

## 2.171 Problem Set 1 Solutions

**Out:** Wednesday, September 13, 2006

Note that any problems that require hand-drawn design plots should be not be done initially in MATLAB. Hand-drawn solutions have been left out of these solutions as a means of keeping file size down.

### Problem 1: FPW 2.9

The problem presents the following system:

$$\ddot{\theta} + 4\theta = T_c + T_d$$

(a) To write the systems of equations, we use the following form (FPW (2.1) and (2.2)):

$$\dot{\mathbf{x}} = \mathbf{F}\mathbf{x} + \mathbf{G}\mathbf{u} \quad (1)$$

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{J}\mathbf{u} \quad (2)$$

The state vector we are to use is

$$\mathbf{x} = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$$

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} T_c + \begin{bmatrix} 0 \\ 1 \end{bmatrix} T_d$$

$$y = \theta = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$$

$$\mathbf{F} = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}$$

$$\mathbf{G} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\mathbf{H} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

(b) The characteristic equation is found by solving for the nontrivial solutions of Eq 1 in the homogeneous case.

$$\dot{\mathbf{x}} = \mathbf{F}\mathbf{x} \quad (3)$$

$$\mathbf{sI}\mathbf{x} = \mathbf{F}\mathbf{x} \quad (4)$$

$$\mathbf{F}\mathbf{x} - \mathbf{sI}\mathbf{x} = (\mathbf{F} - \mathbf{sI})\mathbf{x} = 0 \quad (5)$$

$$\det(\mathbf{F} - \mathbf{sI}) = 0 \quad (6)$$

$$\det(\mathbf{F} - \mathbf{sI}) = 0 \quad (7)$$

$$\det \left( \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} - \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} \right) = \det \begin{bmatrix} -s & 1 \\ -4 & -s \end{bmatrix} = s^2 + 4 = 0$$

(c)

$$\dot{\hat{\mathbf{x}}} = \mathbf{F}\hat{\mathbf{x}} + \mathbf{G}\mathbf{u} + \mathbf{L}(y - \mathbf{H}\hat{\mathbf{x}}) \quad (8)$$

$$\tilde{\mathbf{x}} = (\mathbf{F} - \mathbf{LH})\tilde{\mathbf{x}} \quad (9)$$

$$\det[s\mathbf{I} - (\mathbf{F} - \mathbf{LH})] = 0 \quad (10)$$

$$\begin{aligned} \det\left(\left[\begin{array}{cc} s & 0 \\ 0 & s \end{array}\right] - \left(\left[\begin{array}{cc} 0 & 1 \\ -4 & 0 \end{array}\right] - \left[\begin{array}{c} l_1 \\ l_2 \end{array}\right] \left[\begin{array}{cc} 1 & 0 \end{array}\right]\right)\right) &= \det\left(\left[\begin{array}{cc} s & 0 \\ 0 & s \end{array}\right] - \left[\begin{array}{cc} -l_1 & 1 \\ -4-l_2 & 0 \end{array}\right]\right) \\ &= \det\left[\begin{array}{cc} s+l_1 & -1 \\ 4+l_2 & s \end{array}\right] = s^2 + l_1s + 4 + l_2 = 0 \end{aligned}$$

Just looking at the terms, it is fairly trivial that  $l_1 = 20$  and  $l_2 = 96$ .

$$\dot{\hat{\mathbf{x}}} = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} \hat{\mathbf{x}} + \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \mathbf{u} + \begin{bmatrix} 20 \\ 96 \end{bmatrix} (y - \begin{bmatrix} 1 & 0 \end{bmatrix} \hat{\mathbf{x}})$$

(d)

$$\dot{\hat{\mathbf{x}}} = (\mathbf{F} - \mathbf{GK})\hat{\mathbf{x}} + \mathbf{L}y \quad (11)$$

$$\det[s\mathbf{I} - (\mathbf{F} - \mathbf{LH})] = 0 \quad (12)$$

$$\begin{aligned} \det\left(\left[\begin{array}{cc} s & 0 \\ 0 & s \end{array}\right] - \left(\left[\begin{array}{cc} 0 & 1 \\ -4 & 0 \end{array}\right] - \left[\begin{array}{c} 0 \\ 1 \end{array}\right] \left[\begin{array}{cc} k_1 & k_2 \end{array}\right]\right)\right) \\ = \det\left[\begin{array}{cc} s & -1 \\ 4+k_1 & s+k_2 \end{array}\right] = s^2 + k_2s + 4 + k_1 = 0 \end{aligned}$$

$$s = \frac{-k_2 + \sqrt{k_2^2 - 4(4+k_1)}}{2}$$

$$k_2 = 4$$

$$s = \frac{\sqrt{(16 - 4(4+k_1))}}{2} = \pm 2j$$

$$k_1 = 4$$

(e) See Figure 2.21 in FPW.

(f)

$$\dot{\hat{\mathbf{x}}} = \begin{bmatrix} \mathbf{F} - \mathbf{GK} & \mathbf{GK} \\ \mathbf{0} & \mathbf{F} - \mathbf{LH} \end{bmatrix} + \begin{bmatrix} \mathbf{GN} \\ \mathbf{0} \end{bmatrix} u$$

$$\mathbf{y} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \mathbf{x}$$

Solving for the above system with proper modifications to the "B" vector allows the responses shown in Figure 1 and Figure 2 to be found.

**Additional Work:**  $\frac{Y(s)}{T_c(s)} = \frac{8}{s^2+4s+8}$ . The crossover is at approximately  $\sqrt{8} \frac{rad}{sec}$ . The phase margin is about  $70^\circ$ .

**Problem 2:** FPW 2.17

There are poles at 0, -2, and  $\pm 5j$ . There is a zero at -1. According to root locus rules, the pole at the origin will move towards the zero and the pole at -2 will move towards negative infinity along the real axis. Because there are four poles, the poles on the imaginary axis will move along lines approximately  $45^\circ$  from the real axis as the gain is increased. These characteristics are verified in the MATLAB plot shown in Figure 3.

**Problem 3:** FPW 2.18

Yes, the system is stable, as the phase margin is greater than 0. The phase margin is comfortably above 0, and actually around  $90^\circ$  due to the presence of the DC pole which caused the magnitude of the response to have a -1 slope at low frequencies, and thus the 0dB point is reached at a frequency of  $10^{-3} \frac{rad}{sec}$ . The gain margin is also rather substantial at 46.3dB, which means that the gain can be increased dramatically - by over 1000 - given the current location of the  $-180^\circ$  frequency without driving the system unstable. Refer to Figure 4 for the MATLAB root locus plot.

**Problem 4:** FPW 2.19

Yes, the system is stable. The phase margin is  $44.5^\circ$ . The system can be separated into two main discernable units - what appears to be a lead compensator and a plant resembling a mass. Note that the lead compensator has a zero that occurs at a lower frequency than the pole, which is expected. The crossover frequency of the system lies in between the zero and the pole of the lead compensator so there is a phase margin increase. The phase margin increase is necessary due to the presence of the double poles caused by the mass term, which appears as a constant (in this case 100) divided by a frequency squared term. In a system consisting only of a mass, a lead compensator with a zero-pole pair surrounding the crossover frequency will lead to stability (as the phase margin will certainly be greater than 0). Refer to the Bode plot shown in Figure 5.

**Problem 5:** FPW 3.1

(a) Let us use a lead compensator of the form:

$$K \frac{\alpha\tau s + 1}{\tau s + 1}$$

The overall transfer function is now:

$$K \frac{\alpha\tau s + 1}{s^2(\tau s + 1)}$$

A convenient point to start with this problem are basic root locus rules. From the open-loop compensated transfer function, it is easy to verify that there are two more poles in the system than zeros. According to root locus rules, as the gain in such a system is increased, the overall sum of the real locations of the poles must

remain constant. The pole of the lead compensator will move closer to the zero of the lead compensator as the gain is increased, and the two poles at the origin will move outwards. The lead pole will remain on the real axis, so the two poles that begin at the origin must be the ones that move to  $-4.4 \pm 4.4j$ . To get those poles to move to the right  $-4.4 \approx -4.5$ , it is necessary that the lead pole move to the right by a real magnitude of  $8.8 \approx 9$ , as the lead pole must equal the distance of the two poles beginning at the origin. For the lead pole to move about 9, the difference between the location of the pole and the zero must be 9. One possibility is to place the zero at -1 and the pole at -10. To accomplish this, values of  $\alpha = 10$  and  $\tau = .1$  can be used. The only remaining task is to find the necessary gain. To find the location of the closed loop poles, assuming a sensor transfer function of 1, we can simply apply Black's Equation.

$$\frac{K\alpha\tau s + 1}{K\alpha\tau s + 1 + s^2(\tau s + 1)}$$

The poles are determined by the denominator of the transfer function.

$$K\alpha\tau s + 1 + s^2(\tau s + 1) = 0$$

$$\tau s^3 + s^2 + K\alpha\tau s + 1 = 0$$

Plugging in the known values for  $s$ ,  $\alpha$ , and  $\tau$ , we have:

$$17.037 - 3.4K + (4.4K - 21.683)j = 0$$

Approximating both the real and imaginary terms, one sees that a value of  $K = 5$  suffices. A MATLAB plot of the system shown in Figure 6 confirms the design.

- (b) The delay reduces the damping ratio of the poles with imaginary components. Refer to Figure 7.

**Problem 6: FPW 3.3**

$$D(s) = \frac{U(s)}{E(s)} = 25 \frac{s + 1}{s + 15}$$

$$(s + 15)U(s) = 25(s + 1)E(s)$$

$$\dot{u} + 15u = 25\dot{e} + 25e$$

$$\frac{u(k+1) - u(k)}{T} + 15u(k) = 25 \left( \frac{e(k+1) - e(k)}{T} + e(k) \right)$$

$$u(k+1) = u(k) + T \left( -15u(k) + 25 \left( \frac{e(k+1) - e(k)}{T} + e(k) \right) \right)$$

$$u(k+1) = (1 - 15T)u(k) + 25(T - 1)e(k) + 25e(k+1)$$

$$u(k+1) = \frac{65}{80}u(k) - 25 \left( \frac{79}{80} \right) e(k) + 25e(k+1)$$

$$\begin{aligned}
D(s) &= \frac{U(s)}{E(s)} = 25 \frac{s+1}{s+15} \\
(s+15)U(s) &= 25(s+1)E(s) \\
\dot{u} + 15u &= 25\dot{e} + 25e \\
\frac{u(k) - u(k-1)}{T} + 15u(k) &= 25 \left( \frac{e(k) - e(k-1)}{T} + e(k) \right) \\
u(k) = u(k-1) + T \left( -15u(k) + 25 \left( \frac{e(k) - e(k-1)}{T} + e(k) \right) \right) \\
(1 + 15T)u(k) &= u(k-1) + 25(T+1)e(k) - 25e(k-1) \\
\left( \frac{95}{80} \right) u(k) &= u(k-1) + 25 \left( \frac{95}{80} \right) e(k) - 25e(k-1) \\
u(k) &= \left( \frac{80}{95} \right) u(k-1) + 25e(k) - 25 \left( \frac{80}{95} \right) e(k-1) \\
u(k+1) &= \left( \frac{80}{95} \right) u(k) - 25 \left( \frac{80}{95} \right) e(k) + 25e(k+1)
\end{aligned}$$

The two cases are plotted in Figure 8. Note that the leading derivative case falls below the MATLAB approximation. The lagging derivative case is above the MATLAB approximation. As the frequency of sampling increases, they converge upon the MATLAB approximation.

**Problem 7: FPW 3.5**

Like mentioned in class, rather than doing PID in its traditional form, we will implement a lead-lag compensator. To start, we will work with the lead term. The desired crossover frequency is 100Hz, or  $200\pi \frac{rad}{sec}$ . If we are to use a lead compensator of the form  $K \frac{\alpha\tau s + 1}{\tau s + 1}$ , then  $\sqrt{\alpha}\tau = 200\pi$ . Since we need  $50^\circ$  of phase margin, we will use a value for  $\alpha$  of 10, which will provide  $55^\circ$  of phase margin. The existing phase margin is 0 since the original plant has a double pole at the origin. From this point, we can solve for  $\tau$  simply with the above equations,  $\tau = 5.0329 \times 10^{-4}$ . The gain can be solved simply by finding the gain of the plant at the original point, which is  $\frac{1000}{(200\pi i)^2}$ , finding its reciprocal, and dividing by the gain provided at the point by the lead compensator, which is  $\sqrt{10}$ . Through these steps, it can be found that  $K = \frac{(200\pi)^2}{1000\sqrt{10}} = 125$ .

The lag compensator appears in the form  $\frac{T_i s + 1}{T_i s}$ . Note that the lag compensator does not affect the high frequency gain. Since we added  $55^\circ$  of phase with the lead compensator, we can lose a little bit of phase due to the lag compensator. If we place the lag compensator zero one decade below the crossover frequency, we will lose approximately  $2.5^\circ$  from the phase margin. Thus this suffices as a design consideration.  $T_i = 0.0159$ .

In order to design a digital controller, we will organize the controller terms.

$$G(s) = \frac{p(s)}{r(s)} = K \frac{\alpha\tau s + 1}{\tau s + 1} \frac{T_i s + 1}{T_i s} = K \frac{\alpha\tau s + 1}{\tau s + 1} \left( 1 + \frac{1}{T_i s} \right)$$

$$\begin{aligned}
(\tau s + 1)p(s) &= K(\alpha\tau s + 1) \left(1 + \frac{1}{T_i s}\right) r(s) \\
(\tau s + 1)p(s) &= K \left(\alpha\tau s + \left(1 + \frac{\alpha\tau}{T_i}\right) + \frac{1}{T_i s}\right) r(s) \\
(\tau s^2 + s)p(s) &= K \left(\alpha\tau s^2 + \left(1 + \frac{\alpha\tau}{T_i}\right) s + \frac{1}{T_i}\right) r(s) \\
\tau\ddot{p} + \dot{p} &= K \left(\alpha\tau\ddot{r} + \left(1 + \frac{\alpha\tau}{T_i}\right) \dot{r} + \frac{1}{T_i} r\right) \\
\tau\dot{p} + p &= K \left(\alpha\tau\dot{r} + \left(1 + \frac{\alpha\tau}{T_i}\right) r + \frac{1}{T_i} \int r\right) \\
\tau \frac{p(k) - 2p(k-1) + p(k-2)}{T^2} + \frac{p(k) - p(k-1)}{T} &= \\
K \left( \alpha\tau \frac{r(k) - 2r(k-1) + r(k-2)}{T^2} + \left(1 + \frac{\alpha\tau}{T_i}\right) \frac{r(k) - r(k-1)}{T} + \frac{1}{T_i} r(k) \right) \\
\tau[p(k) - 2p(k-1) + p(k-2)] + T[p(k) - p(k-1)] &= \\
K \left( \alpha\tau r(k) - 2r(k-1) + r(k-2) + \left(1 + \frac{\alpha\tau}{T_i}\right) [r(k) - r(k-1)]T + \frac{T^2}{T_i} r(k) \right) \\
p(k) &= \frac{1}{T + \tau} [(2\tau + T)p(k-1) - \tau p(k-2)] + \\
\frac{K}{T + \tau} \left[ \left( \alpha\tau + T + \frac{T\alpha\tau}{T_i} + \frac{T^2}{T_i} \right) r(k) - \left( 2\alpha\tau + \left[ 1 + \frac{\alpha\tau}{T_i} \right] T \right) r(k-1) + \alpha\tau r(k-2) \right]
\end{aligned}$$

**(i and ii)** See Figure 9. Reducing the sampling rate does not drastically change the accuracy of the model as long as the sampling rate is twice the bandwidth. Decreasing the sampling frequency even further introduces aliasing, which causes deviations from the actual value of the response. The system goes unstable with a sampling time of  $\frac{1}{477}s$ .

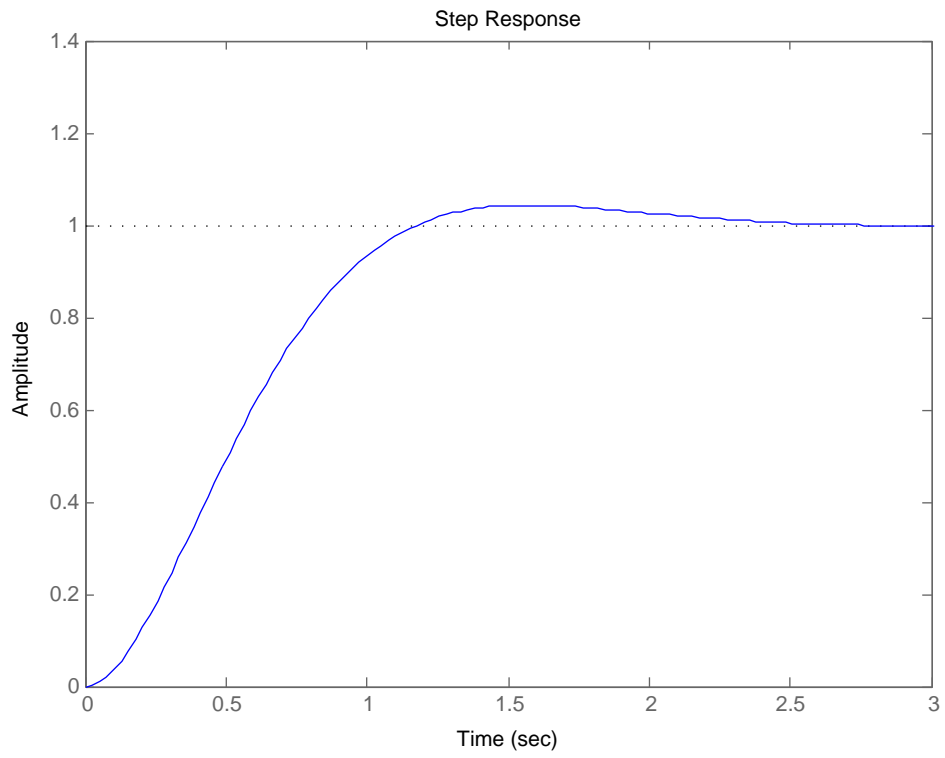


Figure 1: Problem 2.9

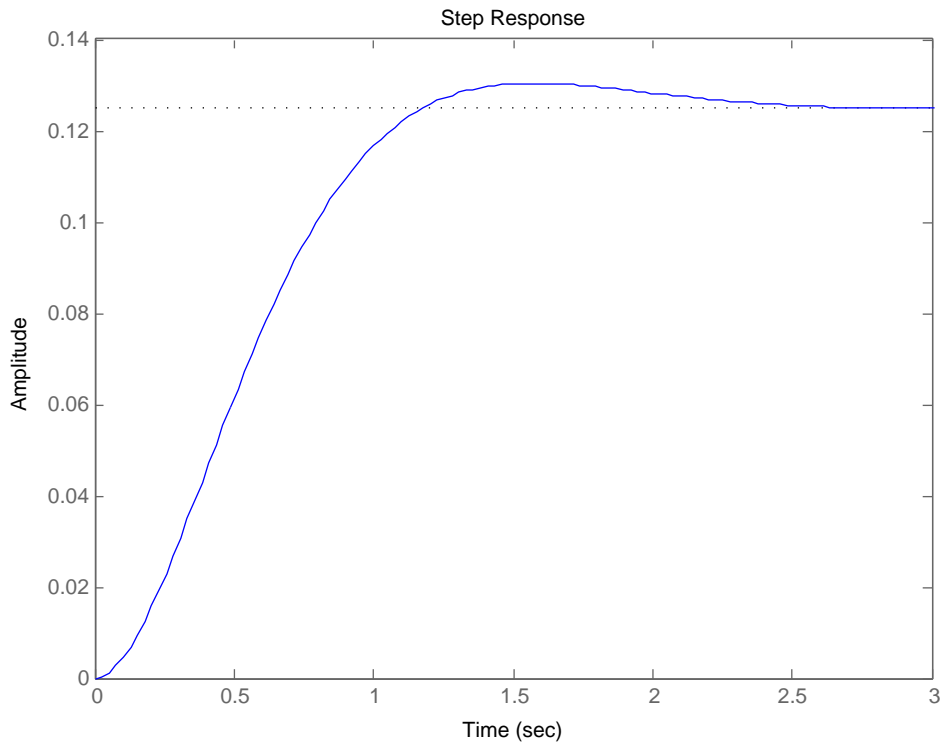


Figure 2: Problem 2.9

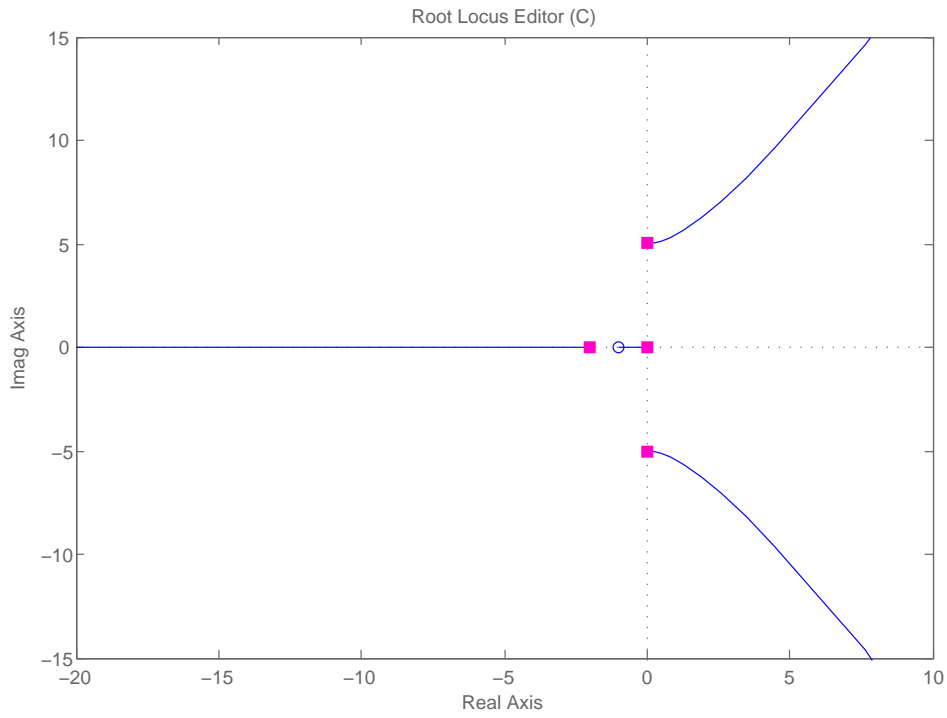


Figure 3: Problem 2.17

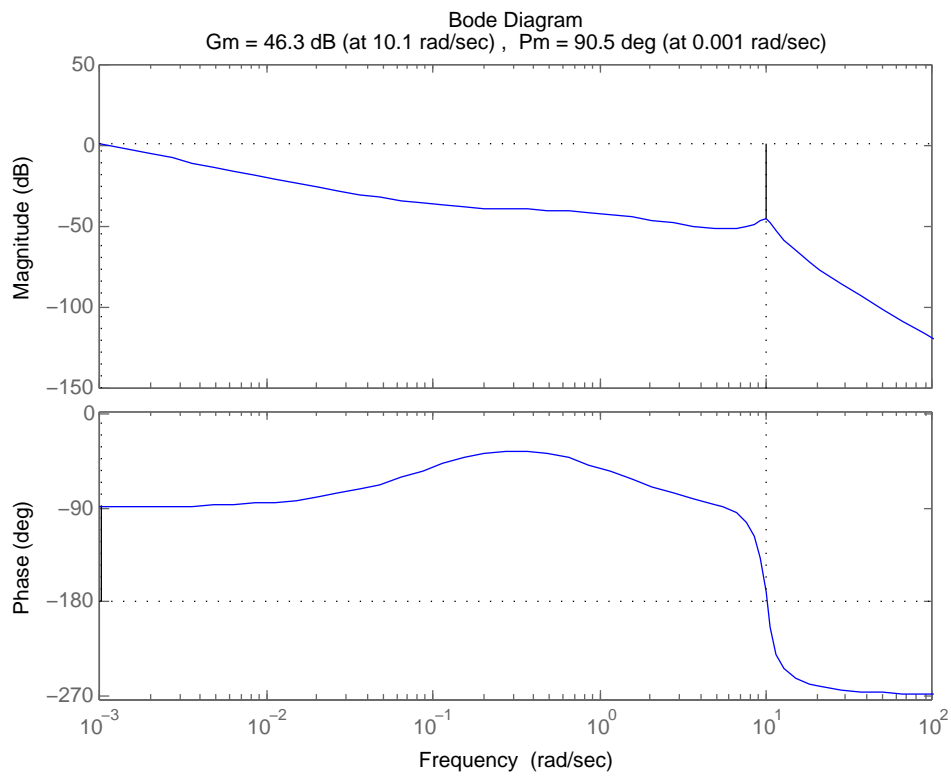


Figure 4: Problem 2.18



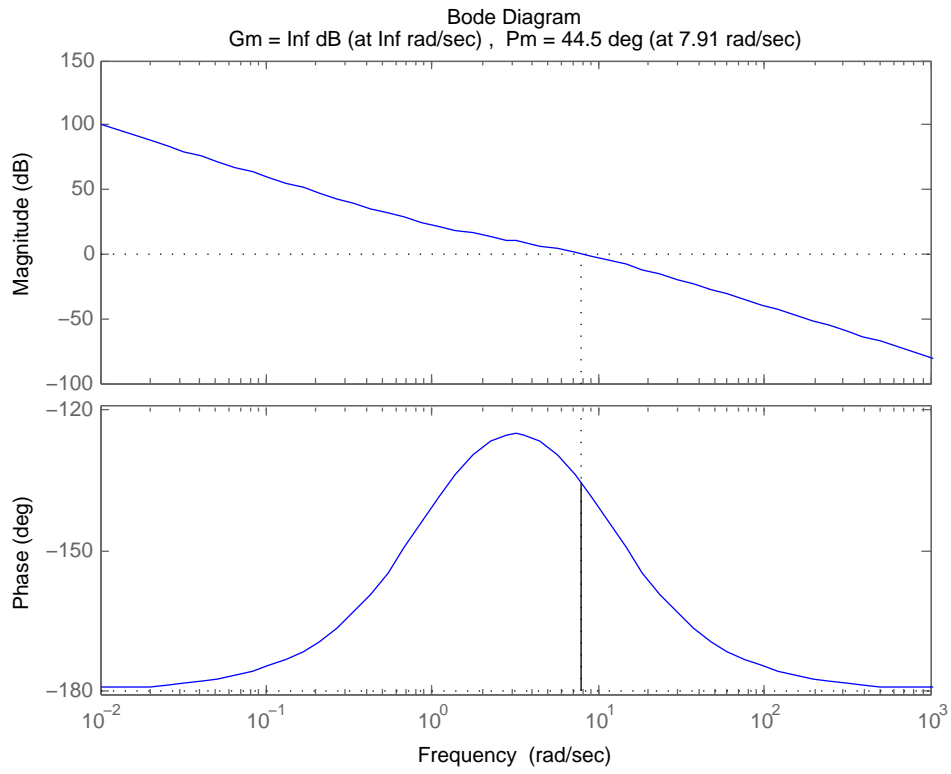


Figure 5: Problem 2.19

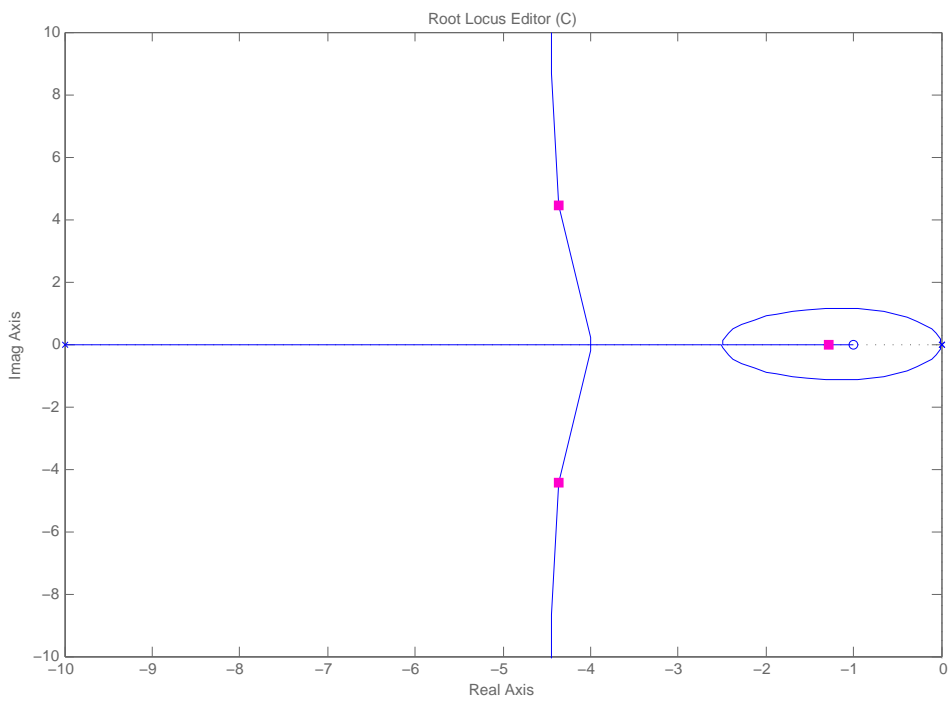


Figure 6: Problem 3.1a

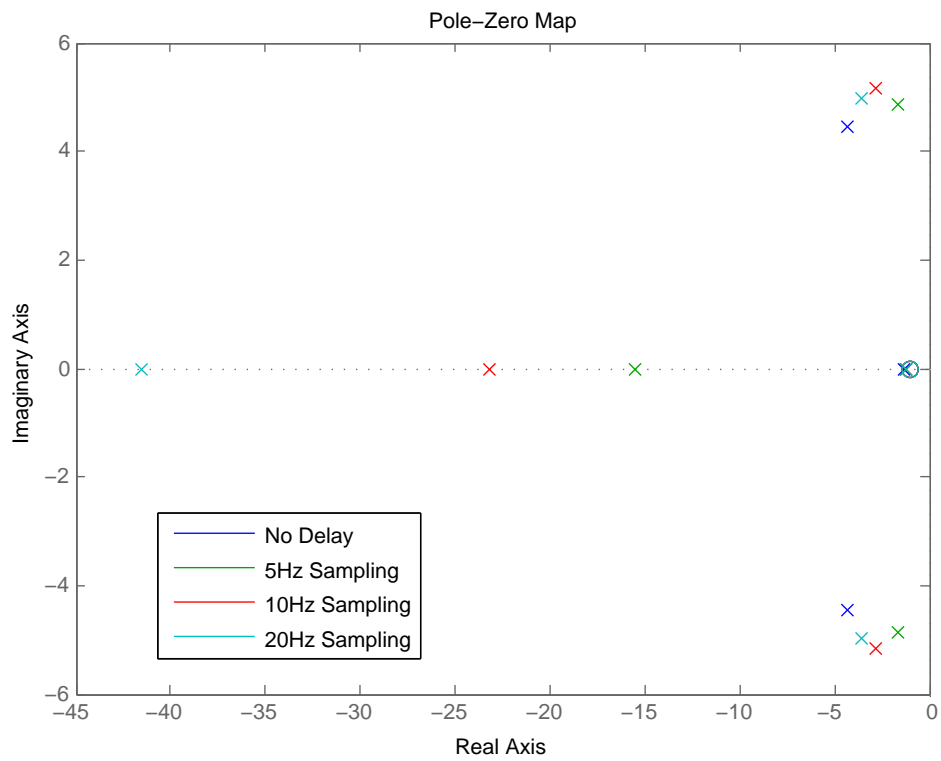


Figure 7: Problem 3.1b

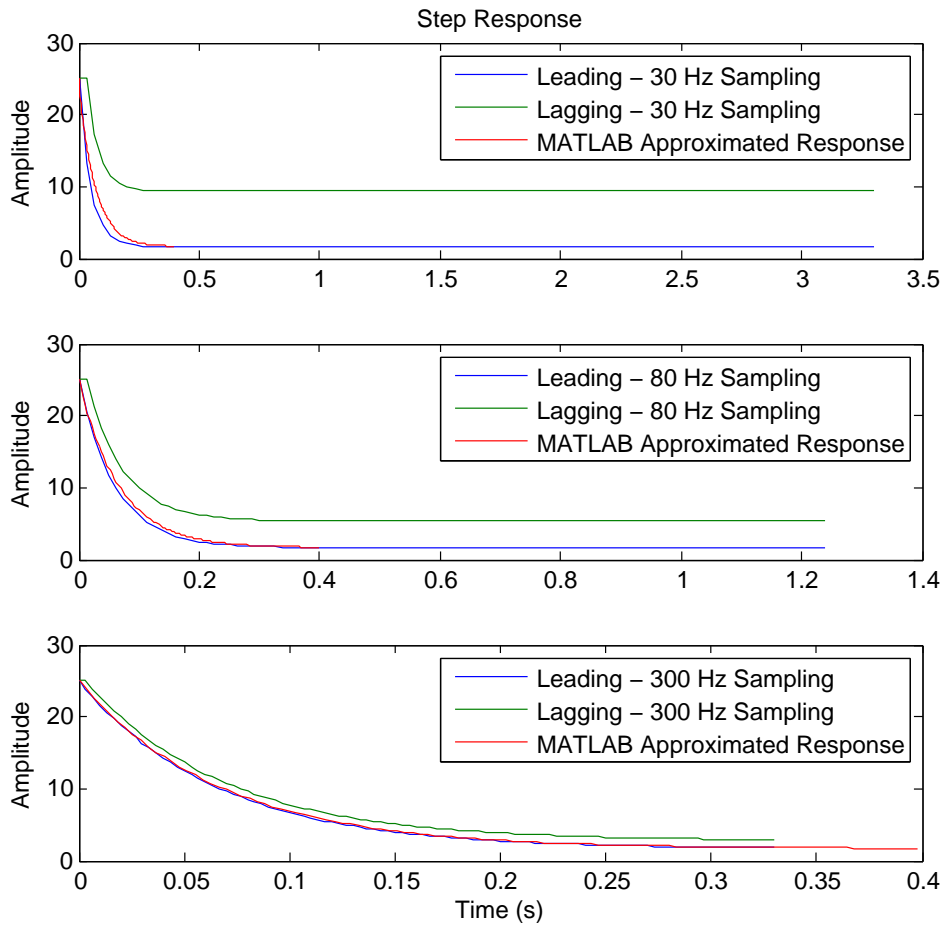


Figure 8: Problem 3.3

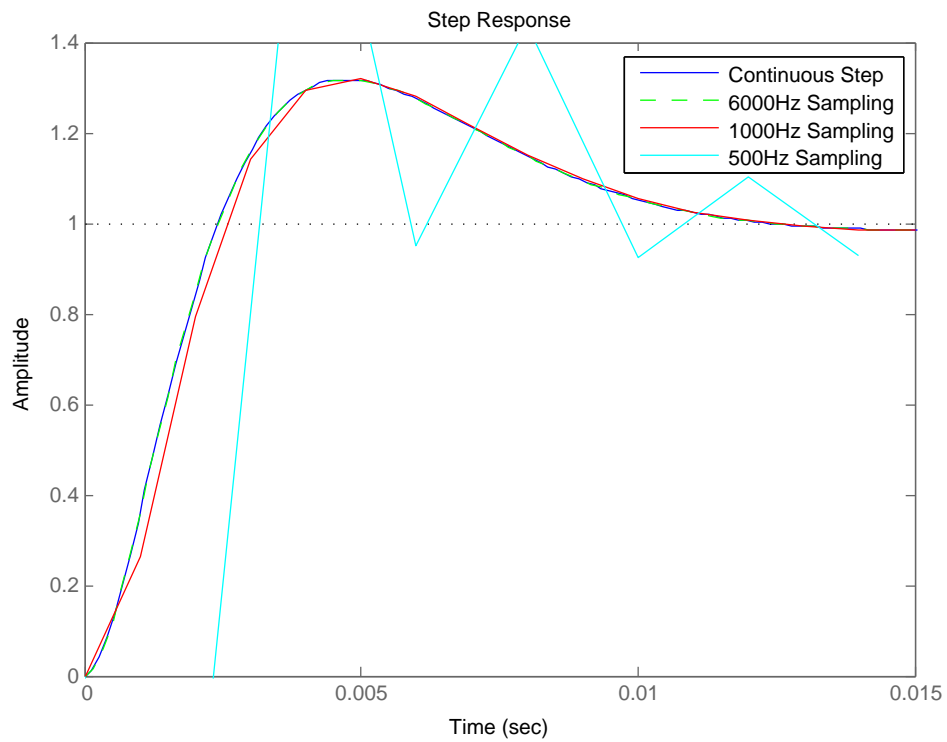


Figure 9: Problem 3.5