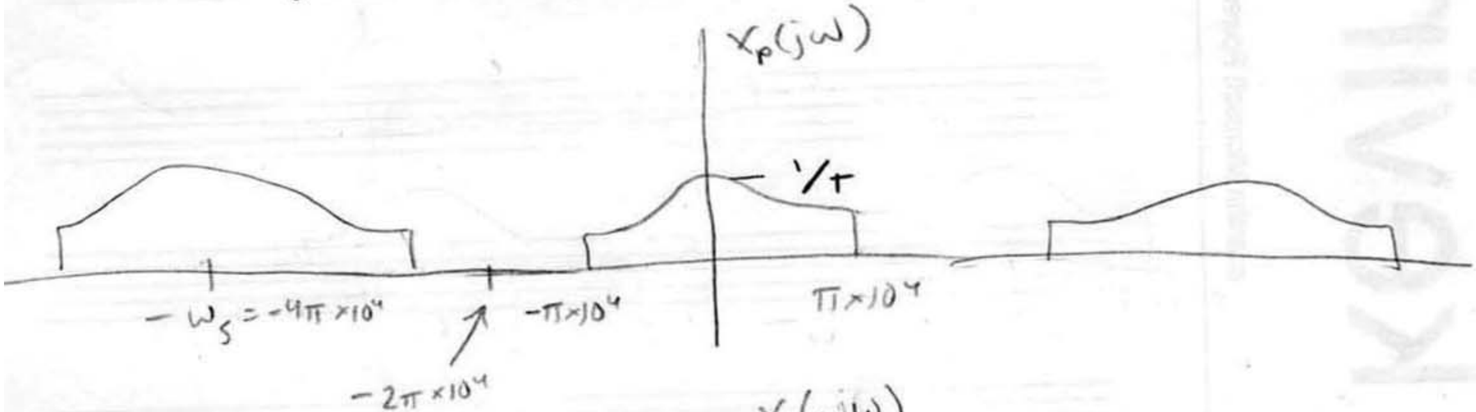
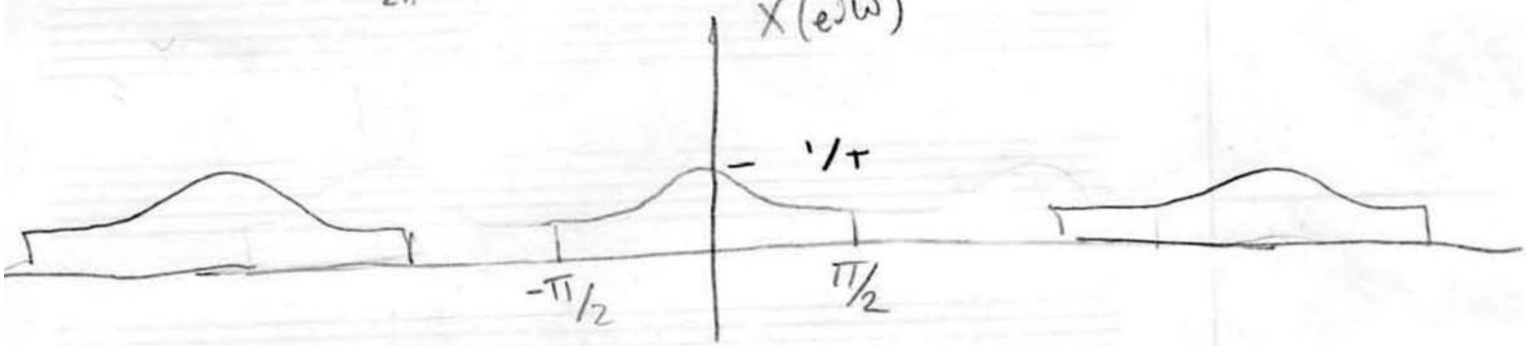
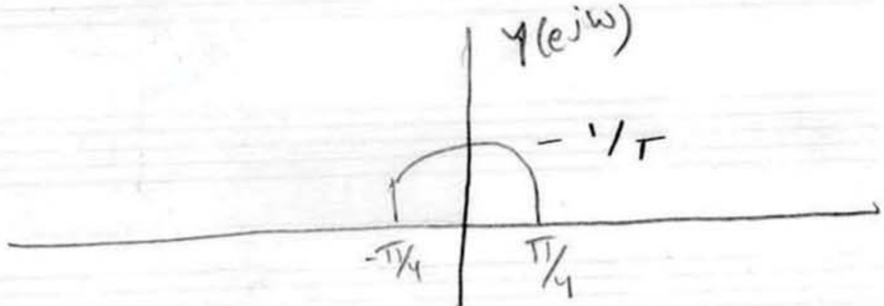
2.171 Problem Set #4

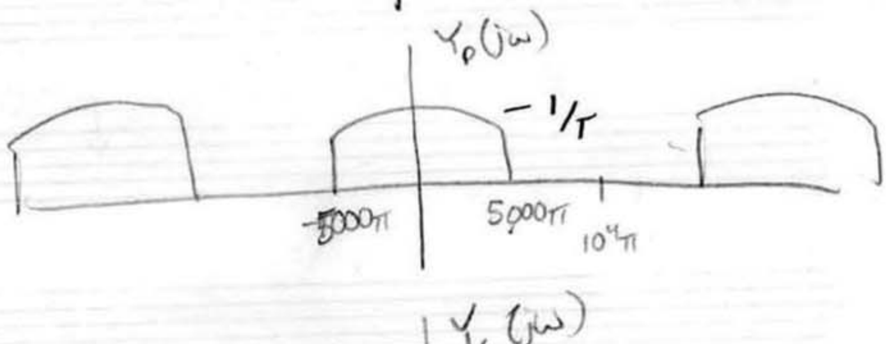
Problem 1: O&W 7.29

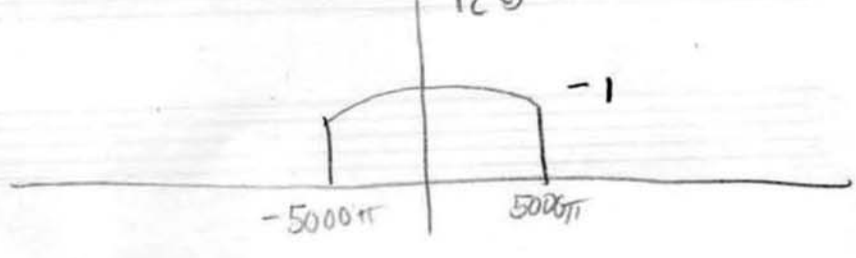
Following the recipe provided on Page 539











Problem 2: O&W 7.31

$$y[n] = \frac{1}{2}y[n-1] + x[n]$$

$$Y(z) = \frac{1}{2}z^{-1}Y(z) + X(z)$$

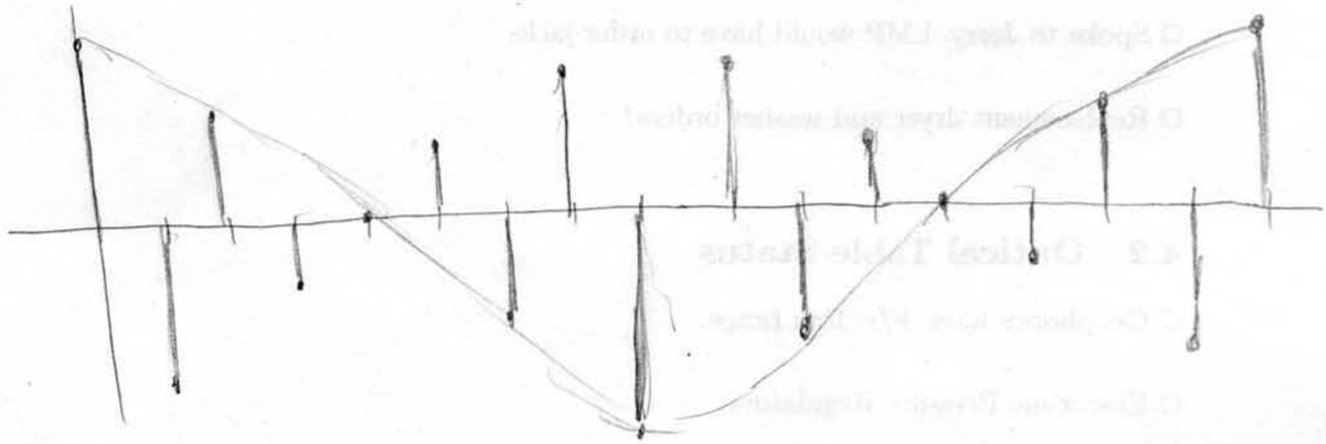
$$\frac{Y(z)}{X(z)} = \frac{1}{1-\frac{1}{2}z^{-1}} = \frac{z}{z-\frac{1}{2}} = H(z) = \frac{z}{z-e^{aT}}, \quad a = \frac{-\ln(2)}{T}$$

$$\boxed{H(j\omega) = \frac{1}{s-\frac{\ln(2)}{T}} = \frac{1}{j\omega-\frac{\ln(2)}{T}}}$$

Problem 3

$y(k) = y_a(kT)$

(a) The waveform is periodic with $N=16$.



beat frequency is $\Delta = \frac{2\pi}{16}$

$\Delta = \pi - \Omega = \frac{2\pi}{16}, \; \Omega = \frac{14\pi}{16}$

$\omega = \frac{\Omega}{T} = \frac{14\pi/16}{T} = \frac{7\pi/8}{T}$

$\phi \equiv$ offset of $90° = \pi/2$

(b) $\Delta = n(\pi \pm \Omega) = 2\pi/16$

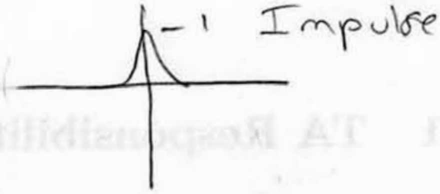
$\Omega = \frac{16\pi n \pm 2\pi}{16} = \pi n \pm \pi/8 = \pi(n \pm \frac{1}{8})$

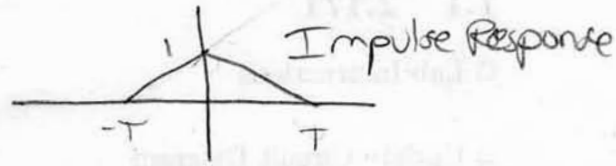
$\omega = \frac{\Omega}{T} = \frac{\pi(n \pm 1/8)}{T}$

$\phi = \pi/2 \pm 2n\pi$

Problem 4: FPW 5.7

(a)
$$p(t) = u_s(t+T)\frac{t+T}{T} - 2u_s(t)\frac{t}{T} + u_s(t-T)\frac{t-T}{T}$$



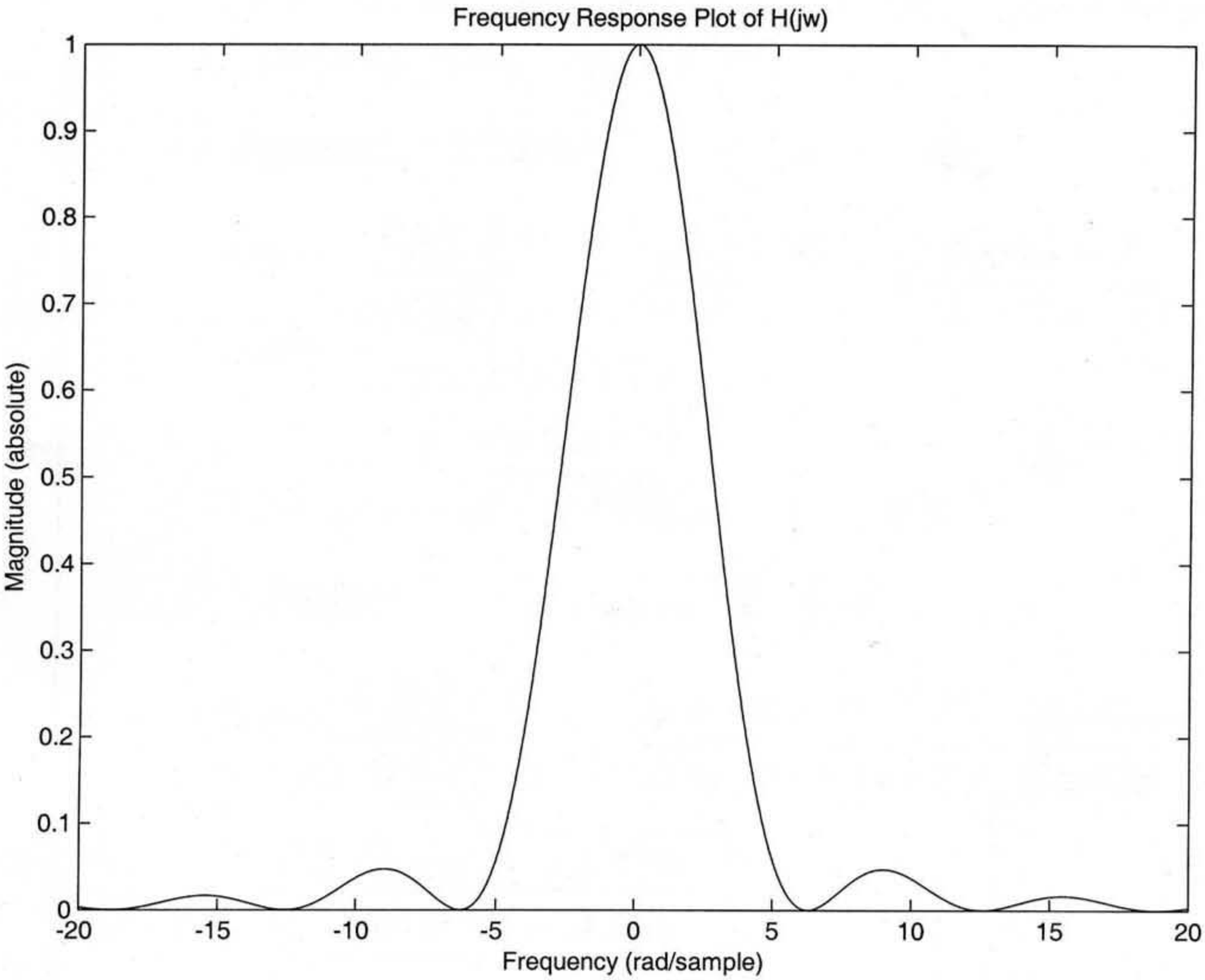


(b)
$$\text{Triangle Hold}(s) = \mathcal{L}\{p(t)\}$$
$$= \int_0^\infty \left[u_s(t+T)\frac{t+T}{T} - 2u_s(t)\frac{t}{T} + u_s(t-T)\frac{t-T}{T}\right] e^{-st}\, dt$$
$$= \frac{e^{Ts} - 2 + e^{-Ts}}{Ts^2}$$

(c) See attached

(d) The delayed triangle hold is $\dfrac{1-2e^{-st}+e^{-2st}}{Ts^2}$

Unlike the previous case the phase is no longer zero. A time delay results.


Frequency Response Plot of H(jw)
Magnitude (absolute)
Frequency (rad/sample)
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-20
-15
-10
-5
0
5
10
15
20

Problem 5: FPW 6.3

$$H(s) = \frac{s+1}{0.1s+1}, \quad \omega_1 = 3 \text{ rad}$$

(a) Calculate phase lead at $z_1 = e^{j\omega_1 T}$

$T = \cancel{0.25s}\ 0.015s$

i. Forward rectangular

$$z = 1 + Ts, \quad s = \frac{z-1}{T}$$

$$H(z) = \frac{\frac{z-1}{T} + 1}{0.1\frac{z-1}{T} + 1} = \frac{z - 1 + T}{0.1(z-1) + T} = \frac{z + (T-1)}{0.1z + (T - 0.1)}$$

Phase = 55.9°

ii Backward Rectangular

$$s = \frac{z-1}{Tz}$$

$$H(z) = \frac{\frac{z-1}{Tz} + 1}{0.1\frac{z-1}{Tz} + 1} = \frac{z - 1 + Tz}{0.1(z-1) + Tz} = \frac{(1+T)z - 1}{(0.1+T)z - 0.1}$$

Phase = 53.8°

iii Tustin Rule

$$z = \frac{1 + Ts/2}{1 - Ts/2}$$

$$H(z) = \frac{\frac{1+Ts/2}{1-Ts/2} + 1}{0.1\,\frac{1+Ts/2}{1-Ts/2} + 1} = \frac{2}{-.9\,Ts/2 + 1.1}$$

Phase = 54.9°

iv Zero Pole Matching

$$H(s) = \frac{s+1}{0.1s+1}$$, Zero at $s = -1$, pole at $s = -10$

$$H(z) = \frac{z - e^{-T}}{z - e^{-10T}}$$

Phase = 54.8°

Problem 6: FPW 7.3

(a) See attached

(b) See attached

(c) See attached

$w_0 = 1$ is unconditionally stable

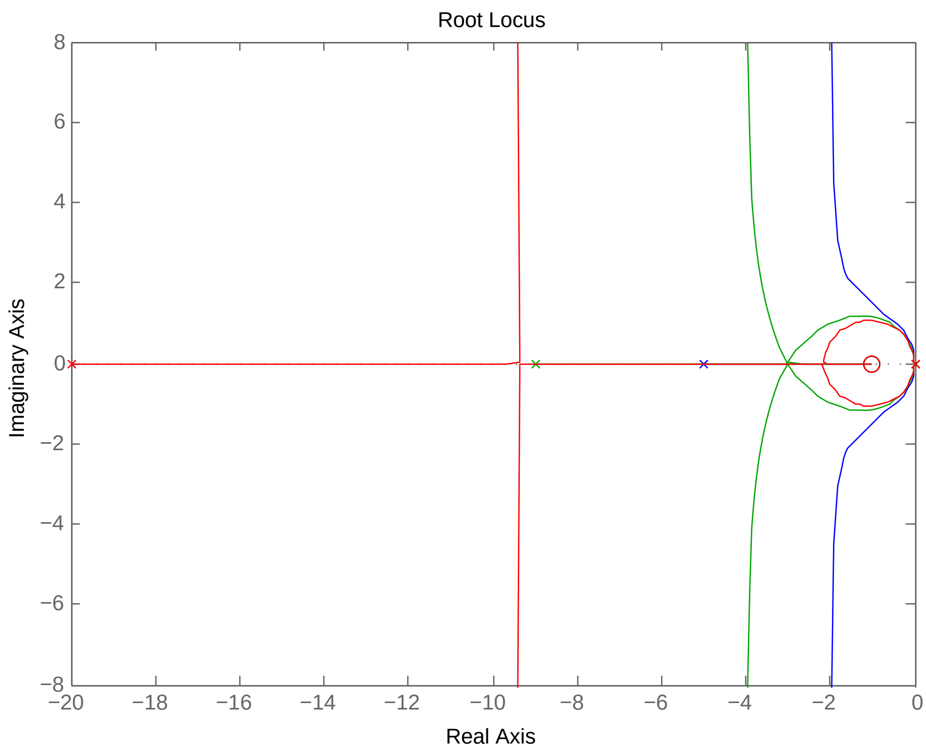
(d)
$$1 + K \frac{s}{(s-p_1)(s-p_2)} = 0$$

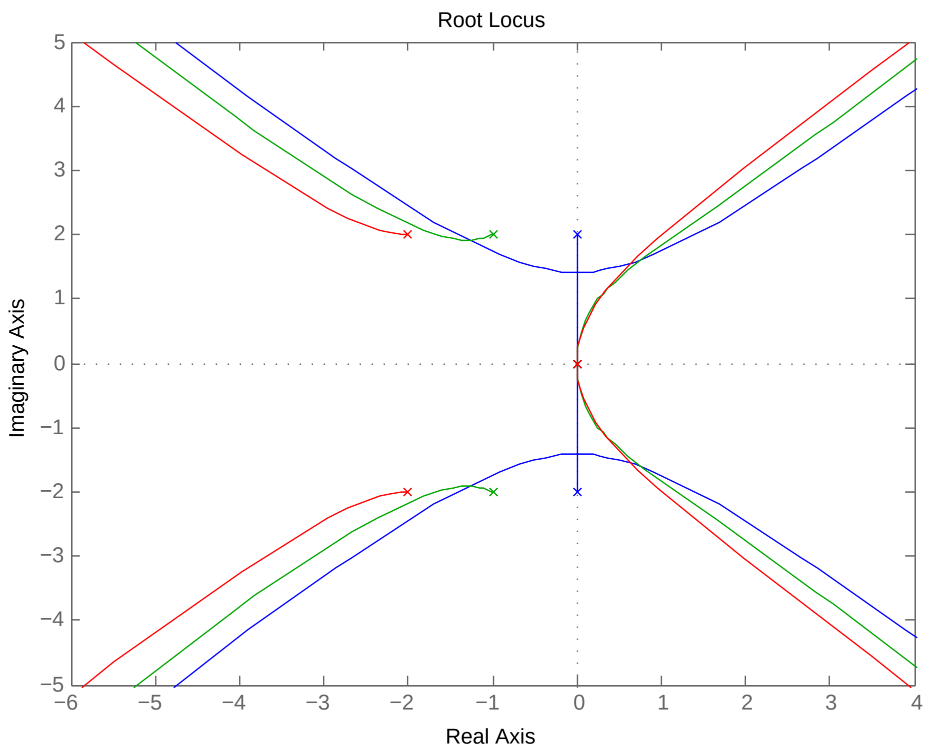
$$(s-p_1)(s-p_2) + Ks = 0$$

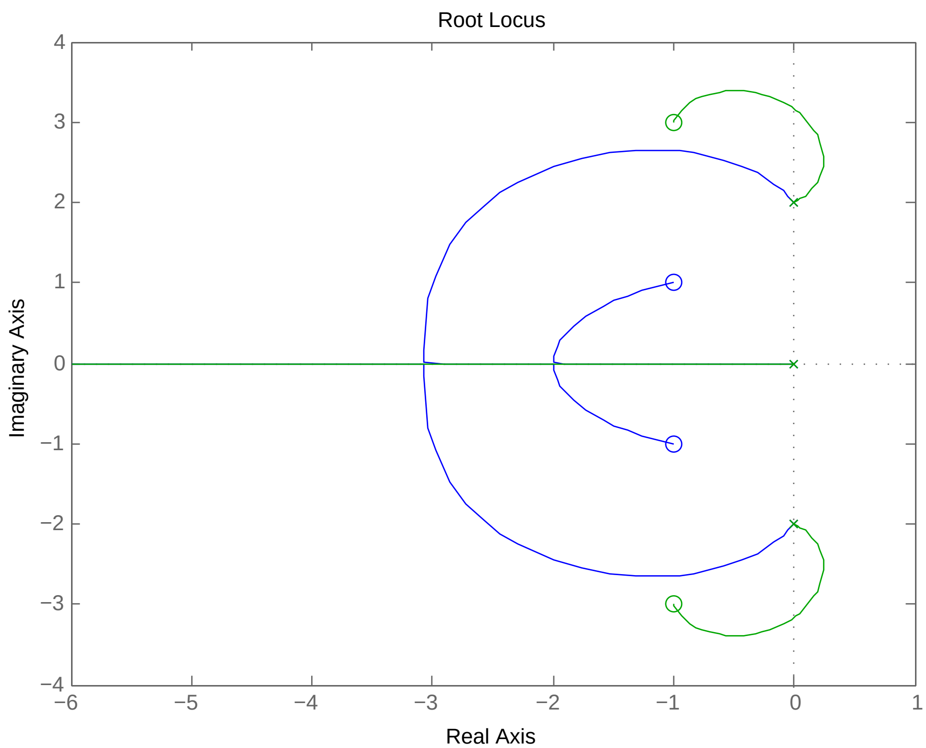
$$s^2 - (p_1+p_2)s + p_1p_2 + Ks = 0$$

$$\boxed{s = \frac{p_1+p_2-K \pm \sqrt{(K-p_1-p_2)^2 - 4p_1p_2}}{2}}$$

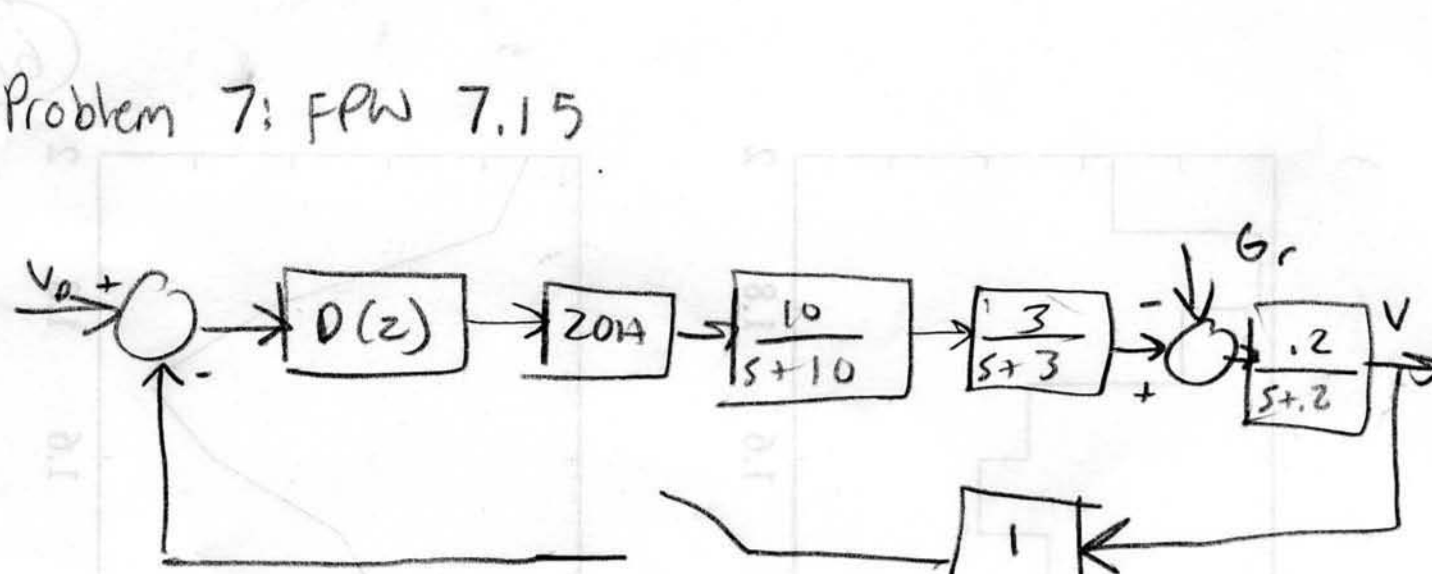
Yes, with negative gain


Root Locus
Imaginary Axis
Real Axis
8
6
4
2
0
−2
−4
−6
−8
−20
−18
−16
−14
−12
−10
−8
−6
−4
−2
0


Root Locus
Imaginary Axis
Real Axis
5
4
3
2
1
0
−1
−2
−3
−4
−5
−6
−5
−4
−3
−2
−1
0
1
2
3
4


Root Locus
Imaginary Axis
Real Axis
4
3
2
1
0
−1
−2
−3
−4
−6
−5
−4
−3
−2
−1
0
1

Problem 7: FPW 7.15



$T = 0.5s$

(a) $t_r = 5\,sec$, no overshoot

$\omega_c \approx \frac{1.8}{t_r}$, $t_r = \frac{1.8}{\omega_c} = 5$, $\omega_c = .36$, round ω_c up to 1 (extra bandwidth does not hurt much here)

$\zeta \approx \frac{\phi_m}{100}$, $\phi_m \approx 100$

$$G(z) = (1-z)^{-1} \mathcal{Z}\left\{\frac{1}{s}\frac{10}{s+10}\frac{3}{s+3}\frac{.2}{s+.2}\right\}$$

$$G(s) = \frac{2/T}{s+2/T}\left(\frac{10}{s+10}\frac{3}{s+3}\frac{.2}{s+.2}\right)$$

Phase at ω_c starts at $\approx -90°$

Add 1 lead compensator

Assume $\alpha = 2$ (don't need much ϕ_m increase)

$$G_{lead}(s) = \frac{K(\alpha\tau s+1)}{\tau s+1}$$

$$\tau = \frac{1}{\sqrt{\alpha}\,\omega_c}, \quad K = \frac{1}{mag|_{\omega_c}}$$

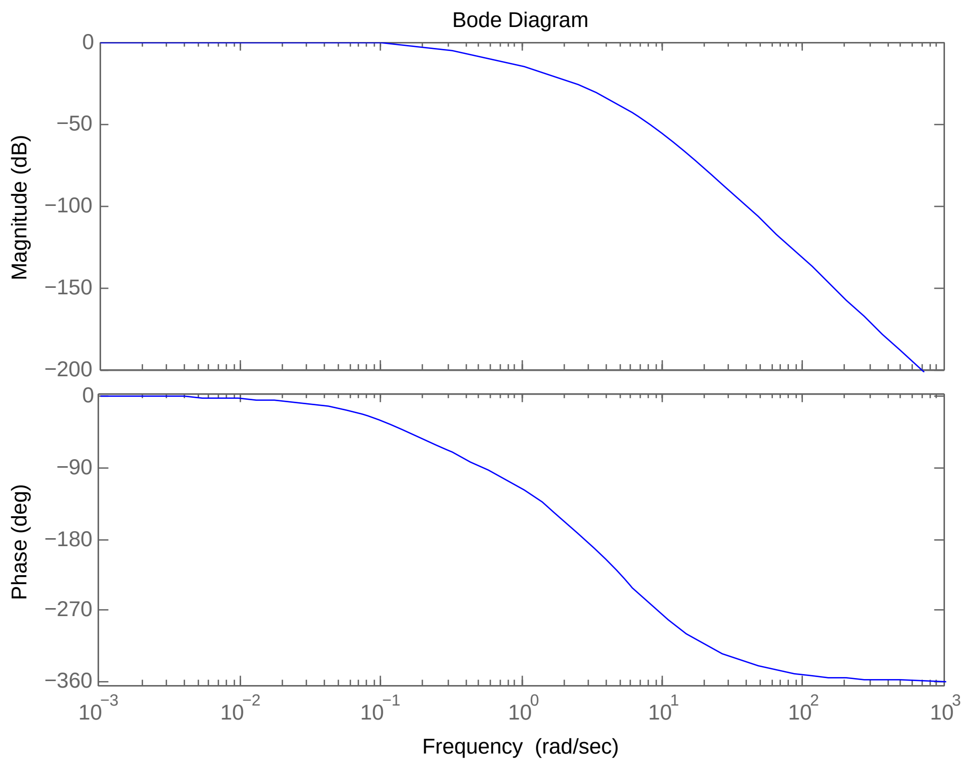
See attached

(b) See attached

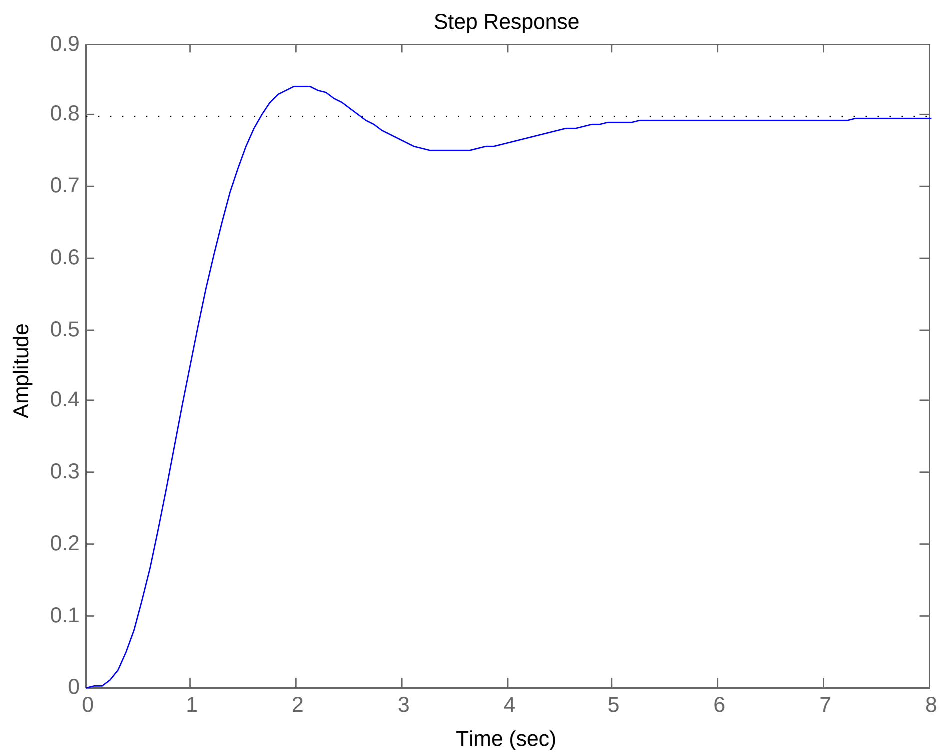
(c) Add a lag compensator

$$T_I = \frac{10}{\omega_c}$$

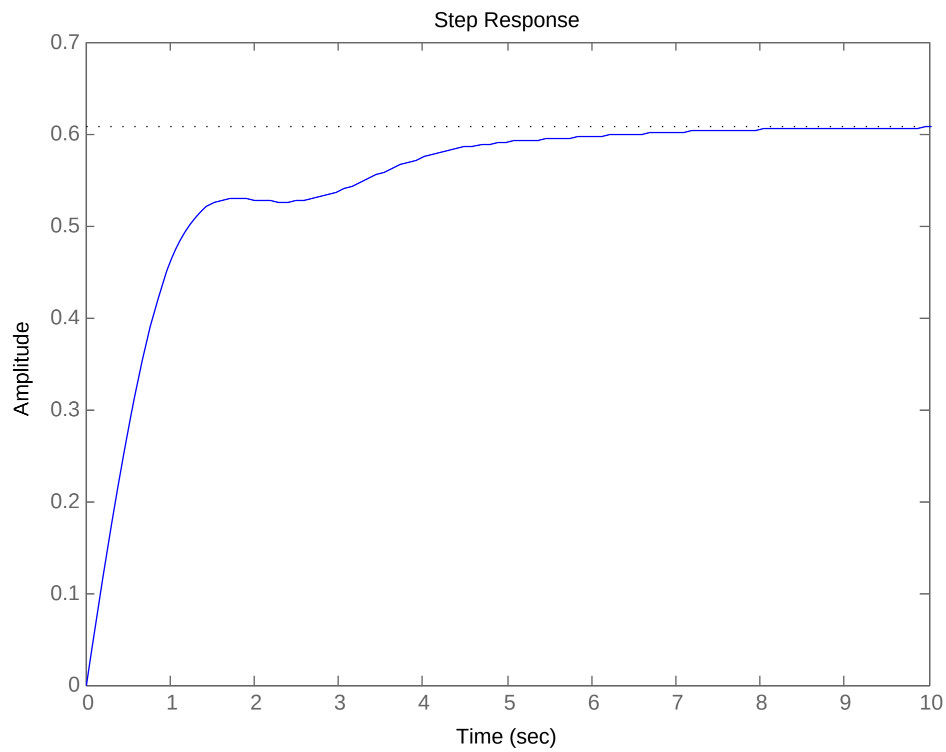
See attached



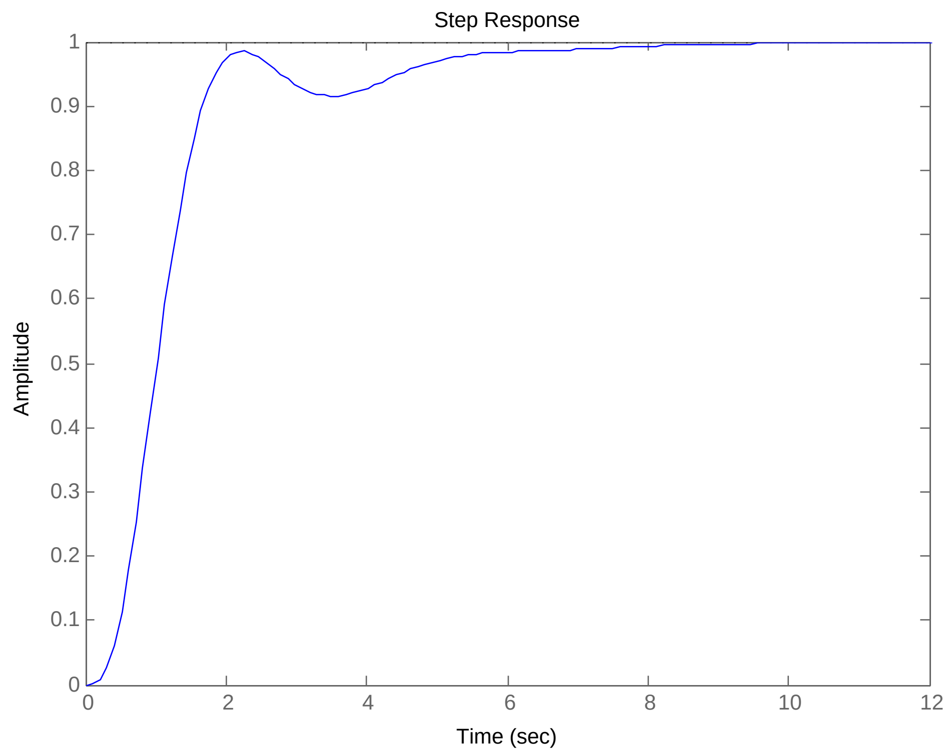
Bode Diagram
Magnitude (dB)
0
−50
−100
−150
−200
Phase (deg)
0
−90
−180
−270
−360
10^{-3}
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
Frequency (rad/sec)



Step Response
Amplitude
Time (sec)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
2
3
4
5
6
7
8



Step Response
Amplitude
Time (sec)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0
1
2
3
4
5
6
7
8
9
10



Step Response
Amplitude
Time (sec)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
2
4
6
8
10
12