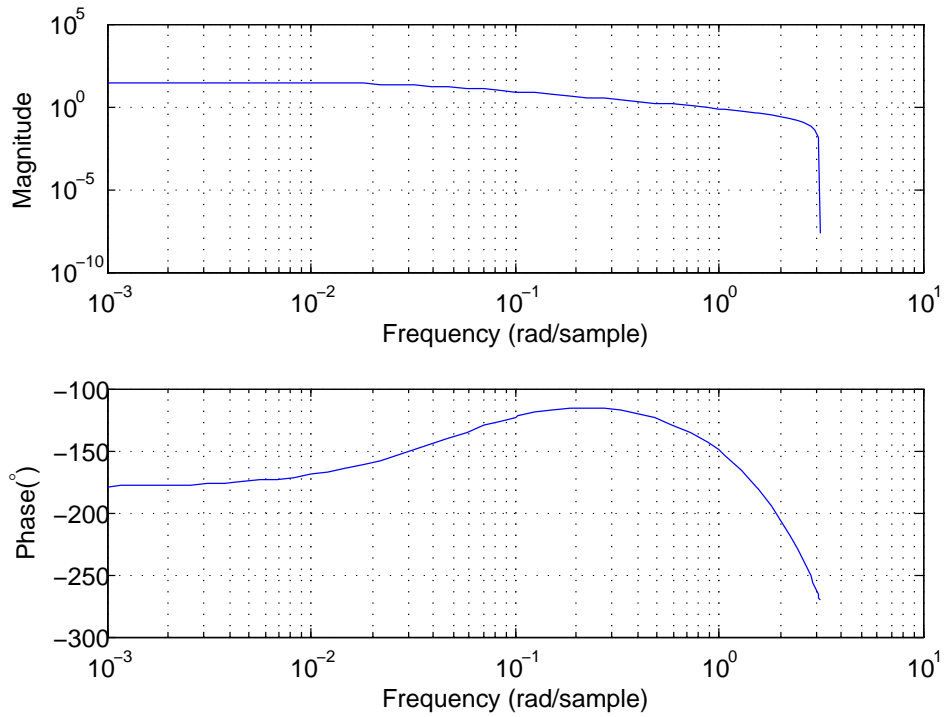


Figure 9: Bode Plot of Compensated System, $K=520$

The design is stable until the frequency $\Omega = \frac{\pi}{2}$ is reached. The difference in gain between this frequency and $\Omega = \frac{\pi}{4}$ is about 2, due to the -1 slope. Thus the system is stable until a gain of approximately $K = 1000$ is used.

- g) As previously solved for, the poles cross the unit circle with a gain of 1000, and they cross at $\pm i$.

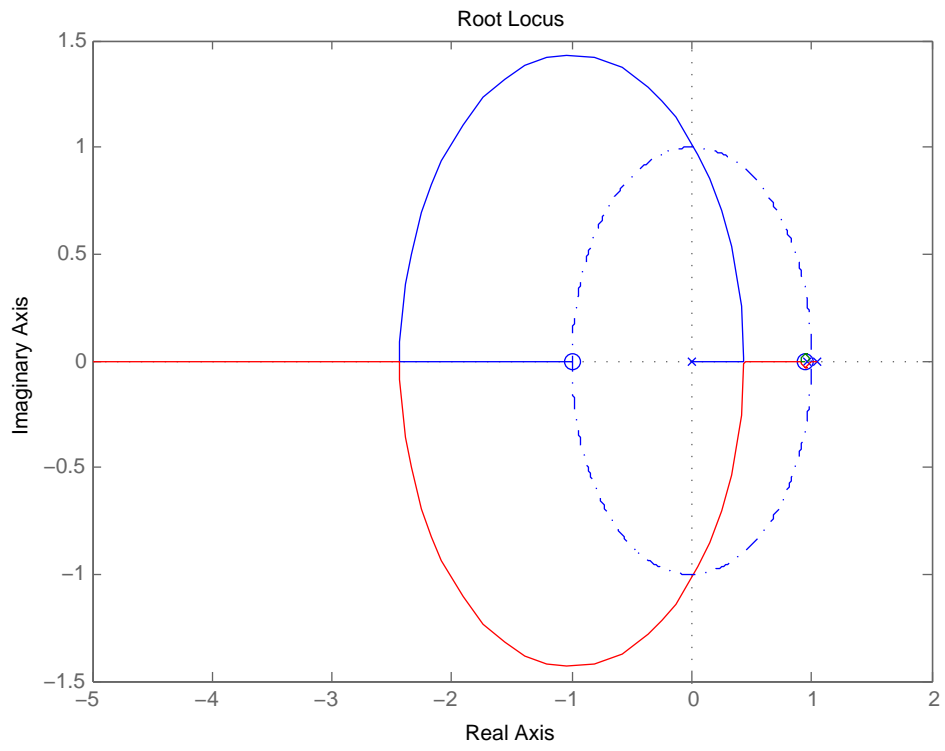


Figure 10: Root Locus of Compensated System